Exercise sheet 4

Exercise 1

$$C = \{ a_0 + a_2 T^2 + \dots + a_n T^n \mid a_i \in \mathbb{C},\ n \in \mathbb{N} \} \subseteq \mathbb{C}[T]$$

Consider the ideal $(T^2, T^3) = \{ a_2 T^2 + \dots + a_n T^n \mid a_i \in \mathbb{C}, n \in \mathbb{N} \} \subseteq C$

(T^2, T^3) is not principle. Note that $(T^2, T^3) \neq C$

Indeed, if $(T^2, T^3) = (f(T))$

Then $\underset{\substack{\uparrow \\ \text{of degree} \geq 2}}{f(T)}$ divides T^2 and T^3 in $C \subseteq \mathbb{C}[T]$ $\Rightarrow$ $(f(T)) = (T^2)$

But one can not write $T^3 = g(T) \cdot T^2$ with $g(T) \in C$

(Actually $g(T) = T \notin C$)

Exercise 2

$r \in R$, $\quad l_r : M \longrightarrow M$, $\quad m \longmapsto r.m$

Since M is an R-modul, $l_r \in \mathrm{End}_{\mathbb{Z}}(M)$ (that is $l_r : M \to M$ is a homomorphism of abelian groups)

It remains to show that l_r is R-linear.

$\forall \alpha \in R,\ \forall m \in M: \quad l_r(\alpha m) = r.(\alpha m) = (r\alpha) m = (\alpha r) m = \alpha (rm) = \alpha\, l_r(m)$

Let $f \in \mathrm{End}_R(M)$ & $l_r \in \mathrm{End}_R(M)$ with $r \in R$

$$l_r \circ f(m) = r f(m)$$
$$f \circ l_r(m) = f(rm) \underset{\substack{\uparrow \\ f\ R\text{-linear}}}{=} r f(m)$$

$\Rightarrow$ $l_r \circ f = f \circ l_r$ in $\mathrm{End}_R(M)$

It follows that $l_r \in \underset{\substack{\uparrow \\ \text{center}}}{Z}(\mathrm{End}_R(M))$

Exercise 3

Consider

$$N \xrightarrow{\psi} N \oplus M \xrightarrow{\varphi} M$$

$$n \overset{\psi}{\longmapsto} (n, 0)$$

$$(n, m) \overset{\varphi}{\longmapsto} (0, m)$$

Both ψ and φ are R-linear.

Let L be a submodule in $N \oplus M$

To show: L is finitely generated

We know that any submodul in N and in M is finitely generated (since N & M are Noetherian)

$\varphi(L)$ submodule in M $\quad \varphi(L) = (m_1, \dots, m_k)$ $\quad m_i \in \varphi(L)$

That is $m_i = \varphi(y_i)$ for some $y_i \in L \subset N \oplus M$.

Let $N' = \underbrace{\operatorname{Im}\psi}_{\overset{\shortparallel}{\operatorname{Ker}\varphi}} \cap L = \cancel{\{(n,0) \in N \oplus M \mid (n,0)}$

$\{(n, 0) \in L\}$.

N' is a submodule of $\operatorname{Im}\psi = N \oplus 0 \simeq N$

$\psi(n_i) = (n_i, 0)$ $\qquad$ Noetherian

$\Rightarrow N' = (\psi(n_1), \dots, \psi(n_e))$ for some $n_i \in N$

Claim: $L = (\psi(n_1), \dots, \psi(n_e), y_1, \dots, y_k)$

Indeed,

Let $y \in L$. Then $\varphi(y) \in \varphi(L)$ and $\varphi(y) = a_1 m_1 + \dots + a_k m_k =$ for some $a_i \in R$

$= \cancel{a_1 \varphi(\tilde{m}_1) + \dots + a_k \varphi(\tilde{m}_k)}$ $a_1 \varphi(y_1) + \dots + a_k \varphi(y_k) = \varphi(a_1 y_1 + \dots + a_k y_k)$

Then $y - (a_1 y_1 + \dots + a_k y_k) \in L \cap \operatorname{Ker}\varphi = \operatorname{Im}\psi \cap L = N' = (\psi(n_1), \dots, \psi(n_e))$

$\Rightarrow y - (a_1 y_1 + \dots + a_k y_k) = b_1 \psi(n_1) + \dots + b_e \psi(n_e)$ for some $b_e \in R$

$\Rightarrow y \in (\psi(n_1), \dots, \psi(n_e), y_1, \dots, y_k)$.

Remark:

More generally: for any exact sequence of R-moduls

$$0 \to M' \xrightarrow{\psi} M \xrightarrow{\varphi} M'' \to 0$$

holds (M Noetherian $\Leftrightarrow$ M' & M'' Noetherian)

In our case: $M' = N$, $M = N \oplus M$, $M'' = M$.

Exercise 4

We know from Algebra 1 Lecture (see Satz 6.37) that C admits small (co)-limits $\iff$ C admit (co)-products (*) & C admits (co)-equalizers (**)

For $C = Mod(R)$ (*) was proven in the lecture (see Example 6.2 (10))

Let M and M' two R-modules

(**) Let $f, g \in Hom_R(M, M')$

Then $Eq(f,g) = Ker(f-g) \in Mod(R)$

$$CoEq(f,g) = M'/_{Im(f-g)} \in Mod(R)$$

Exercise 5

Denote by $(*)$ the initial = terminal Object in C.

