Algebra 2

Exercise Sheet 8

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Exercise 1. Let K be a field. Show that a finite subgroup G of the multiplicative group K^* is cyclic.
Hint: Consider the abelian group G as a $\mathbb{Z}$-module and use Corollary 6.76 from the lecture.

Exercise 2. Let R be a PID and M a finitely generated R-module. Recall that by Corollary 6.76 we have

$$M \simeq M' \oplus \bigoplus_{i=1}^{m} R/(a_i)$$

where M' is free of finite rank and $a_i \in R$ are non-zero and non-units with the divisibility relation $a_1 \mid a_2 \mid \dots \mid a_m$. Show that the ideals (a_i), $i = 1, \dots, m$, are uniquely determined by M.

Exercise 3. Consider the following $\mathbb{Z}$-module

$$M = \prod_{p \in \mathrm{Spec}\,(\mathbb{Z})} \mathbb{Z}/(p)\,.$$

(1) Describe the submodule $\mathrm{Tors}(M)$ of M.
(2) Show that $\mathrm{Tors}(M)$ is not a direct summand of M (that is M does not decompose as $\mathrm{Tors}(M) \oplus M'$ for some submodule M' of M).
Remark: Note that the $\mathbb{Z}$-module M is not finitely generated and is not isomorphic to a direct sum of simple modules (compare to Corollary 6.76).

Exercise 4. Let R be a commutative ring and f an element in the intersection of all prime ideals of R. From the lecture we know that f is nilpotent (see Proposition 4.5).
Give another proof of this statement using localization.
Hint: Consider R_f.

Exercise 5. Let R be a Noetherian ring and S any multiplicatively closed subset of R. Show that $S^{-1}R$ is Noetherian.
Remark: The converse in general is not true.

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Exercise 1

By Corollary 6.76

$G \underset{\uparrow}{\simeq} \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_m)$ for some integers $a_1 | a_2 | \ldots | a_m$

as multiplicative group

Without loss of generality Assume $a_i > 1$. (positive)

Clearly $x^{a_m} = 1$ for every $x \in G$.

$|G| = a_1 \cdot \ldots \cdot a_m$ and $G \leq K^*$

So the polynomial $X^{a_m} - 1 \in K[X]$ has

at least $|G| = a_1 \cdot \ldots \cdot a_m \geq a_m = \deg(X^{a_m} - 1)$ roots in K

~~From~~ It follows from Algebra 1 Lecture,

that $|G| = a_m \Rightarrow m = 1 \Rightarrow G$ is cyclic

Exercise 2

Using Chinese Remainder Theorem we can assume that a_i are powers of primes.

By taking $\otimes R/(p^N)$ with big enough N

we can reduce the problem to the case:

all a_i are the powers of the same prime p.

We have to prove

$$\underbrace{R/(p^{e_1}) \oplus R/(p^{e_2}) \oplus \dots \oplus R/(p^{e_m})}_{N} \simeq \underbrace{R/(p^{k_1}) \oplus \dots \oplus R/(p^{k_r})}_{M}$$

$0 < e_1 \leq \dots \leq e_m$ $\qquad$ $0 < k_1 \leq \dots \leq k_r$

$$\Downarrow$$

$$m = k \quad \& \quad e_i = k_i$$

By Taking $\otimes R/p$ we get $m = r$

(Indeed, $R/p^{e_i} \otimes R/p \simeq R/p$)

Now induction on m.

Assume $k_1 \geq e_1$

If $N \simeq M$ as R-modules, the $p^{e_1} N \simeq p^{e_1} M$

We get $R/(p^{e_2 - e_1}) \oplus \dots \oplus R/(p^{e_m} - e_1) \simeq R/p^{k_1 - e_1} \oplus \dots \oplus R/p^{k_r - e_1}$

It follows $k_1 = e_1$ & by Induction hypothesis

$e_i = k_i$ for $i = 2, \dots, m$

Exercise 3

(1)
Denote by $\{p_i\}_{i\in N}$ the sequence of positive prime numbers

$M = \prod_{i\in N} \mathbb{Z}/(p_i)$ Claim: $\text{Tors}\, M = \bigoplus_{i\in N} \mathbb{Z}/(p_i) \subseteq M$

"$\subseteq$"

Let $m = (m_i)_{i\in N} \in \text{Tors}\, M$, $m_i \in \mathbb{Z}/(p_i)$

Then for some $a \neq 0$, $a \in \mathbb{Z}$ $am = (am_i)_{i\in N} = 0$

$\exists\, k \in N$ such that
Starting from some k , ~~all p_i~~ for $i \geq k$ we have $p_i > a$.

