Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 2

PD Dr. Maksim Zhykhovich Summer Semester 2025, 08.05.2025

Exercise 1. Let (V, q) be a quadratic form over a field F. Recall that the vector subspace $U \subset V$ is called *totally isotropic* if $q(x) = 0$ for every $x \in U$.
(1) Show that

$$i(q) = \max\{\dim U \mid U \text{ is a totally isotropic vector subspace in } (V, q)\} ,$$

where $i(q)$ denotes the Witt index of q.
Hint: Proceed by induction on $\dim q$.
(2) Let W be a vector subspace of V, such that $\dim W + i(q) > \dim V$. Show that the quadratic form $q_{|W}$ is isotropic.

Exercise 2. Let V be a vector space over a field F. Let $V^* = \mathrm{Hom}_F(V, F)$ be the dual space. Consider the map $q : V \oplus V^* \to F, (v, f) \mapsto f(v)$.
(1) Show that $(V \oplus V^*, q)$ is a non-degenerate quadratic form.
(2) Show that q is hyperbolic.
Hint: Use Exercise 1.1.

Exercise 3. Let F be a field. Let c be an element in $F^\times$, such that $c = a^2 + b^2$ for some $a, b \in F$.
(1) Show that the 4-dimensional form $\langle 1, 1, -c, -c\rangle$ is hyperbolic.
Hint: First show that the form $\langle 1, 1, -c, -c\rangle$ is isotropic.
(2) Show that the quadratic form $\langle 1, -c, -c\rangle$ is isotropic.
Hint: Use Exercise 1.2.

Exercise 4. Let (V_1, q_1), (V_2, q_2) be two quadratic forms over a field F and set $V = V_1 \otimes V_2$.
Show that there exists a unique bilinear form $B : V \times V \to F$ satisfying

$$B(v_1 \otimes v_2, v_1' \otimes v_2') = B_{q_1}(v_1, v_1')B_{q_2}(v_2, v_2') .$$

for every $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$.