

Exercise sheet 4

Exercise 1

$$C = \{ a_0 + a_2 T^2 + \dots + a_n T^n \mid a_i \in \mathbb{C}, n \in \mathbb{N} \} \subseteq \mathbb{C}[T]$$

Consider the ideal $(T^2, T^3) = \{ a_2 T^2 + \dots + a_n T^n \mid a_i \in \mathbb{C}, n \in \mathbb{N} \} \stackrel{\subseteq C}{\subseteq} C$

(T^2, T^3) is not principle. Note that $(T^2, T^3) \neq C$

Indeed, if $(T^2, T^3) = (f(T))$

Then $f(T)$ divides T^2 and T^3 in $C \subseteq \mathbb{C}[T]$ $\Rightarrow (f(T)) = (T^2)$
 \uparrow
of degree ≥ 2

But one can not write $T^3 = g(T) \cdot T^2$ with $g(T) \in C$
(Actually $g(T) = T \notin C$)

Exercise 2

$$r \in R, \quad \ell_r : M \longrightarrow M$$

$$m \longmapsto rm$$

Since M is an R -module, $\ell_r \in \text{End}_{\mathbb{Z}}(M)$ (that is $\ell_r : M \rightarrow M$ is a homomorphism of abelian groups)

It remains to show that

ℓ_r is R -linear.

$$\forall d \in R, \forall m \in M: \ell_r(dm) = r(dm) = (rd)m = (dr)m = d(rm) = d \ell_r(m)$$

Let $f \in \text{End}_R(M)$ & $\ell_r \in \text{End}_R(M)$ with $r \in R$

$$\ell_r \circ f(m) = r f(m)$$

$$\Rightarrow \ell_r \circ f = f \circ \ell_r \text{ in } \text{End}_R(M)$$

$$f \circ \ell_r(m) = f(rm) \stackrel{\uparrow}{=} r f(m)$$

\uparrow
 f R -linear

It follows that

$$\ell_r \in Z(\text{End}_R(M))$$

\uparrow
center

Exercise 3

Consider

$$\begin{array}{ccccc} N & \xrightarrow{\Psi} & N \oplus M & \xrightarrow{\varphi} & M \\ n & \xrightarrow{\Psi} & (n, 0) & & \\ & & (n, m) & \xrightarrow{\varphi} & (0, m) \end{array}$$

Both Ψ and φ are R -linear.

Let L be a submodule in $N \oplus M$

To show: L is finitely generated

We know that any submodule in N and in M is finitely generated (since N & M are Noetherian)

$\varphi(L)$ submodule in M $\varphi(L) = (m_1, \dots, m_k)$ $m_i \in \varphi(L)$

That is $m_i = \varphi(y_i)$ for some $y_i \in L \subset N \oplus M$.

$$\text{Let } N' = \underbrace{\text{Im } \Psi}_{\text{Ker } \varphi} \cap L = \{ (n, 0) \in N \oplus M \mid (n, 0) \in L \}$$

N' is a submodule of $\text{Im } \Psi = N \oplus 0 \cong N$

$$\Psi(n_i) = (n_i, 0)$$

$\Rightarrow N' = (\Psi(n_1), \dots, \Psi(n_e))$ for some $n_i \in N$ Noetherian

Then Claim: $L = (\Psi(n_1), \dots, \Psi(n_e), y_1, \dots, y_k)$

Indeed,

Let $y \in L$. Then $\varphi(y) \in \varphi(L)$ and $\varphi(y) = a_1 m_1 + \dots + a_k m_k =$ for some $a_i \in R$

$$= \cancel{\varphi(a_1 \tilde{m}_1 + \dots + a_k \tilde{m}_k)} \quad a_1 \varphi(y_1) + \dots + a_k \varphi(y_k) = \varphi(a_1 y_1 + \dots + a_k y_k)$$

Then $y - (a_1 y_1 + \dots + a_k y_k) \in \text{Ker } \varphi = \text{Im } \Psi \cap L = N' = (\Psi(n_1), \dots, \Psi(n_e))$

$\Rightarrow y - (a_1 y_1 + \dots + a_k y_k) = b_1 \Psi(n_1) + \dots + b_e \Psi(n_e)$ for some $b_i \in R$

$\Rightarrow y \in (\Psi(n_1), \dots, \Psi(n_e), y_1, \dots, y_k)$.

Remark:

More generally: For any exact sequence of R -modules

$$0 \rightarrow M' \xrightarrow{\Psi} M \xrightarrow{\Psi} M'' \rightarrow 0$$

holds (M Noetherian $\Leftrightarrow M'$ & M'' Noetherian)

In our case: $M' = N$, $M = N \oplus M$, $M'' = M$.

Exercise 4

We know from ALGEBRA 1 Lecture (see Satz 6.37)

that \mathcal{C} admits small (co)-limits $\Leftrightarrow \mathcal{C}$ admit (co)-products (*) & \mathcal{C} admits (co)-equalizers (**)

For $\mathcal{C} = \text{Mod}(R)$ (*) was proven in the lecture (see Example 6.2 (10))

Let M and M' two R -modules

(**) Let $f, g \in \text{Hom}_R(M, M')$

Then $\text{Eq}(f, g) = \text{Ker}(f-g) \in \text{Mod}(R)$

$\text{CoEq}(f, g) = M' / \text{Im}(f-g) \in \text{Mod}(R)$

Exercise 5

Denote by $(*)$ the initial = terminal object in \mathcal{C} .

