Exercise 1

$R \neq 0$, $\Sigma = \{ S \subset R \mid S \text{ mult. closed in } R \}$

$\Sigma \neq \emptyset$, since $\{1\} \in \Sigma$

(a) To apply Zorn Lemma for Σ we need to show that every totally ordered subset $\{S_i\}_{i \in I}$ from Σ has an upper bound.

Upper bound $= \bigcup_{i \in I} S_i \in \Sigma$

- $1 \in \bigcup_{i \in I} S_i$
- Let $x, y \in \bigcup_{i \in I} S_i$

  Then $x \in S_i$ and $y \in S_j$ for some $i, j \in I$

  And without loss of generality $S_i \subseteq S_j$

  Then $x, y \in S_j \Rightarrow xy \in S_j \subseteq \bigcup_{i \in I} S_i$

Then, by Zorn lemma, Σ has a max. element

(b) $S \in \Sigma$ maximal $\Longleftrightarrow$ $R \setminus S$ minimal Prime ideal.

"$\Longrightarrow$" ~~By~~ Since $0 \notin S$, $S^{-1}R \neq 0$ (Exercise 3, Tutorium 9)

Then $\emptyset \neq \operatorname{Spec} S^{-1}R = \{ S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R, \mathfrak{p} \cap S = \emptyset \}$

Corollary 7.10 (2)

$\Rightarrow \exists \mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \emptyset \Rightarrow R \setminus \mathfrak{p} \in \Sigma$ and $S \subseteq R \setminus \mathfrak{p}$

By maximality of S: $S = R \setminus \mathfrak{p}$

Once again by max. of S: $\mathfrak{p}$ is minimal prime ideal

If $\mathfrak{p}' \subsetneq \mathfrak{p}$ then $R \setminus \mathfrak{p} \subsetneq R \setminus \mathfrak{p}' \in \Sigma$

"$\Leftarrow$"

~~Assume~~ Let $S = R \setminus \mathfrak{p}$, where $\mathfrak{p}$ minimal prime ideal.

Assume S is not maximal in $\Sigma \Rightarrow S \subsetneq \tilde{S}$ in Σ for some $\tilde{S}$.

$\tilde{S}^{-1}R \neq 0 \implies \exists\, \mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{q} \cap \tilde{S} = \emptyset$

That is $\mathfrak{q} \subseteq R \setminus \tilde{S} \subset R \setminus S = \mathfrak{p}$ $\lightning$ since $\mathfrak{p}$ minimal prime ideal.

Exercise 2

(1) $$S^{-1}(N+P) = \left\{ \frac{n+p}{s} \,\middle|\, \begin{array}{l} n \in N, \\ p \in P \end{array}, s \in S \right\} \subseteq S^{-1}N + S^{-1}P = \left\{ \frac{n}{s_1} + \frac{p}{s_2} \,\middle|\, \begin{array}{l} n \in N, p \in P \\ s_1, s_2 \in S \end{array} \right\}$$

But $$\frac{n}{s_1} + \frac{p}{s_2} = \frac{\overbrace{s_2 n}^{\in N} + \overbrace{s_1 p}^{\in P}}{\underbrace{s_1 s_2}_{\in S}} \in S^{-1}(N+P)$$

Hence, "$\supseteq$" also holds

(2) $$S^{-1}(N \cap P) = \left\{ \frac{m}{s} \,\middle|\, m \in N \;\&\; m \in P, s \in S \right\} \subseteq S^{-1}N \cap S^{-1}P$$

Such $\frac{m}{s} \in S^{-1}N$ & $\frac{m}{s} \in S^{-1}P$

Let $x \in S^{-1}N \cap S^{-1}P$, so $x = \frac{n}{s_1}$ and $x = \frac{p}{s_2}$ for some $n \in N$, $p \in P$ & $s_1, s_2 \in S$.

$$\frac{n}{s_1} = \frac{p}{s_2} \Rightarrow s'(s_2 n - s_1 p) = 0$$

$$\Rightarrow s' s_2 n = s' s_1 p \in N \cap P$$

and $$x = \frac{s' s_2 n}{s' s_2 s_1} \in S^{-1}(N \cap P).$$

$$x = \frac{s' s_1 p}{s' s_1 s_2}$$

(3)

