Algebra 2

Exercise Sheet 9

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Exercise 1. Let $R \neq 0$ be a commutative ring and Σ the set of all multiplicatively closed subsets S of R with $0 \notin S$.
Show that:
a) Σ has a maximal element (with respect to inclusion).
b) $S \in \Sigma$ is maximal $\iff$ $A \backslash S$ is a minimal prime ideal.

Exercise 2. Let R be a commutative ring and S a multiplicatively closed subset of R. Let M be an R-modules and N, P submodules of M. We consider $S^{-1}N$, $S^{-1}P$ as $S^{-1}R$-submodules of $S^{-1}M$. Show that
(1) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$
(2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$
(3) $S^{-1}(M/N) \simeq S^{-1}M/S^{-1}N$ as $S^{-1}R$-submodules.

Exercise 3. Let R be a commutative ring. Denote by $\mathrm{Max}(R)$ the set of maximal ideal of R. Show that the R-module

$$\bigoplus_{\mathfrak{m} \in \mathrm{Max}(R)} R_{\mathfrak{m}}$$

is faithfully flat.

Exercise 4. Let R be a commutative ring and S a multiplicatively closed subset of R. Let M be an R-module.
Consider S as a category with $\mathrm{Hom}(s, s') = \{x \in R \mid xs = s'\}$ for any $s, s' \in S$. Consider the following functor $F : S \to Mod(R)$, $s \mapsto M_s := M$, which associate $x \in \mathrm{Hom}(s, s')$ to $\mu_x : M_s \to M_{s'}$, where μ_x is the multiplication by x (that is $\mu_x(m) = xm$). Show that the colimit of F is isomorphic to $S^{-1}M$ with maps $\alpha_s : M_s \to S^{-1}M, m \mapsto \frac{m}{s}$.