Algebra 2

Tutorium 6

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Exercise 1. Let A be a commutative ring, M, N two A-modules. Assume that N is free with A-basis $\{n_i\}_{i\in I}$.
a) Show that:

$$\sum_{i\in I} m_i \otimes n_i = 0 \text{ in } M \otimes_A N \iff m_i = 0 \text{ for all } i \in I\,,$$

where $m_i \in M$ and $m_i \neq 0$ only for finitely many $i \in I$.
Hint: Find an appropriate A-linear morphisms $f : M \to M'$ and $g : N \to N'$ and apply $f \otimes g$.
b) Show that every free A-module is flat over A.
c) Assume that M is also free with A-Basis $\{m_j\}_{j\in J}$.
Show that: $M \otimes_A N$ ist a free A-module with basis $\{m_j \otimes n_i \mid j \in J, i \in I\}$.

Exercise 2. Let A be a commutative ring, M a flat A-modules, a an element in A which is not zero divisor. Assume $am = 0$ for some $m \in M$. Show that $m = 0$.

Exercise 3. Let A be a commutative ring and let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be an exact sequence of A-modules. Show that the following conditions are equivalent:
1) There exists an A-linear map $i : M \to M'$ such that $i \circ f = id_{M'}$.
2) There exists an A-linear map $j : M'' \to M$ such that $g \circ j = id_{M''}$.
3) There exists an isomorphism $h : M \to M' \oplus M''$, such that $h \circ f$ is a natural injection of M' into the direct sum, and $g \circ h^{-1}$ is the natural projection of the direct sum onto M''.