Algebra 2

Exercise Sheet 6

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Exercise 1. Show that over a commutative ring every projective module is flat.

Exercise 2. Let R be a commutative ring and n, m positive integers.
(1) Let M be a finitely generated R-module and $\varphi : M \to M$ a surjective R-linear map. Show that φ is an isomorphism.
Hint: Consider M as an $R[X]$-module (via $F(X) \cdot m := F(\varphi)m$) and use Corollary 6.48 from the lecture.
(2) Show that any n generators of the free module R^n form an R-basis.
Hint: Use question (1).
(3) Show that: $R^n \simeq R^m$ (as R-modules) implies $n = m$.

Exercise 3. Let K be a field, $V = \oplus_{i\in\mathbb{N}} K$ a K-vector space and $R = \mathrm{End}_K(V)$. Note that R is a ring, but not commutative.
Show that: $R \simeq R^2$ as left R-modules.

Exercise 4. Let $K \subset L$ be a finite Galois field extension and $G = \mathrm{Gal}(L/K)$. Show that the map

$$\begin{aligned} L \otimes_K L &\longrightarrow \prod_{\sigma\in G} L \\ a \otimes b &\longmapsto (a \cdot \sigma(b))_{\sigma\in G} \end{aligned}$$

is an isomorphism of L-algebras, where the L-algebra structure on $L \otimes_K L$ is given by the multiplication on the first factor.