Algebra 2

Exercise Sheet 4

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Exercise 1. Let $C \subseteq \mathbb{C}[T]$ be the sub $\mathbb{C}$-algebra generated by T^2 and T^3. Show that C is not isomorphic to $\mathbb{C}[X]$.
Hint: Show that C is not a PID.

Exercise 2. Let R be a commutative ring and M an R-module. Show that the scalar multiplication map $R \to \mathrm{End}_{\mathbb{Z}}(M)$ factors through the forgetful map $\mathrm{End}_R(M) \to \mathrm{End}_{\mathbb{Z}}(M)$, that is, scalar multiplication by an element of R determines an R-linear endomorphism ℓ_r of M. Moreover, show that the image of R in $\mathrm{End}_R(M)$ consists of central elements, that is, for all $r \in R$ and $f \in \mathrm{End}_R(M)$, we have $\ell_r \circ f = f \circ \ell_r$.

Exercise 3. Let R be a commutative ring and let M and N be Noetherian R-modules. Show that $M \oplus N$ is again Noetherian.

Exercise 4. Let R and S be commutative rings. Show that the category $\mathrm{Mod}(R)$ admits all small limits and colimits, and that for any ring homomorphism $S \to R$, the restriction of scalars functor $\mathrm{Mod}(R) \to \mathrm{Mod}(S)$ commutes with all small limits and colimits.

Exercise 5. Let $\mathcal{C}$ be a category with finite coproducts and finite products in which the inital object is terminal. Let I be a finite set and for each $i \in I$, let X_i be an object of $\mathcal{C}$.

(1) Construct a canonical map $\coprod_{i \in I} X_i \to \prod_{i \in I} X_i$, such that this map is an isomorphism if $\mathcal{C} = \mathrm{Mod}(R)$ for R a commutative ring.
(2) Assume that the map of (1) are isomorphisms. Show that for all objects X, Y of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ carries a canonical structure of an abelian monoid and that composition with a morphism in $\mathcal{C}$ is a monoid homomorphism.