LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Dr. Maksim Zhykhovich

Summer Term 2021
13.07.2021

Algebraic theory of quadratic forms

Exam (Hausarbeit)

Family Name: ______________ First name: ______________

Student ID: ______________ Term: ______________

Degree course: Bachelor, PO ❑ 2010 ❑ 2011 ❑ 2015 Master, PO ❑ 2010 ❑ 2011

Lehramt Gymnasium: ❑ modularisiert ❑ nicht modularisiert

❑ Diplom ❑ Other: ______________

Major subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ ________

Minor subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ ________

Credit Points to be used for ❑ Hauptfach ❑ Nebenfach (Bachelor / Master)

Please switch off your mobile phone and do not place it on the table; place your identity and student ID cards on the table so that they are clearly visible.
Please check that you have received all **four problems**.
Please do not write with the colours red or green. Write **on every page** your **family name and your first name**.
Write your solutions on the page marked with the appropriate problem number. If you run out of space, use the empty pages at the end of the examination paper ensuring that each problem is clearly marked.
Please make sure to submit only one solution for each problem; cross out everything that should not be graded.
You have **110 minutes** in total to complete this examination.

Good luck!

1	2	3	4	$\sum$
/12	/12	/18	/12	/54

Name: ____________________

Problem 1. **[12 Points]**

1) Find the Witt index of the quadratic form $\langle 1, 1, -2, -3\rangle$ over a field F in the following cases:
a) $F = \mathbb{R}$.
b) $F = \mathbb{Q}$.
c) $F = \mathbb{F}_5$, where $\mathbb{F}_5$ is the finite field with 5 elements. [2+2+2=6 Points]

2) Show that the following quadratic forms are isometric over $\mathbb{Q}$.
a) $\langle 1, 1, 2\rangle \simeq \langle 1, 3, 6\rangle$,
b) $\langle 1, 1, -5\rangle \simeq \langle 2, -2, 5\rangle$. [3+3=6 Points]

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Problem 2. **[12 points]**

Let F be a field and let $a, b, c \in F^\times$.
Show that the quadratic form

$$\langle a, b, c, ab, ac, bc \rangle$$

is hyperbolic if and only if $-abc$ is a square in $F^\times$.

Name: ______________________________

Problem 3. **[18 Points]**

Let F be a field.
1) Assume that $s(F) = +\infty$ and $|F^\times / F^{\times 2}| = 2$. Show that $W(F) \simeq \mathbb{Z}$. In particular, $W(F)$ is an integral domain. [8 Points]

2) Assume that $F^\times \neq F^{\times 2}$ and $W(F)$ is an integral domain. Show that $s(F) = +\infty$ and $|F^\times / F^{\times 2}| = 2$. [10 Points]

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Problem 4. **[12 Punkte]**

Let F be a field. Let π be an n-fold Pfister form over F and let φ be an m-fold Pfister form over F with $m > n > 0$.
Assume that $\pi \subseteq \varphi$ (π is a subform of φ). Show that $\varphi \simeq \pi \otimes \pi'$ for some Pfister form π'.
Hint: Start with the case $m - n = 1$ and then proceed by induction on $m - n$. Use function fields of quadratic forms.

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