Algebra 2

Tutorium 9

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Exercise 1. We know from the lecture that $\mathbb{Q}$ is flat over $\mathbb{Z}$. Show that $\mathbb{Q}$ is not faithfully flat over $\mathbb{Z}$.

Exercise 2. Let p, q be two different prime numbers. Consider $\mathbb{Z}/(p)$ as a module over $\mathbb{Z}/(pq)$. Show that $\mathbb{Z}/(p)$ is projective but not faithfully flat.

Exercise 3. Let R be a commutative ring, $S \subset R$ a multiplicatively closed subset. Show that $S^{-1}R = 0$ if and only if $0 \in S$.

Exercise 4. Let R be a commutative ring, $S \subset R$ a multiplicatively closed subset. Let $\varphi : R \to S^{-1}R$, $a \mapsto \frac{a}{1}$, be the canonical ring homomorphism.
Show that φ is an isomorphism if and only if $S \subset R^{\times}$.
Remark: Give two proofs: one, by directly showing that φ is injective and surjective if and only if $S \subset R^{\times}$, and another proof by using the universal property of $S^{-1}R$.

Exercise 5. Let R be a commutative ring, $S \subset R$ a multiplicatively closed subset and $I \subset R$ an ideal of R. Let $\varphi : R \to S^{-1}R$, $a \mapsto \frac{a}{1}$, be the canonical ring homomorphism.
(1) Show that $S^{-1}I := \{\frac{i}{s} \mid i \in I, s \in S\} \subset S^{-1}R$ is an ideal in $S^{-1}R$.
(2) Show that the ideal in $S^{-1}R$ generated by the set $\varphi(I)$ is equal to $S^{-1}I$.
(3) Assume I is prime in R and $I \cap S = \varnothing$. Show that $S^{-1}I$ is prime in $S^{-1}R$.