Algebra 2

Exercise Sheet 6

Prof. Markus Land
Dr. Maksim Zhykhovich

Summer Semester 2023
07.06.2023

Exercise 1. Show that over a commutative ring every projective module is flat.

Exercise 2. Let R be a commutative ring and n, m positive integers.
(1) Let M be a finitely generated R-module and $\varphi : M \to M$ a surjective R-linear map. Show that φ is an isomorphism.
Hint: Consider M as an $R[X]$-module (via $F(X) \cdot m := F(\varphi)m$) and use Corollary 6.48 from the lecture.
(2) Show that any n generators of the free module R^n form an R-basis.
Hint: Use question (1).
(3) Show that: $R^n \simeq R^m$ (as R-modules) implies $n = m$.

Exercise 3. Let K be a field, $V = \oplus_{i\in\mathbb{N}} K$ a K-vector space and $R = \mathrm{End}_K(V)$. Note that R is a ring, but not commutative.
Show that: $R \simeq R^2$ as left R-modules.

Exercise 4. Let $K \subset L$ be a finite Galois field extension and $G = \mathrm{Gal}(L/K)$. Show that the map

$$\begin{aligned} L \otimes_K L &\longrightarrow \prod_{\sigma\in G} L \\ a \otimes b &\longmapsto (a \cdot \sigma(b))_{\sigma\in G} \end{aligned}$$

is an isomorphism of L-algebras, where the L-algebra structure on $L \otimes_K L$ is given by the multiplication on the first factor.

Exercise sheet 6

Ex 1

Recall: P, Q, M R-modules

$$M \otimes (P \oplus Q) \xrightarrow{\sim} (M \otimes P) \oplus (M \otimes Q)$$
$$m \otimes (x, y) \longmapsto (m \otimes x, m \otimes y)$$

In particular, $(*)$ $P \otimes M \xhookrightarrow{i \otimes id} (P \oplus Q) \otimes M$ is injective

(induced by inclusion $\begin{matrix} P \hookrightarrow P \oplus Q \\ x \longmapsto (x, 0) \end{matrix}$)

R comm. ring

Let P be a projective module$/_R$ and let $N' \xhookrightarrow{f} N$ injection.

By Prop 6.62 $\exists$ module Q, such that $P \oplus Q$ is free

By Ex 3, Tutorium 5 every free module is flat

We get the commutative diagramm:

$$\begin{tikzcd}
N' \otimes P \arrow[r, "f \otimes id"] \arrow[d] & N \otimes P \arrow[d] \\
N' \otimes (P \oplus Q) \arrow[r] & N \otimes (P \oplus Q)
\end{tikzcd}$$

injective by $(*)$

injective, since $P \oplus Q$ flat

It follows that $f \otimes id$ is injective

Therefore, P is flat over R.

Ex 2

(1) M finitely gen. R-module

$\varphi : M \longrightarrow M$ surjective

Consider the ring hom.
$$\begin{aligned} R[X] &\longrightarrow \mathrm{End}_{\mathbb{Z}}(M) \\ X &\longmapsto \varphi \\ f(X) &\longmapsto f(\varphi) \end{aligned}$$

$\rightsquigarrow$ we can consider M as $R[X]$-module with

$$f(X).\, m = f(\varphi)(m) \quad \forall\ f \in R[X],\ m \in M$$

Let $I = (X) \subset R[X]$

Since φ is surjective, we have $M = IM$

By Corollary 6.48 $\exists\ a \in I$, s.t. $m = am \quad \forall\ m \in M$

$a = X \cdot g(X) \overset{= g(X)\cdot X}{} $ for som $g \in R[X]$.

We have $m = \varphi \circ g(\varphi)(m) \quad \forall\ m \in M$

$$\Downarrow$$

$$g(\varphi) \circ \varphi = \mathrm{id} = \varphi \circ g(\varphi) \quad \text{in } \mathrm{End}(M)$$

$\Rightarrow$ $g(\varphi)$ is an inverse of φ in $\mathrm{End}(M)$

$\Rightarrow$ φ is bijective

(2) Let $e_1, \dots, e_n$ be an R-basis of R^n

and $v_1, \dots, v_n$ generate R^n

Then $\exists\ \varphi : R^n \longrightarrow R^n$, $e_i \longmapsto v_i$, φ is surjective, since v_i generate R^n

By (1) φ is also injective $\Rightarrow$ v_i is a basis.

(3) Assume $m < n$ (without loss of generality)

Then R^n is generated by $v_1, \ldots, v_m$

Set: $v_{m+1} = v_{m+2} = \ldots = v_n = 0$

$\{v_i\}_{i=1}^n$ generate R^n. Hence by (2) they form a basis

(thus all $v_i \neq 0$) $\Rightarrow$ $n = m$.

