Algebra 2

Tutorium 10

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Exercise 1. Let R be a commutative ring, $S \subset R$ a multiplicatively closed subset of R.
(1) Show that $S^{-1}\mathcal{N}_R = \mathcal{N}_{S^{-1}R}$.
(2) Deduce from (1) that R is reduced if and only if $R_\mathfrak{m}$ is reduced for every maximal ideal $\mathfrak{m}$ of R.

Exercise 2. Let K be a field and let $n > 0$ be an integer. Consider the ring $R = \prod_{i=1}^n K$. Recall that $\operatorname{Spec} R = \{\mathfrak{m}_1, ..., \mathfrak{m}_n\}$, where $\rho_i = \varphi_i^{-1}(0)$ and $\varphi_i : R \to K$ is the projection to the i-th component (see Exercise 5.2, Exercise Sheet 2).
Show that $R_{\mathfrak{m}_i} \simeq K$ for every $i = 1, ..., n$.
Hint: Use the universal property of the localization $R_{\mathfrak{m}_i}$.
Remark: Note that R is a regular von Neumann ring and this exercise is a particular case of Exercise 1.1 (Exercise Sheet 10).

Exercise 3. Let R be a commutative ring. Let M be an Artinian R-module and $\varphi : M \to M$ an injective endomorphism. Show that φ is an isomorphism.

Exercise 4. Let R be an integral domain. For a prime ideal $\rho \in \operatorname{Spec} R$ we consider R_ρ as a subring in the quotient field $Q(R)$ of R.
Show that the following equality holds in $Q(R)$:

$$R = \bigcap_{\rho \in \operatorname{Spec} R} R_\rho \,.$$