Algebraic theory of quadratic forms and central simple algebras

Exercise Sheet 1

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All quadratic forms are supposed to be non-degenerate (except in Exercise 1).

Exercise 1. Let (V, q) be a quadratic form over a field F. Show that the map $\widetilde{q} : V/\operatorname{Rad} q \to F$, $\widetilde{q}(\bar{v}) := q(v)$, is a well-defined non-degenerate quadratic form over F.

Exercise 2. (1) Let (V, q), (V', q') be two quadratic forms over a field F and let $\varphi : (V, q) \to (V', q')$ be an isometry. Recall that $\varphi : V \to V'$ is an isomorphism of vector spaces, such that $q(x) = q'(\varphi(x))$ for every $x \in V$. Let B_q and $B_{q'}$ be the bilinear forms associated to q and q' respectively.
(1) Show that $B_q(x, y) = B_{q'}(\varphi(x), \varphi(y))$ for every $x, y \in V$.
Deduce that: $x \perp y \iff \varphi(x) \perp \varphi(y)$.
(2) Let U be a vector subspace of V. Show that $\varphi(U^\perp) = \varphi(U)^\perp$.
(3) Let $\psi : (V', q') \to (V'', q'')$ be another isometry. Show that the composition $\psi \circ \varphi : (V, q) \to (V'', q'')$ is also an isometry.
(4) Show that $O(V, q) = \{\text{isometries of } (V, q)\}$ is a group. This group is called the *orthogonal group* of q.

Exercise 3. Let (V, q) be a quadratic form over a field F. Let $y \in V$ be an anisotropic vector (i.e. $q(y) \neq 0$). Recall that the *reflection* $\tau_y : V \to V$ is a linear map given by a formula

$$\tau_y(x) = x - \frac{2B_q(x, y)}{q(y)} y \qquad (1)$$

for every $x \in V$, where B_q is the bilinear form associated to q.
Let now $x, y \in V$, such that $q(x) = q(y)$ and $q(x - y) \neq 0$. Using formula (1) show that the reflection τ_{x-y} sends x to y.

Exercise 4. Let F be a field. Recall that a quadratic form (V, q) *represents* $a \in F$ if there exists $v \in V$, $v \neq 0$, such that $q(v) = a$. We say that q is *isotropic* if it represents 0.
(1) Let q be an isotropic quadratic form over F. Show that q represents every $a \in F$.
Hint: Let $v \in V$, $v \neq 0$, with $q(v) = 0$. Choose $w \in V$, such that $B_q(v, w) \neq 0$ and consider $q(\alpha v + w)$, where $\alpha \in F^\times$.

(2) Show that a quadratic form f over F represents $a \in F^\times$ if and only if the quadratic form $f \perp \langle -a \rangle$ is isotropic.