<u>Ex 3</u> $V = \{(a_n)_{n\in\mathbb{N}} \mid a_n \in K\}$

K-Basis of V: $e_1, e_2, e_3, \ldots$

$$e_k = (\delta_{nk})_{n\in\mathbb{N}} \qquad \delta_{nk} = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{if } n \neq k \end{cases}$$

Define f: $e_{2k-1} \longmapsto 0$ $\quad\forall k \in \mathbb{N}$

$\qquad e_{2k} \longmapsto e_k$

$$\left(\begin{array}{l} e_1 \mapsto 0 \\ e_2 \mapsto e_1 \\ e_3 \mapsto 0 \\ e_4 \mapsto e_2 \\ \vdots \end{array}\right)$$

g: $e_{2k-1} \longmapsto e_k$ $\quad\forall k \in \mathbb{N}$

$\qquad e_{2k} \longmapsto 0$

$$\left(\begin{array}{l} e_1 \mapsto e_1 \\ e_2 \mapsto 0 \\ e_3 \mapsto e_2 \\ e_4 \mapsto 0 \end{array}\right)$$

To show: $\{f, g\}$ is an R-basis of R (considered as left R-module)

- Assume f, g are R-lin. dependent (that is $\alpha f + \beta g = 0$ in R for some $\alpha, \beta \in R$)

$\forall k \in \mathbb{N}$, ~~then~~ we have

$$0 = (\alpha f + \beta g)(e_{2k}) = \alpha(\underset{\overset{\shortparallel}{e_k}}{f(e_{2k})}) + \beta(\underbrace{g(e_{2k})}_{0}) = \alpha(e_k) \Rightarrow \alpha = 0$$

$$0 = (\alpha f + \beta g)(e_{2k-1}) = \beta(e_k) \Rightarrow \beta = 0$$

Now take $\alpha, \beta \in \mathrm{End}_k(V) = R$ with $\alpha(e_k) = e_{2k}$

&

$\beta(e_k) = e_{2k-1}$

Then

$$\begin{array}{c} 1 \in R \\ \parallel \\ \mathrm{id} \end{array} = \alpha f + \beta g$$

$$\alpha f(e_{2k}) + \beta g(e_{2k}) = \alpha(e_k) + \beta(0) = e_{2k} = \mathrm{id}(e_{2k})$$

$$(\alpha f + \beta g)(e_{2k-1}) = \alpha(0) + \beta(e_k) = e_{2k-1} = \mathrm{id}(e_{2k-1})$$

$\mathrm{id} = \alpha f + \beta g \Rightarrow f, g$ generate left R-module R

$\Rightarrow \{f, g\}$ form an R-basis.

$\Rightarrow R \simeq R^2$.

<u>Ex 4</u> $L = K(\alpha_1)$ (since L/K separable)

Let $f \in K[X]$ be the min. polynomial of α_1 over K.

L/K Galois $\Rightarrow$ $f(x) = (x-\alpha_1)\cdot\ldots\cdot(x-\alpha_n)$ in $L[X]$

where all α_i are different

Take $a \otimes b \in L \otimes_K L$, $b = g(\alpha_1)$ for some $g \in K[X]$

$$a \otimes g(\alpha) \longmapsto a \otimes \overline{g(x)} \longmapsto \overline{ag(x)}$$

$$L \otimes_K L \simeq L \otimes_K K[X]/(f) \underset{\uparrow}{\simeq} L[X]/(f) = L[X]/(x-\alpha_1)\cdot\ldots\cdot(x-\alpha_n) \simeq$$

Ex 2.2
Sheet 5

$$\longmapsto \prod_{i=1}^{n} (ag \bmod x-\alpha_i) \longmapsto \prod_{i=1}^{n} ag(\alpha_i) = \prod_{i=1}^{n} ag(\sigma_i(\alpha_1))$$

$$\underset{\uparrow}{\simeq} L[X]/(x-\alpha_1) \times \ldots \times L[X]/(x-\alpha_n) \simeq \overbrace{L \times \ldots \times L}^{n \text{ times}}$$

Chinese
Remainder
theorem

$$\prod_{i=1}^{n} ag(\sigma_i(\alpha_1)) = \prod_{i=1}^{n} a\,\sigma_i(g(\alpha_1)) = \prod_{i=1}^{n} a\,\sigma_i(b)$$

isomorphism of K-algebras.

$|Gal(L/K)| = n$ $\qquad Gal(L/K) = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$,

where $\sigma_i(\alpha_1) = \alpha_i$

Clearly $a \otimes b \longmapsto \prod_{i=1}^{n} a\,\sigma_i(b)$ preserves

L-algebra structure.