Algebra 2

Tutorium 4

Prof. Markus Land | Summer Semester 2023
Dr. Maksim Zhykhovich | 18.05.2023

Exercise 1. Let A be a commutative ring. Consider $Spec\, A$ with the Zariski topology. Recall that every closed subset in $Spec\, A$ is of the form $V(I) := \{\mathfrak{p} \in Spec\, A \mid I \subset \mathfrak{p}\}$ for some subset I in A.
(1) Assume that $Spec\, A$ is not connected. Show that there exists a non-trivial idempotent $e \in A$ (that is $e^2 = e$, $e \neq 0, 1$) and, therefore, $A \simeq A_1 \times A_2$, where A_1 and A_2 are non-zero rings.
(2) Let A be a local ring. Show that $Spec\, A$ is connected.

Exercise 2. Let A be a commutative finite ring. Show that $\mathcal{N}_A = \mathcal{J}_A$.

Exercise 3. (1) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.
(2) Let $n, m \in \mathbb{Z}$. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$, where d is the greatest common divisor of n and m.