Algebra 2 Exercise Sheet 3

Exercise 1 $A = \{ a_0 + a_1 X + \dots + a_n X^n \in \mathbb{Q}[X] \mid a_0 \in \mathbb{Z} \}$

Let $I_n := \left(\frac{1}{2^n} X \right)$

Then $(*)$ $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots \; I_n \subsetneq I_{n+1} \subseteq \dots$

$$\uparrow$$

$$\frac{1}{2^n} X = 2 \cdot \left(\frac{1}{2^{n+1}} X \right)$$

But we can not write

$$\frac{1}{2^{n+1}} X = \frac{1}{2^n} X \cdot f \quad , \; f \in A$$

(Otherwise $\deg f = 0$, $f = a_0 \in \mathbb{Z}$

and $\frac{1}{2^{n+1}} = \frac{1}{2^n} \cdot a_0$ $\lightning$).

The sequence $(*)$ does not stabilize $\Longrightarrow$ A is not Noetherian.

Exercise 2

(1) Assume $S \neq \emptyset$.

Consider a chain (totally ordered subset in S)

$(I_j)_{j \in J}$ (that is $I_{j_1} \subseteq I_{j_2}$ or $I_{j_2} \subseteq I_{j_1}$ $\forall\, j_1, j_2 \in J$).

Every I_j, $j \in J$, is not finitelly generated.

Set $I := \bigcup_{j \in J} I_j$ $\left(\neq \cancel{\sum_{j \in J} I_j} \right)$.

I is an ideal in A. To show: $I \in S$.

Assume $I = (x_1, \dots, x_n)$ finitely generated.

Then $x_k \in I = \bigcup_{j \in J} I_j \Rightarrow x_k \in I_{j_k}$ for some $j_k \in J$.

Since $(I_j)_{j \in J}$ totally ordered, take a maximal I_j among $I_{j_1}, \dots, I_{j_n}$

Then all $x_1, \dots, x_n \in I_j \Rightarrow I = (x_1, \dots, x_n) \subseteq I_j \Rightarrow I_j = I$

Contradiction, since $I_j \in S$ is not finitely generated.

It follows that $I \in S$ is an upper bound for $(I_j)_{j \in J}$

By Zorn Lemma S contains a max. element.

(2) Assume A is not Noetherian.

Then $S \neq \emptyset$ and by (1) $\exists$ a max element $\mathfrak{p}$ in S.

~~Claim: $\mathfrak{p}$ is a prime ideal~~.

~~Assume~~ Since every prime ideal is finitely generated, $\mathfrak{p}$ is not prime.

$\exists\, a, b \in A$, s.t. $a \notin \mathfrak{p}$, $b \notin \mathfrak{p}$ but $ab \in \mathfrak{p}$.

$$\mathfrak{p} \begin{array}{l} \subsetneq \mathfrak{p} + (a) \\ \subsetneq \mathfrak{p} + (b) \end{array} \xrightarrow[\text{by maximality of } \mathfrak{p} \text{ in } S]{} \mathfrak{p} + (a) \text{ is finitely generated.}$$

with generators

$$y_1 + x_1 a, \ldots, y_n + x_n a$$

where $y_i \in \mathfrak{p}$ & $x_i \in A$.

Consider the ideal $\mathfrak{p}' := \{ x \in A \mid xa \in \mathfrak{p} \}$

We have $\mathfrak{p} \subsetneq \mathfrak{p} + (b) \subseteq \mathfrak{p}' \Rightarrow \mathfrak{p}'$ finitely generated

Then $\mathfrak{p}' \cdot (a)$ is also finitely generated.

<u>Claim</u>: $\mathfrak{p} = (y_1) + (y_2) + \ldots + (y_n) + \mathfrak{p}'(a) \leftarrow$ finitely generated

"$\supseteq$" is clear

Let $x \in \mathfrak{p}$, then $x = \sum c_i (y_i + x_i a)$ for some $c_i \in A$

Then $$x = \underbrace{c_1 y_1 + \ldots + c_n y_n}_{\in (y_1, \ldots, y_n)} + \overbrace{(\underset{\substack{\cap \\ \mathfrak{p}'}}{\textstyle\sum c_i x_i})\, a}^{\in \mathfrak{p}' a}$$

Since $\mathfrak{p} \in S$ and $\mathfrak{p}$ is finitely gen. $\Rightarrow$ Contradiction ↯.

