Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 4

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Exercise 1. Let p be a prime number. Recall that the ring of p-adic integers is defined as follows:

$$\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \{(x_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}\colon\ x_n \in \mathbb{Z}/p^n\mathbb{Z},\ x_{n+1} \equiv x_n \bmod p^n\}.$$

(1) Show that $\mathbb{Z}_p^\times = \{(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p \mid x_1 \neq 0\}$.
(2) Show that $\mathbb{Z} \to \mathbb{Z}_p$, $a \mapsto (a \bmod p^n)_{n\in\mathbb{N}}$, is an injective ring homomorphism. We identify the image of this map with $\mathbb{Z}$.
(3) Consider the ring homomorphism $e_m : \mathbb{Z}_p \to \mathbb{Z}/p^m\mathbb{Z}$, $(x_n)_{n\in\mathbb{N}} \mapsto x_m$. Show that the kernel of e_m is the principal ideal (p^m) in $\mathbb{Z}_p$.
(4) Denote by $\mathbb{Q}_p$ the field of fractions of $\mathbb{Z}_p$. Show that $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$.
(5) Show that every element $x \in \mathbb{Q}_p$ can be uniquely written in the form $x = p^k u$, where $u \in \mathbb{Z}_p^\times$. Define $v : \mathbb{Q}_p^\times \to \mathbb{Z}$, $v(p^k u) = k$. Show that v is a discrete valuation on $\mathbb{Q}_p$.
(6) Show that $\mathbb{Q}_p$ is complete with respect to the p-adic metric $d(x, y) = 1/p^{v(x-y)}$. What is the residue field of $\mathbb{Q}_p$?
(7) Show that the sequence $(\sum_{k=0}^{n} p^k)_{n\in\mathbb{N}}$ converges to $\frac{1}{1-p}$ with respect to p-adic metric.
(8) Show that $\mathbb{Q}$ is not complete with respect to p-adic metric.

Exercise 2. (1) Find all odd prime numbers p, such that the quadratic form $\langle 3, 7, -15\rangle$ is isotropic over $\mathbb{Q}_p$.
(2) For which odd prime numbers p does the quadratic form $\langle 3, 3, 11\rangle$ represent 2 over $\mathbb{Q}_p$?
(3) Let $K = \mathbb{R}((t))$. Determine whether the quadratic form $\langle t, -t^2, \frac{t-1}{t+1}, -t^2 + t^3\rangle$ is isotropic or anisotropic over K?

Exercise 3. Let p be an odd prime number. Let $u \in \mathbb{Q}_p$ be such that $u \in U$ (that is $v_p(u) = 0$) and $\bar{u} \notin \mathbb{F}_p^{\times 2}$. Show that the form $q := \langle 1, -u, -p, pu\rangle$ is the unique (up to an isometry) 4-dimensional anisotropic form over $\mathbb{Q}_p$.