Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 10

PD Dr. Maksim Zhykhovich Summer Semester 2025, 11.07.2025

Exercise 1. Let F be a field and $a, b, c \in F^\times$. Assume that the quaternion algebras $(\frac{a,b}{F})$ and $(\frac{a,c}{F})$ are split. Show that $(\frac{a,bc}{F})$ is also split.
Remark: Give two proofs: one using Wedderburn's theorem and one using norm forms.

Exercise 2. (Frobenius theorem)
(1) Show that any finite dimensional division $\mathbb{R}$-algebra A is isomerphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathcal{H}$, where $\mathcal{H} = (\frac{-1,-1}{\mathbb{R}})$ denotes the algebra of Hamilton's quaternions.
Hint: Observe that any $\mathbb{R}$-subalgebra of A is also division. Consider the center $Z(A)$ of A and show that $Z(A) \simeq \mathbb{R}$ or $Z(A) \simeq \mathbb{C}$. In case $Z(A) = \mathbb{R}$ proceed as in Exercise 2, Sheet 9.
(2) Describe all central simple algebras over $\mathbb{R}$ up to isomorphism.

Exercise 3. Let F be a field and $A = (\frac{a,b}{F}) \otimes (\frac{c,d}{F})$ a biquaternion algebra over F, where $a, b, c, d \in F^\times$. Let $q = \langle -a, -b, ab, c, d, -cd \rangle$ be the associated Albert form of A. Show the following implications:
(1) q is hyperbolic $\Longleftarrow$ A is split (that is $A \simeq M_4(F)$).
Hint: Consider $(\frac{a,b}{F}) \otimes_F A$ and use the "uniqueness" from Wedderburn's theorem.
(2) q is isotropic $\Longleftarrow$ A is not division.
Hint: Assume A is not division. Using Wedderburn's theorem show that A is split over some quadratic extension of F. Then use (1).
Remark: The direction $\Longrightarrow$ in (1) and (2) is already proven in Exercise Sheet 9, Exercise 3.

Exercise 4. Let A and B be two F-algebras (not necessarily finite dimensional).
(1) Show that $Z(A \otimes_F B) = Z(A) \otimes_F Z(B)$.
Hint: Let $\{x_i\}_{i \in I}$ be an F-basis of B. Write an element from $Z(A \otimes_F B)$ in the form $\sum_{i \in I} a_i \otimes x_i$ for some $a_i \in A$. What can one say about the elements a_i?
(2) Assume that A is central simple. Show that the two-sided ideals in $A \otimes_F B$ are of the form $A \otimes_F J$, where J is a two-sided ideal in B.
Hint: Let $\widetilde{J}$ be a two-sided ideal in $A \otimes_F B$. Observe that $J := \widetilde{J} \cap (1 \otimes B)$ is a two-sided ideal in B and show that $\widetilde{J} = A \otimes_F J$.