Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 3

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Exercise 1. Let F be a field. Show that the Witt ring $W(F)$ is finite if and only if -1 is a sum of squares in F and $F^\times/F^{\times 2}$ is finite.
Hint: To prove the implication "$\Longrightarrow$" use Remark 3.8 from the lecture: two anisotropic quadratic forms over F are equal in $W(F)$ if and only if they are isometric.

Exercise 2. Let p be an odd prime number and let $F = \mathbb{F}_p$ be the field with p elements.
(1) Show that $|F^\times/F^{\times 2}| = 2$.
(2) Show that every quadratic form of dimension 3 over F is isotropic.
(3) Find all isometry classes of anisotropic quadratic forms over F. Compute $W(F)$.
Hint: In the questions (2) and (3) consider two cases: -1 is a square in F and -1 is not a square in F.

Exercise 3. Let F be a field and let $K = F(t)$ be the rational function field.
(1) Show that a quadratic form q over F is anisotropic if and only if q_K is anisotropic over K.
Hint: Assume $q = \langle a_1, ..., a_n\rangle$, $a_i \in F^\times$. If q_K is isotropic then write a solution in the form $(f_1(t), ..., f_n(t))$, where $f_1, ..., f_n \in F[t]$ and $gcd(f_1, ..., f_n) = 1$.
(2) Deduce from (1) that $i(q) = i(q_K)$ for every quadratic form q over F and that the ring homomorphism $W(F) \longrightarrow W(K)$ is injective.