(1) To define

$$\phi: \coprod_{i \in I} X_i \longrightarrow \prod_{i \in I} X_i$$

\Downarrow

To define $X_i \longrightarrow X_j \quad \forall i, j \in I$

$$(X_i \longrightarrow X_j)_i = \begin{cases} \text{id}, & \text{if } i=j \\ X_i \longrightarrow * \longrightarrow X_j, & \text{if } i \neq j \end{cases}$$

If $\mathcal{C} = \text{Mod}(R)$, then $(*) = \text{zero Module } (0)$.

and $X_i \longrightarrow (0) \longrightarrow X_j$ is a zero map

so

$$\bigoplus_{i=1}^n M_i \longrightarrow \prod_{i=1}^n M_i \quad \text{is an isomorphism.}$$

$$(m_1, \dots, m_n) \longmapsto m_1 \times m_2 \times \dots \times m_n$$

(2) Let $f, g \in \text{Hom}_{\mathcal{C}}(X, Y)$

We define $f+g$ as the composition

$f+g$

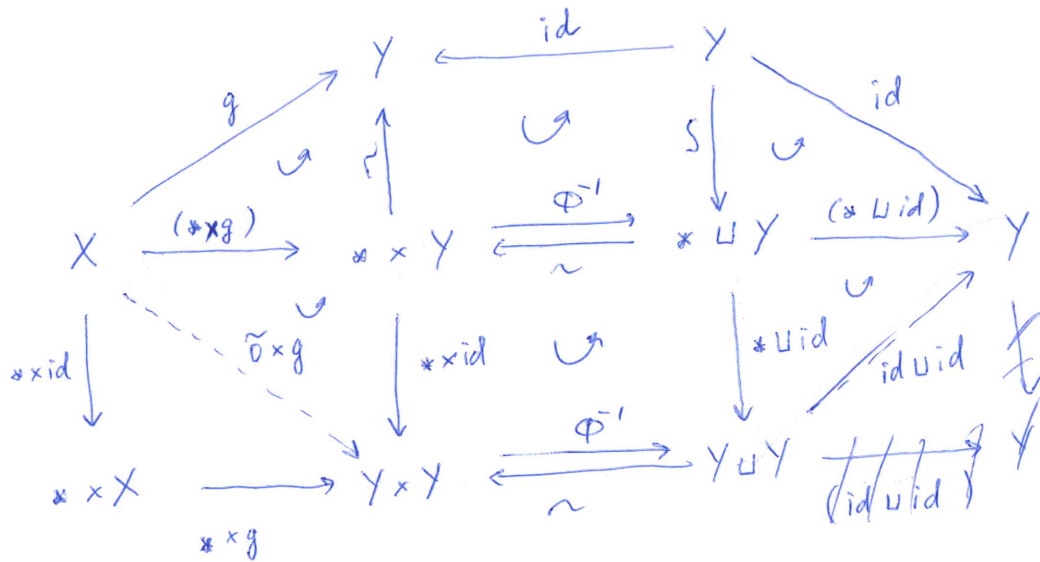
$$X \xrightarrow{f+g} Y \times Y \xrightleftharpoons[\phi^{-1}]{\phi} Y \sqcup Y \xrightarrow{\text{id} \sqcup \text{id}} Y$$

To see that $\text{Hom}_C(X, Y)$ is an abelian monoid one needs several iso for direct products/coproducts

- $X \times Y \cong Y \times X$
 - $X \times (Y \times Z) \cong (X \times Y) \times Z$
 - $X \times * \cong * \times X \cong X$
- (some properties hold also for coproducts)

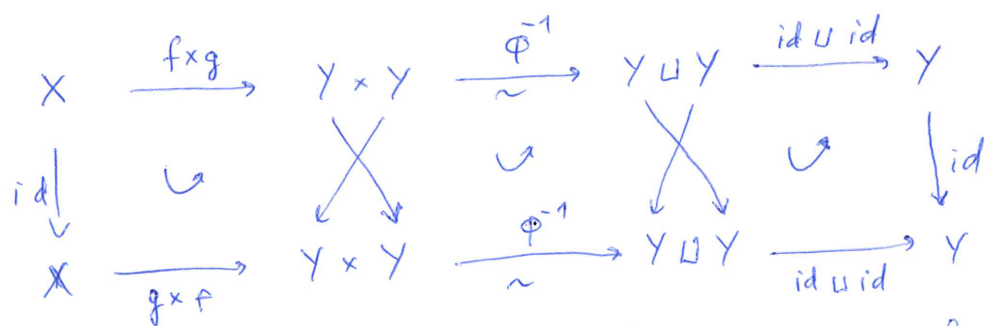
Zero map $\tilde{0} \in \text{Hom}_C(X, Y) := X \rightarrow * \rightarrow Y$

To show: $\tilde{0} + g = g$



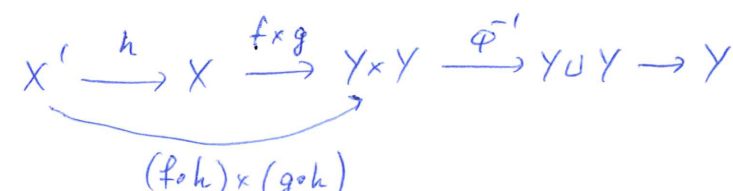
The diagram commutes. Above composition = g
Below composition = $\tilde{0} + g$.

To show: $f + g = g + f$



To show that: $(f + g) \circ h = f \circ h + g \circ h$, where $h: X' \rightarrow X$

$\text{Hom}_C(X, Y) \rightarrow \text{Hom}_C(X', Y)$
 $f \mapsto f \circ h$



Monoid hom.

To show that : $h \circ (f+g) = h \circ f + h \circ g$, where $h: Y \rightarrow Y'$

