Algebra 2

Tutorium 7

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Exercise 1. Let A be a commutative ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of A-modules. Show that the following conditions are equivalent:
(1) There exists an A-linear map $i : M \to M'$ such that $i \circ f = id_{M'}$.
(2) There exists an A-linear map $j : M'' \to M$ such that $g \circ j = id_{M''}$.
(3) There exists an isomorphism $h : M \to M' \oplus M''$, such that $h \circ f$ is a natural injection of M' into the direct sum, and $g \circ h^{-1}$ is the natural projection of the direct sum onto M''.
Remark: The exact sequence satisfying one of the above conditions is called *split*.

Exercise 2. Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1 + \sqrt{-5})$ an ideal of R.
(1) Show that I is projective over R.
Hint: Show that I is a retract (direct summand) of R^2 by considering the map $I \to R^2, a \mapsto (-a, (\frac{1-\sqrt{-5}}{2})a)$.
(2) Show that I is not free as R-module.

Exercise 3. Let R be a commutative ring. Let $I \subset R$ be a finitely generated ideal of R, such that $I = I^2$. Show that I is principal and generated by an idempotent (that is $I = (e)$, where $e^2 = e$).

Exercise 4. Let R be a commutative ring, $I \subset R$ a finitely generated ideal of R. Show that the following conditions are equivalent:
(1) R/I is flat over R.
(2) I is principal and generated by an idempotent.
(3) R/I is projective over R.