Then $\gcd(a, p_i) = 1$

And $am_i = 0$ in $\mathbb{Z}/(p_i)$ $\Rightarrow$ $m_i = 0$.

Hence in $m = (m_i)_{i\in N}$ all coordinates $m_i = 0$ for $i \geq k$

$\Rightarrow m \in \bigoplus_{i\in N} \mathbb{Z}/(p_i)$

"$\supseteq$" Let $m \in \bigoplus_{i\in N} \mathbb{Z}/(p_i)$

Then ~~for some~~ there exists $k \in N$, such that all $m_i = 0$ if $i \geq k$.

Take $a = \prod_{j=1}^{k-1} p_j \in \mathbb{Z}$. Then $am = 0$

(2)

(2) Assume Tors M is a direct summand of M

Then $0 \longrightarrow \mathrm{Tors}\, M \longrightarrow M \longrightarrow M/\mathrm{Tors}\, M \longrightarrow 0$

is split $\Rightarrow$ $M \simeq \mathrm{Tors}\, M \oplus M/\mathrm{Tors}\, M$

Note that any element $\bar{x} \in M/\mathrm{Tors}\, M$ is divisible by all p_i, $i \in \mathbb{N}$.

Indeed, $\bar{x} = \overline{(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots)}$

$\quad \| \text{ in } M/\mathrm{Tors}\, M$

$\overline{(0, 0, \ldots, 0, \underset{p \cdot y_{i+1}}{x_{i+1}}, \underset{p \cdot y_{i+2}}{x_{i+2}}, \ldots)} = p_i \cdot \bar{y}$

& p_i is invertible in $\mathbb{Z}/(p_k)$ $\forall k > i$

$\Rightarrow$ $\bar{x} = p_i \bar{y}$ for some $\bar{y} \in M/\mathrm{Tors}\, M$.

But no element $\neq 0$ in M has such a property:

Let $m \in M$, if $m = p_i y$ for some $y \in M$

then $\{m_i\} = 0$.

Hence, if m is divisible by all primes, then $m = 0$ ∎

<u>Exercise 4</u> Let $f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$, consider the ring R_f.

By Corollary 7.10,

$$\operatorname{Spec} R_f = \{ S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ with } \mathfrak{p} \cap S = \emptyset \},$$

where $S = \{f^n\}_{n \in \mathbb{N}}$

Since $f \in \mathfrak{p}$ for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, then the condition $\mathfrak{p} \cap S = \emptyset$ does not ~~see~~ hold for any $\mathfrak{p}$.

Hence, $\operatorname{Spec} R_f = \emptyset \implies R_f = 0 \xRightarrow[\text{Tutorium 9}]{\text{Exercise 3}} 0 \in \{f^n\}_{n \in \mathbb{N}}$

$\implies f$ is nilpotent

<u>Exercise 5</u> Assume R is Noetherian (every ideal in R is finitely generated)

Let $J \subset S^{-1}R$ be an ideal

By Corollary 7.10 ~~this~~ $J = S^{-1}I$ for some I ideal in R

R Noetherian $\implies I$ is generated by $i_1, \dots, i_n$

Let $\frac{i}{s} \in S^{-1}I$ with $i \in I \implies i = a_1 i_1 + \dots + a_n i_n$

$$\frac{i}{s} = \frac{a_1}{s} \cdot \frac{i_1}{1} + \dots + \frac{a_n}{s} \cdot \frac{i_n}{1} \implies S^{-1}I \text{ is generated by } \frac{i_1}{1}, \dots, \frac{i_n}{1}$$