Algebra 2

Exercise Sheet 7

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Exercise 1. Let R be a commutative ring and

$$0 \to M' \to M \to M'' \to 0$$

an exact sequence of R-modules. Assume that M'' is flat over R.
(1) Let N be an R-module. Show that the sequence

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

of R-modules is exact.
Hint: There exists an exact sequence of R-modules $0 \to K \to L \to N \to 0$, where L is a free module. Then use Snake-Lemma (Lemma 6.24) and the fact that every free module is flat.
(2) Show that M is flat if and only if M' is flat.
Hint: Use Snake-Lemma (Lemma 6.24).

Exercise 2. Let R be a commutative ring.
Recall that an R-module M is called *faithfully flat* if the exactness of any sequence $N' \to N \to N''$ of R-modules is equivalent to the exactness of the induced sequence $N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M$.
Show that the following conditions are equivalent.
a) M faithfully flat.
b) M is flat and $M \otimes_A N = 0$ implies $N = 0$ for any R-module N.

Exercise 3. Let R be a commutative ring and M a flat R-module.
(1) Show that M is faithfully flat if and only if $M \neq \mathfrak{m}M$ for every maximal ideal $\mathfrak{m}$ of R.
Hint: Use Exercise 2.
(2) Deduce from (1) that every nonzero finitely generated flat module over a local ring is faithfully flat.