# Algebra 2

## Exercise Sheet 3

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**Exercise 1.** Show that the ring $\{P \in \mathbb{Q}[X] \mid P(0) \in \mathbb{Z}\}$ is not Noetherian.

**Exercise 2.** Let $A$ be a commutative ring.

(1) Show that the set $S = \{I \subseteq R \mid I \text{ not finitely generated ideal }\}$ is either empty or contains a maximal element.
(2) Show that $A$ is Noetherian if and only if every prime ideal in $A$ is finitely generated.

**Exercise 3.** Let $R$ be a commutative ring and $f = a_0 + a_1X + ... + a_nX^n \in R[X]$. Show that
(1) $f \in R[X]^\times \iff a_0 \in R^\times$ and $a_1, ..., a_n \in \mathcal{N}_R$.
*Hint:* Let $f^{-1} = b_0 + b_1X + ... + b_mX^m$. Show that $a_n^{i+1}b_{m-i} = 0$ for every $i = 0, 1, ..., m$.
(2) $f \in \mathcal{N}_{R[X]} \iff a_0, a_1, ..., a_n \in \mathcal{N}_R$.

**Exercise 4.** Let $K$ be a field and $n \geq 1$. Show that there are infinitely many irreducible elements (even up to units) in $K[X_1, \dots, X_n]$.

**Exercise 5.** Let $A$ be a commutative ring and $\mathfrak{a}$ a strict ideal of $A$. Show that

$$\sqrt{\mathfrak{a}} = \bigcap \{\mathfrak{p} \in \operatorname{Spec}_{\mathrm{rab}}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$