# Algebra 2

# Tutorium 11

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Let $R$ be a commutative ring and $M$ an $R$-module. The *support* of $M$ is the set

$$\mathrm{Supp}(M) := \{\rho \in \mathrm{Spec}\,(R) \mid M_\rho \neq 0\}\,.$$

The *annihilator* of $M$ is the following ideal in $R$

$$\mathrm{Ann}(M) := \{x \in R \mid xM = 0\}\,.$$

**Exercise 1.** Let $R$ be a commutative ring, $S$ a multiplicatively closed subset of $R$ and let $M$ be an $R$-module.
Show that:
(1) $S^{-1}M = 0$ if $\mathrm{Ann}(M) \cap S \neq \emptyset$.
(2) Assume that $M$ is finitely generated. Show that

$$S^{-1}M = 0 \iff \mathrm{Ann}(M) \cap S \neq \emptyset\,.$$

(3) Show that $\mathrm{Supp}(M) \subseteq V(\mathrm{Ann}(M))$.
(4) Show that $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ if $M$ is finitely generated.
(5) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $R$-modules.
Show that $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.
(6) Let $\mathfrak{m}$ be a maximal ideal of $R$. Show that $\mathrm{Supp}(R/\mathfrak{m}) = \{\mathfrak{m}\}$.

**Exercise 2.** Let $B$ be an integral domain and $A$ a subring of $B$ such that $B$ is integral over $A$. Show that $B$ is a field if and only if $A$ is a field.

**Exercise 3.** Show that the ring $\mathbb{Z}[X]/(X^2-3)$ is integral over $\mathbb{Z}$, and that the ring $\mathbb{Z}[X]/(2X^2-3)$ is not integral over $\mathbb{Z}$.
More generally, let $A$ be an integral domain and $P(X) \in A[X]$ a polynomial. Show that $A[X]/(P(X))$ is integral over $A$ if and only if the leading coefficient of $P(X)$ is a unit in $A$ (an element in $A$ is called a unit if it is invertible in $A$).