

Exercise 1

$$R \neq 0, \quad \Sigma = \{ S \subset R \mid S \text{ mult. closed in } R \}$$

$$\Sigma \neq \emptyset, \text{ since } \{1\} \in \Sigma$$

(a) To apply Zorn Lemma for Σ we need to show that every totally ordered subset $\{S_i\}_{i \in I}$ from Σ has an upper bound.

$$\text{Upper bound} = \bigcup_{i \in I} S_i \in \Sigma$$

$$\bullet 1 \in \bigcup_{i \in I} S_i$$

$$\bullet \text{ Let } x, y \in \bigcup_{i \in I} S_i$$

Then $x \in S_i$ and $y \in S_j$ for some $i, j \in I$

And without loss of generality $S_i \subseteq S_j$

$$\text{Then } x, y \in S_j \Rightarrow xy \in S_j \subseteq \bigcup_{i \in I} S_i$$

Then, by Zorn lemma, Σ has a max. element

$$(b) \quad S \in \Sigma \text{ maximal} \iff R \setminus S \text{ minimal Prime ideal.}$$

" \implies " ^{By} Since $0 \notin S$, $S^{-1}R \neq 0$ (Exercise 3, Tutorium 3)

$$\text{Then } \emptyset \neq \text{Spec } S^{-1}R \stackrel{\uparrow}{=} \{ S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R, \mathfrak{p} \cap S = \emptyset \}$$

Corollary 7.10(2)

$$\implies \exists \mathfrak{p} \in \text{Spec } R \text{ with } \mathfrak{p} \cap S = \emptyset \implies R \setminus \mathfrak{p} \in \Sigma \text{ and } S \subseteq R \setminus \mathfrak{p}$$

$$\text{By maximality of } S: \quad S = R \setminus \mathfrak{p}$$

Once again by max. of S : \mathfrak{p} is minimal prime ideal
If $\mathfrak{p}' \subsetneq \mathfrak{p}$ then $R \setminus \mathfrak{p} \subsetneq R \setminus \mathfrak{p}' \in \Sigma$

" \Leftarrow "

(2)

Let Assume $S = R \setminus \mathfrak{p}$, where \mathfrak{p} minimal prime ideal.

Assume S is not maximal in $\Sigma \Rightarrow S \neq \tilde{S}$ in Σ for some \tilde{S} .

$$\tilde{S}^{-1}R \neq 0 \Rightarrow \exists \mathfrak{q} \in \text{Spec } R \text{ with } \mathfrak{q} \cap \tilde{S} = \emptyset$$

That is $\mathfrak{q} \in R \setminus \tilde{S} \subset R \setminus S = \mathfrak{p} \downarrow$ since \mathfrak{p} minimal prime ideal.

Exercise 2

$$(1) \quad \tilde{S}^{-1}(N+P) = \left\{ \frac{n+p}{s} \mid \begin{array}{l} n \in N, \\ p \in P, \\ s \in S \end{array} \right\} \subseteq \tilde{S}^{-1}N + \tilde{S}^{-1}P = \left\{ \frac{n}{s_1} + \frac{p}{s_2} \mid \begin{array}{l} n \in N, p \in P \\ s_1, s_2 \in S \end{array} \right\}$$

$$\text{But } \frac{n}{s_1} + \frac{p}{s_2} = \frac{\overbrace{s_2 n}^{\in N} + \overbrace{s_1 p}^{\in P}}{\underbrace{s_1 s_2}_{\in S}} \in \tilde{S}^{-1}(N+P)$$

Hence, " \supseteq " also holds

$$(2) \quad \tilde{S}^{-1}(N \cap P) = \left\{ \frac{m}{s} \mid m \in N \& m \in P, s \in S \right\} \subseteq \tilde{S}^{-1}N \cap \tilde{S}^{-1}P$$

\uparrow such $\frac{m}{s} \in \tilde{S}^{-1}N$ & $\frac{m}{s} \in \tilde{S}^{-1}P$

Let $x \in \tilde{S}^{-1}N \cap \tilde{S}^{-1}P$, so $x = \frac{n}{s_1}$ and $x = \frac{p}{s_2}$ for some $n \in N, p \in P$ & $s_1, s_2 \in S$.

