Algebra 2

Exercise Sheet 5

Prof. Markus Land — Summer Semester 2023
Dr. Maksim Zhykhovich — 23.05.2023

Exercise 1. Let R be a commutative ring, I an ideal of R and M an R-module. Show that there is a canonical isomorphism $R/I \otimes_R M \to M/IM$ of R/I-modules.

Exercise 2. Let A be a commutative ring, B a commutative A-algebra.
(1) Show that: $B \otimes_A A[X] \simeq B[X]$ as B-algebras.
(2) Let I be an ideal of $A[X]$. Show that $B \otimes_A (A[X]/I) \simeq B[X]/IB[X]$.

Exercise 3. Let A be a commutative ring, I, J ideals in A. We consider $\mathbb{Q}[y]$ as $\mathbb{Q}[x]$-algebra via the homomorhism $\mathbb{Q}[x] \to \mathbb{Q}[y]$, $x \mapsto y^2$.
Compute explicitely the following tensor products:
a) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$
b) $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$
c) $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2})$
d) $A/I \otimes_A A/J$
e) $\mathbb{Q}[y] \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x)$
f) $\mathbb{Q}[y] \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x-1)$
g) $\mathbb{Q}[y] \otimes_{\mathbb{Q}[x]} \mathbb{Q}[x]/(x+1)$

Exercise 4. Consider a commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
\downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D'
\end{array}$$

of R-modules and R-linear maps in which the rows are exact. Suppose m and p are injective and l is surjective.
Show: n is injective.