(1) To define

$$\Phi : \coprod_{i \in I} X_i \longrightarrow \prod_{i \in I} X_i$$

$$\Updownarrow$$

To define $X_i \longrightarrow X_j \quad \forall\, i,j \in I$

$$(X_i \longrightarrow X_j) := \begin{cases} id, & \text{if } i=j \\ X_i \to * \longrightarrow X_j, & \text{if } i \neq j \end{cases}$$

If $C = Mod(R)$, then $(*) =$ zero Module (0).

and $X_i \longrightarrow (0) \longrightarrow X_j$ is a zero map

so

$$\bigoplus_{i=1}^{n} M_i \longrightarrow \prod_{i=1}^{n} M_i \quad \text{is an isomorphism.}$$

$$(m_1, \dots, m_n) \longmapsto m_1 \times m_2 \times \dots \times m_n$$

(2) Let $f, g \in$ ~~Mor~~ $Hom_C(X, Y)$

We define $f+g$ as the composition
~~$f \times g$~~

$$X \xrightarrow{f \times g} Y \times Y \underset{\Phi^{-1}}{\overset{\Phi}{\underset{\longrightarrow}{\overset{\sim}{\longleftarrow}}}} Y \sqcup Y \xrightarrow{id \sqcup id} Y$$

To see that $Hom_C(X,Y)$ is an abelian monoid one needs several iso for direct products/coproducts

- $X \times Y \simeq Y \times X$
- $X \times (Y \times Z) \simeq (X \times Y) \times Z$
- $X \times * \simeq * \times X \simeq X$

(Some properties hold also for coproducts)

Zero map $\tilde{o} \in Hom_C(X,Y) := \quad X \longrightarrow * \longrightarrow Y$

~~$\tilde{o} + f = f$~~ To show: $\tilde{o} + g = g$

$$\begin{tikzcd}
 & Y & Y \arrow[l, "id"'] \arrow[d, "\wr"] \arrow[dr, "id"] & \\
X \arrow[ur, "g"] \arrow[r, "(*\times g)"] \arrow[d, "*\times id"'] \arrow[dr, dashed, "\tilde{o}\times g"] & *\times Y \arrow[u, "\wr"] \arrow[r, shift left, "\varphi^{-1}"] \arrow[d, "*\times id"] & *\sqcup Y \arrow[l, shift left, "\sim"] \arrow[r, "(*\sqcup id)"] \arrow[d, "*\sqcup id"] & Y \\
\times X \arrow[r, "\times g"'] & Y\times Y \arrow[r, shift left, "\varphi^{-1}"] & Y\sqcup Y \arrow[l, shift left, "\sim"] \arrow[ur, "id\sqcup id"'] &
\end{tikzcd}$$

The diagramm commutes. Above composition $= g$

Below composition $= \tilde{o} + g$.

To show: $f + g = g + f$

$$\begin{tikzcd}
X \arrow[r, "f\times g"] \arrow[d, "id"'] & Y\times Y \arrow[r, "\varphi^{-1}", "\sim"'] \arrow[d] & Y\sqcup Y \arrow[r, "id\sqcup id"] \arrow[d] & Y \arrow[d, "id"] \\
X \arrow[r, "g\times f"'] & Y\times Y \arrow[r, "\varphi^{-1}", "\sim"'] & Y\sqcup Y \arrow[r, "id\sqcup id"'] & Y
\end{tikzcd}$$

To show that: $(f+g)\circ h = f\circ h + g \circ h$, where $h : X' \longrightarrow X$

$Hom_C(X,Y) \longrightarrow Hom_C(X',Y)$

$f \longmapsto f \circ h$

Monoid hom.

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[rr, bend right, "(f\circ h)\times(g\circ h)"'] & X \arrow[r, "f\times g"] & Y\times Y \arrow[r, "\varphi^{-1}"] & Y\sqcup Y \arrow[r] & Y
\end{tikzcd}$$

To show that : $h \circ (f+g) = h \circ f + h \circ g$, where $h : Y \to Y'$

$$
\begin{tikzcd}
X \arrow[r, "f \times g"] \arrow[dr, "h \circ f \times h \circ g"'] \arrow[rrr, bend left=30, "h \circ (f+g)"] \arrow[drrr, bend right=40, "h \circ f + h \circ g"'] & Y \times Y \arrow[r, "\sim \varphi^{-1}"] \arrow[d, "h \times h"] \arrow[dl, phantom, "\circlearrowright"] & Y \sqcup Y \arrow[r, "id \sqcup id"] \arrow[d, "h \sqcup h"] \arrow[dl, phantom, "\circlearrowright"] & Y \arrow[d, "h"] \arrow[dl, phantom, "\circlearrowright"] \\
 & Y' \times Y' \arrow[r, "\sim"', "\varphi^{-1}"'] & Y' \sqcup Y' \arrow[r, "id \sqcup id"'] & Y'
\end{tikzcd}
$$