Algebra 2

Tutorium 2

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Exercise 1. Let A be a commutative ring and $\mathfrak{m}$ a maximal ideal of A. Show that A is a local ring if and only if $1 + \mathfrak{m} \subseteq A^\times$, where $1 + \mathfrak{m} = \{1 + x \mid x \in \mathfrak{m}\}$.

Exercise 2. Let A be a commutative ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

Exercise 3. Let K be a field and $a_1, ..., a_n \in K$.
Show that the ideal $(X_1 - a_1, ..., X_n - a_n)$ is maximal in the ring $K[X_1, ..., X_n]$.
Prove or disprove that every maximal ideal in $K[X_1, ..., X_n]$ has such a form.

Exercise 4. Let A be a ring. Consider $Spec\,A$ with the Zariski topology. Recall that every closed subset in $Spec\,A$ is of the form $V(I) := \{\mathfrak{p} \in Spec\,A \mid I \subset \mathfrak{p}\}$ for some subset I in A.
Assume that there exists a non-trivial idempotent $e \in A$, that is $e^2 = e$, $e \neq 0, 1$.
Show that
(1) $Spec\,A$ is not connected.
(2) $A \simeq A_1 \times A_2$, where neither of the rings A_1 and A_2 is the zero ring.