Algebra 2

Exercise Sheet 8

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Exercise 1. Let K be a field. Show that a finite subgroup G of the multiplicative group K^* is cyclic.
Hint: Consider the abelian group G as a $\mathbb{Z}$-module and use Corollary 6.76 from the lecture.

Exercise 2. Let R be a PID and M a finitely generated R-module. Recall that by Corollary 6.76 we have

$$M \simeq M' \oplus \bigoplus_{i=1}^{m} R/(a_i)$$

where M' is free of finite rank and $a_i \in R$ are non-zero and non-units with the divisibility relation $a_1 \mid a_2 \mid \ldots \mid a_m$. Show that the ideals (a_i), $i = 1, \ldots, m$, are uniquely determined by M.

Exercise 3. Consider the following $\mathbb{Z}$-module

$$M = \prod_{p \in \operatorname{Spec}(\mathbb{Z})} \mathbb{Z}/(p)\,.$$

(1) Describe the submodule $\operatorname{Tors}(M)$ of M.
(2) Show that $\operatorname{Tors}(M)$ is not a direct summand of M (that is M does not decompose as $\operatorname{Tors}(M) \oplus M'$ for some submodule M' of M).
Remark: Note that the $\mathbb{Z}$-module M is not finitely generated and is not isomorphic to a direct sum of simple modules (compare to Corollary 6.76).

Exercise 4. Let R be a commutative ring and f an element in the intersection of all prime ideals of R. From the lecture we know that f is nilpotent (see Proposition 4.5).
Give another proof of this statement using localization.
Hint: Consider R_f.

Exercise 5. Let R be a Noetherian ring and S any multiplicatively closed subset of R. Show that $S^{-1}R$ is Noetherian.
Remark: The converse in general is not true.