$$\frac{n}{s_1} = \frac{p}{s_2} \Rightarrow s'(s_2 n - s_1 p) = 0$$

$$\Rightarrow s' s_2 n = s' s_1 p \in N \cap P$$

$$\text{and } x = \frac{s' s_2 n}{s' s_2 s_1} \in \tilde{S}^{-1}(N \cap P).$$

$$x = \frac{s' s_1 p}{s' s_1 s_2}$$

(3)

$$0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} M/N \rightarrow 0 \quad \text{exact}$$

\Downarrow Proposition 7.16 : \bar{S}^{-1} is an exact functor
 $\text{Mod}(R) \rightarrow \text{Mod}(\bar{S}^{-1}R)$

$$0 \rightarrow \bar{S}^{-1}N \xrightarrow{\bar{S}^{-1}f} \bar{S}^{-1}M \xrightarrow{\bar{S}^{-1}g} \bar{S}^{-1}(M/N) \rightarrow 0$$

\Downarrow

~~isomorphism~~

$$\bar{S}^{-1}(M/N) \cong \bar{S}^{-1}M / \text{Ker } \bar{S}^{-1}g = \bar{S}^{-1}M / \text{Im } \bar{S}^{-1}f = \bar{S}^{-1}M / \bar{S}^{-1}N$$

Exercise 3

(4)

Lemma $M := \bigoplus_{i \in I} M_i$ is flat $\Leftrightarrow \forall i \in I: M_i$ is flat

Proof Let $N \xrightarrow{\varphi} N'$ be an injective map of modules.

Since \otimes commutes with direct sums we have the following commutative diagram:

$$\begin{array}{ccc} \{m_i\}_{i \in I} \otimes n & \xrightarrow{\quad} & (m_i)_{i \in I} \otimes \varphi(n) \\ \bigoplus_{i \in I} M_i \otimes N & \xrightarrow{\text{id}_M \otimes \varphi} & \bigoplus_{i \in I} M_i \otimes N' \\ \parallel & & \parallel \\ \bigoplus_{i \in I} (M_i \otimes N) & \xrightarrow{(id_{M_i} \otimes \varphi)_{i \in I}} & \bigoplus_{i \in I} (M_i \otimes N') \\ \downarrow & & \downarrow \\ (m_i \otimes n)_{i \in I} & \xrightarrow{\quad} & (m_i \otimes \varphi(n))_{i \in I} \end{array}$$

$\text{id}_M \otimes \varphi$ is injective $\Leftrightarrow (id_{M_i} \otimes \varphi)_{i \in I}$ is injective

$\Leftrightarrow \forall i \in I: id_{M_i} \otimes \varphi$ is injective

It follows $\bigoplus_{i \in I} M_i$ is flat $\Leftrightarrow \forall i \in I: M_i$ is flat

□

By Corollary 7.20 every R_m is flat over R

Then by Lemma $M := \bigoplus_{m \in \text{Max}(R)} R_m$ is flat over R .

By Exercise 2 (Ex. sheet 7) M is faithfully flat

$$\Leftrightarrow (M \otimes_R N = 0 \Rightarrow N = 0).$$

Let N be an R -module

$$\text{Then if } 0 = M \otimes_R N = \left(\bigoplus_{m \in \text{Max}(R)} R_m \right) \otimes_R N \cong$$

$$\cong \bigoplus_{m \in \text{Max}(R)} (R_m \otimes_R N) \Rightarrow N_m = 0 \quad \forall m \in \text{Max}(R)$$

$\stackrel{\text{Proposition 7.19}}{\cong} N_m$

N_m

$$\implies N = 0$$

Proposition 7.24

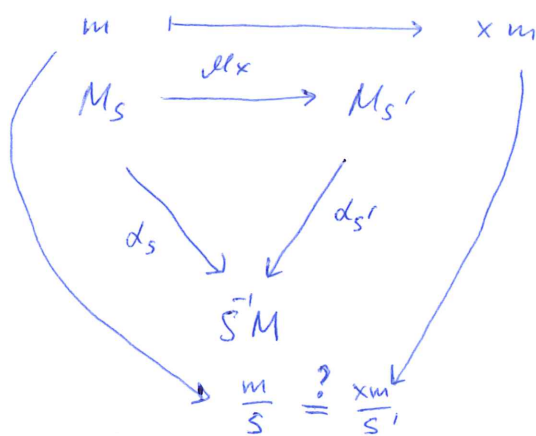
Exercise 4

Let us show that $\bar{S}^{-1}M$ with maps $d_s: M_s \rightarrow \bar{S}^{-1}M$
 $m \mapsto \frac{m}{s}$

is a "Kegel" of F (see Algebra 1 Lecture for notations)