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} M/N \longrightarrow 0 \quad \text{exact}$$

$\Downarrow$ Proposition 7.16 : S^{-1} is an exact functor $Mod(R) \longrightarrow Mod(S^{-1}R)$

$$0 \longrightarrow S^{-1}N \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}(M/N) \longrightarrow 0$$

$\Downarrow$

$$S^{-1}(M/N) \simeq S^{-1}M/\operatorname{Ker} S^{-1}g = S^{-1}M/\operatorname{Im} S^{-1}f = S^{-1}M/S^{-1}N$$

Exercise 3

Lemma $M := \bigoplus_{i \in I} M_i$ is flat $\iff$ $\forall i \in I$: M_i is flat

Proof Let $N \xrightarrow{\varphi} N'$ be an injective map of modules.

Since $\otimes$ commutes with direct sums we have the following commutative diagram:

$$\begin{tikzcd}
\{m_i\}_{i\in I} \otimes n \arrow[r, mapsto] \arrow[dddd, bend right, mapsto] & (m_i)_{i\in I} \otimes \varphi(n) \arrow[dddd, bend left, mapsto] \\
\bigoplus_{i\in I} M_i \otimes N \arrow[r, "\mathrm{id}_M \otimes \varphi"] \arrow[d, phantom, "\cong"] & \bigoplus_{i\in I} M_i \otimes N' \arrow[d, phantom, "\cong"] \\
{} & {} \\
\bigoplus_{i\in I} (M_i \otimes N) \arrow[r, "(\mathrm{id}_{M_i} \otimes \varphi)_{i\in I}"] & \bigoplus_{i\in I} (M_i \otimes N') \\
(m_i \otimes n)_{i\in I} \arrow[r, mapsto] & (m_i \otimes \varphi(n))_{i\in I}
\end{tikzcd}$$

$\mathrm{id}_M \otimes \varphi$ is injective $\iff$ $(\mathrm{id}_{M_i} \otimes \varphi)_{i\in I}$ is injective

$\iff$ $\forall i \in I$: $\mathrm{id}_{M_i} \otimes \varphi$ is injective

It follows $\bigoplus_{i\in I} M_i$ is flat $\iff \forall i \in I: M_i$ is flat ∎

By Corollary 7.20 every $R_{\mathfrak{m}}$ is flat over R

Then by Lemma $M := \bigoplus_{\mathfrak{m} \in Max(R)} R_{\mathfrak{m}}$ is flat over R.

By Exercise 2 (Ex. sheet 7) M is faithfully flat

$\iff \left(M \otimes_R N = 0 \implies N = 0 \right)$.

Let N be an R-module

Then if $0 = M \otimes_R N = \left(\bigoplus_{\mathfrak{m} \in Max(R)} R_{\mathfrak{m}} \right) \otimes_R N \simeq$

$$\simeq \bigoplus_{\mathfrak{m} \in Max(R)} \underbrace{\left(R_{\mathfrak{m}} \otimes_R N \right)}_{\substack{\text{IS Proposition 7.19} \\ N_{\mathfrak{m}}}} \implies N_{\mathfrak{m}} = 0 \quad \forall \mathfrak{m} \in Max(R)$$

$\xRightarrow[\text{Proposition 7.24}]{} N = 0$

$\blacksquare$

Exercise 4

Let us show that $S^{-1}M$ with maps
$$\begin{aligned} \alpha_s : M_s &\longrightarrow S^{-1}M \\ m &\longmapsto \frac{m}{s} \end{aligned}$$
is a "Kegel" of F (see Algebra 1 Lecture for notations)

We need to check that $\forall\, x \in \mathrm{Hom}_S(s, s')$, $s' = xs$, the following diagram commutes:

$$\begin{tikzcd}
m \arrow[rr, mapsto] \arrow[ddr, bend right] & & xm \arrow[ddl, bend left] \\
M_s \arrow[r, "\mu_x"] \arrow[dr, "\alpha_s"'] & M_{s'} \arrow[d, "\alpha_{s'}"] & \\
& S^{-1}M & \\
& \frac{m}{s} \overset{?}{=} \frac{xm}{s'} &
\end{tikzcd}$$

