Algebraic theory of quadratic forms and Kaplansky's problem

Exercise Sheet 1

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Exercise 1. (1) Let (V, q), (V', q') be two quadratic forms over a field F and let $\varphi : (V, q) \to (V', q')$ be an isometry. Recall that $\varphi : V \to V'$ is an isomorphism of vector spaces, such that $q(x) = q'(\varphi(x))$ for every $x \in V$. Let B_q and $B_{q'}$ be the bilinear forms associated to q and q' respectively.
(1) Show that $B_q(x, y) = B_{q'}(\varphi(x), \varphi(y))$ for every $x, y \in V$.
Deduce that: $x \perp y \iff \varphi(x) \perp \varphi(y)$.
(2) Let U be a vector subspace of V. Show that $\varphi(U^\perp) = \varphi(U)^\perp$.
(3) Let $\psi : (V', q') \to (V'', q'')$ be another isometry. Show that the composition $\psi \circ \varphi : (V, q) \to (V'', q'')$ is also an isometry.
(4) Show that $O(V, q) = \{\text{isometries of } (V, q)\}$ is a group. This group is called the *orthogonal group* of q.

Exercise 2. (1) Let q_1 and q_2 be two 2-dimensional quadratic forms over a field F. Show that $q_1 \simeq q_2$ if and only if both q_1, q_2 represent the same element $a \in F$ and $\det q_1 = \det q_2$.
(2) Find a field F and two 2-dimensional quadratic forms q_1 and q_2 over F, such that $\det q_1 = \det q_2$ but q_1 and q_2 are not isometric.

Exercise 3. (1) Let q_1 and q_2 be two isotropic 3-dimensional quadratic forms over a field F. Show that $q_1 \simeq q_2$ if and only if $\det q_1 = \det q_2$.
(2) Find a field F and two 3-dimensional anisotropic quadratic forms q_1 and q_2 over F, such that $\det q_1 = \det q_2$ but q_1 and q_2 are not isometric.

Exercise 4. Let F be a field. Recall that a quadratic form (V, q) *represents* $a \in F$ if there exists $v \in V$, $v \neq 0$, such that $q(v) = a$. We say that q is *isotropic* if it represents 0.
(1) Let q be an isotropic quadratic form over F. Show that q represents every $a \in F$.
Hint: Let $v \in V$, $v \neq 0$, with $q(v) = 0$. Choose $w \in V$, such that $B_q(v, w) \neq 0$ and consider $q(\alpha v + w)$, where $\alpha \in F^\times$.
(2) Show that a quadratic form f over F represents $a \in F^\times$ if and only if the quadratic form $f \perp \langle -a \rangle$ is isotropic.