We need to check that $\forall x \in \text{Hom}_S(s, s')$ the following

diagram commutes:



Let $m \in M = M_s$

• $d_s(m) = \frac{m}{s}$

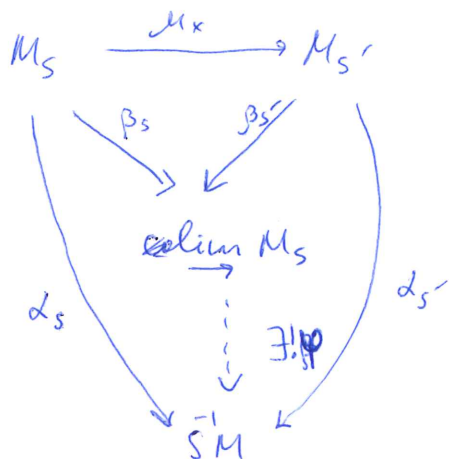
• $d_{s'} \circ \mu_x(m) = \frac{xm}{s'} = \frac{xm}{xs} \stackrel{\text{in } \bar{S}^{-1}M}{=} \frac{m}{s} = d_s(m)$

Then by universal property of $\varinjlim M_s := \text{colim}_S F$ (with $M_s \xrightarrow{\beta_s} \varinjlim M_s$)

we have a morphism of R -modules $\varinjlim M_s \rightarrow \bar{S}^{-1}M$,

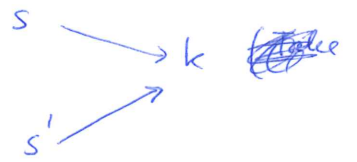


such that $\forall s, s' \in S, \forall x \in \text{Hom}(s, s')$ the diagram commutes

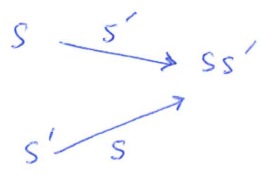


Observe that S is a filtered category,

that is $\forall s, s' \in S \exists k \in S$ with



Take $k = ss'$



$\forall x, y \in \text{Hom}(s, s')$, $\exists t \in \text{Hom}(s', s'')$ for some $s'' \in S$
 s.t. $xt = yt$ (Take $s'' = ss'$, $t = s$).

For a filtered category S :

canonical map β_s

$$M = M_s \xrightarrow{\beta_s} \varinjlim M_s = \coprod_{s \in S} M_s / \sim$$

where $m_1 \in M_s \sim m_2 \in M_{s'}$ if $\exists k \in S$, with $s \xrightarrow{x} k$
 $s' \xrightarrow{y} k$

such that $\mu_x(m_1) = \mu_y(m_2)$ in M_k

~~So every element $m \in \varinjlim M_s$ lifts to some $m' \in M_s$~~

Surjectivity of β : Let $\frac{m}{s} \in \varinjlim M_s$, we have

$$\begin{array}{ccc} M_s & \xrightarrow{\beta_s} & \varinjlim M_s \\ & \searrow \alpha_s & \downarrow \varphi \\ & & \varinjlim M_s \end{array} \Rightarrow \varphi(\beta_s(m)) = \frac{m}{s}$$

$\Rightarrow \varphi$ surjective.

Injectivity of β : Let $m \in \varinjlim M_s$, then $m' \in M_s$ for some $s \in S$

Then $\beta_s(m') = \frac{m'}{s} = 0 \Rightarrow s'm' = 0$ for some $s' \in S$

But then $\mu_{s'}(m') = 0 \Rightarrow$ class of $m' = 0$ in $\varinjlim M_s$