Let $m \in M = M_s$

- $\alpha_s(m) = \dfrac{m}{s}$
- $\alpha_{s'} \circ \mu_x(m) = \dfrac{xm}{s'} = \dfrac{xm}{xs} \underset{\text{in } S^{-1}M}{=} \dfrac{m}{s} = \alpha_s(m)$

Then by universal property of $\varinjlim M_s := \underset{S}{\mathrm{colim}}\, F$ $\left(\text{with } M_s \xrightarrow{\beta_s} \varinjlim M_s\right)$ we have a morphism of R-modules $\varinjlim M_s \longrightarrow S^{-1}M$,

$$M_s \longrightarrow M_s$$

such that $\forall\, s, s' \in S$, $\forall\, x \in \mathrm{Hom}(s, s')$ the diagram commutes

$$\begin{tikzcd}
M_s \arrow[rr, "\mu_x"] \arrow[dr, "\beta_s"] \arrow[ddr, bend right, "\alpha_s"'] & & M_{s'} \arrow[dl, "\beta_{s'}"'] \arrow[ddl, bend left, "\alpha_{s'}"] \\
& \varinjlim M_s \arrow[d, dashed, "\exists!\varphi"] & \\
& S^{-1}M &
\end{tikzcd}$$

Observe that S is a filtered category,

that is • $\forall\, s, s' \in S \quad \exists\, k \in S$ with
$$\begin{tikzcd} s \arrow[dr] & \\ & k \\ s' \arrow[ur] & \end{tikzcd}$$

Take $k = ss'$
$$\begin{tikzcd} s \arrow[dr, "s'"] & \\ & ss' \\ s' \arrow[ur, "s"'] & \end{tikzcd}$$

• $\forall\, x, y \in \mathrm{Hom}(s, s')$, $\exists\, t \in \mathrm{Hom}(s', s'')$ for some $s'' \in S$

s.t. $xt = yt$ (Take $s'' = ss'$, $t = s$).

For a filtered category S:

$$M = M_s \xrightarrow[\text{canonical map } \beta_s]{} \varinjlim M_s = \bigsqcup_{s \in S} M_s \Big/ \sim$$

where $\overset{\in M_s}{m_1} \sim \overset{\in M_{s'}}{m_2}$ if $\exists\, k \in S$, with
$$\begin{tikzcd} s \arrow[r, "x"] & k \\ s' \arrow[ur, "y"'] & \end{tikzcd}$$

such that $\mu_x(m_1) = \mu_y(m_2)$ in M_k

~~So every element $m \in \varinjlim M_s$ lifts to some $m' \in M_s$~~

Surjectivity of β: Let $\frac{m}{s} \in S^{-1}M$, we have

$$\begin{tikzcd} M_s \arrow[r, "\beta_s"] \arrow[dr, "\alpha_s"'] & \varinjlim M_s \arrow[d, "\varphi"] \\ & S^{-1}M \end{tikzcd}$$
$\Rightarrow \varphi(\beta_s(m)) = \frac{m}{s}$

$\Rightarrow \varphi$ surjective.

& $\varphi(m) = 0$

Injectivity of β: Let $m \in \varinjlim M_s$, then m lifts to some $m' \in M_s$ for some $s \in S$

$$\begin{tikzcd} m' \arrow[r, mapsto, "\alpha_s"] \arrow[dr, bend right, mapsto] & m \arrow[d, bend left, mapsto] \\ M_s \arrow[r, "\alpha_s"] \arrow[dr, "\beta_s"'] & \varinjlim M_s \arrow[d, "\varphi"] \\ & S^{-1}M \\ & \frac{m'}{s} = 0 \end{tikzcd}$$

Then $\beta_s(m') = \frac{m'}{s} = 0 \Rightarrow s'm' = 0$ for some $s' \in S$

But then $\mu_{s'}(m') = 0 \Rightarrow$ class of $m' = 0$ in $\varinjlim M_s$