Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 5

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Let q, φ be quadratic forms over a field F. Recall that the form φ is called a *subform* of q (we write $\varphi \subseteq q$) if $q \simeq \varphi \perp q'$ for some quadratic form q' over F.

Exercise 1. Let F be a field and let q, φ be quadratic forms over F.
(1) Show that $\varphi \subseteq q$ if and only if $i(q \perp -\varphi) \geq \dim \varphi$.
Hint: To show the direction "$\Longleftarrow$" denote by $\widetilde{q}$ the anisotropic part of $q \perp -\varphi$ and write $q \perp -\varphi = \widetilde{q}$ in $W(F)$. Then use Remark 3.8 from the lecture: if f and g are two quadratic forms over F of the same dimension and $f = g$ in the Witt ring $W(F)$, then $f \simeq g$.
(2) Let f, g be two anisotropic quadratic forms over F. Assume $i(f \perp -g) \geq n$. Show that there exists a quadratic form φ of dimension n, such that $\varphi \subseteq f$ and $\varphi \subseteq g$.

Exercise 2. Let F be a field and let $K = F(\alpha)$ be a quadratic field extension, where $\alpha^2 = a \in F^\times$.
(1) Let q be an anisotropic quadratic form over F. Show that q_K is isotropic over K if and only if $b\langle 1, -a\rangle$ is a subform of q for some $b \in F^\times$.
(2) Show that the kernal of the ring homomorphism $W(F) \longrightarrow W(K)$ is the principal ideal in $W(F)$ generated by the form $\langle 1, -a\rangle$.
Hint: Use question (1).

Exercise 3. Let F be a field. For a quadratic form q over F denote by $D(q)$ the set of non-zero values of q.
(1) Let $K = F(\alpha)$ be a quadratic field extension, where $\alpha^2 = d \in F^\times$. Find the norm $N_{K/F}(x + \alpha y)$, where $x, y \in F$.
(2) Let $a \in F^\times$ and let $q = \langle 1, a\rangle$ be the 2-dimensional form over F. Show that $D(q)$ is a subgroup in $F^\times$.
Hint: Use the question (1) and recall that the norm $N_{K/F}$ is multiplicative, i.e. $N_{K/F}(zz') = N_{K/F}(z)N_{K/F}(z')$ for every $z, z' \in L$.

Let A be a quaternion algebra over F, i.e. A is a 4-dimensional algebra over F with a basis $\{1, i, j, k\}$, where the multiplicative structure is given by: $i^2 = a$, $j^2 = b, k = ij = -ji$. Note that this algebra is not commutative, in particular, any two different elements from $\{i, j, k\}$ anticommute.
(3) For $u = x + yi + zj + wk \in A$, $x, y, z, w \in F$ define $\bar{u} = x - yi - zj - wk$.

Show that $\overline{uv} = \bar{v}\bar{u}$ for every $u, v \in A$.
Hint: Write $u = (x + yi) + (z + wi)j$.
(4) For $u \in A$ define the norm $N(u) = u\bar{u}$. Show that N is multiplicative, i.e. $N(uv) = N(u)N(v)$ for every $u, v \in A$. Compute explicitely $N(x+yi+zj+wk)$.
(5) Show that $D(q)$ is a subgroup in $F^{\times}$ for every 4-dimensional form

$$q = \langle 1, -a \rangle \otimes \langle 1, -b \rangle$$

over F, where $a, b \in F^{\times}$.