It follows that A is Noetherian.

Exercise 3

Lemma (Aufgabe 4, Übungsblatt 3, Algebra 1)

R commutative ring, $x \in \mathcal{N}_R \Rightarrow 1+x \in R^*$

More generally, $a+x \in R^*$ for any $a \in R^*$.

(1) "$\Leftarrow$" $a_1, \dots, a_n \in \mathcal{N}_R \Rightarrow a_1X, \dots, a_nX^n \in \mathcal{N}_{R[X]} \Rightarrow a_1X + \dots + a_nX^n \in \mathcal{N}_{R[X]}$

Then by Lemma, $a_0 + \underbrace{a_1X + \dots + a_nX^n}_{\in \mathcal{N}_{R[X]}} \in R[X]^\times$

"$\Rightarrow$" $f^{-1} = b_0 + b_1X + \dots + b_mX^m$

$$1 = ff^{-1} = a_0b_0 + (a_1b_0 + a_0b_1)X + \dots + a_nb_mX^{n+m}$$

$\Downarrow$ $a_0b_0 = 1 \Rightarrow a_0 \in R^*, b_0 \in R^\times$

$\Downarrow$ $a_nb_m = 0$ (*)

X^{n+m-1} : $\{ a_nb_{m-1} + a_{n-1}b_m = 0$

$$0 = a_n(a_nb_{m-1} + a_{n-1}b_m) = a_n^2b_{m-1} + a_{n-1}\underbrace{a_nb_m}_{\overset{(*)}{=}0}$$

We get $a_n^2b_{m-1} = 0$ (**)

By induction on i one can show

that $a_n^{i+1}b_{m-i} = 0 \quad \forall\, i = 0, 1, \dots, m$

(*) (**)

For $i=m$ we get $a_n^{m+1}\underset{\in R^*}{b_0} = 0 \Rightarrow a_n^{m+1} = 0 \Rightarrow a_n \in \mathcal{N}_R$.

Consider $f - a_nX^n \in R[X]$. By Lemma $f \in R[X]^*$ and $\deg f \leq n-1$

By Induction on $\deg f$, $\Longrightarrow$ one can show

that $a_1, a_2, \dots, a_{n-1}, a_n \in \mathcal{N}_R$

(2) "$\Leftarrow$"

$$a_i X^i \in \underset{\substack{\uparrow \\ \text{ideal}}}{\mathcal{N}_{R[X]}} \implies f = a_0 + a_1 X + \dots + a_n X^n \in \mathcal{N}_{R[X]}$$

"$\Rightarrow$"

$$f \text{ nilpotent} \implies \exists\, r \in \mathbb{N}, \text{ s.t. } f^r = 0$$

$$\Downarrow$$

$$f^r = (a_0^r + \dots + a_n^r X^{nr}) = 0$$

$$\Downarrow$$

$$a_n^r = 0 \implies a_n \in \mathcal{N}_R.$$

$\implies$ $f - a_n X^n \in \mathcal{N}_{R[X]}$

~~Lemma~~

$\mathcal{N}_{R[X]}$ ideal

has degree $\leq n-1$

Induction on $\deg f \implies a_0, a_1, \dots, a_n \in \mathcal{N}_R$.

Exercise 4

<u>Remark</u> A domain

Then $(f \in A \text{ irreducible}) \Rightarrow (f \in A[x] \text{ irreducible})$

Hence, it is enough to consider the case $n = 1$.

- If K is infinite, then $X - a$, $a \in K$, are irreducible

- If K is finite, $|K| = q$.

Then $\forall\ n \geq 1$, $\exists$ a field extension L_n/K with $[L_n : K] = n$.

Namely, $L_n =$ splitting field of $X^{q^n} - X$

L_n/K separable (since K finite) $\xRightarrow[\text{Algebra 1}]{} L_n = K(\alpha_n)$ for some $\alpha_n \in L_n$

Then the min. polynomial of α_n has degree n and is irr. ~~over~~ in $K[X]$.

Exercise 5

<u>Recall</u>: (Definition 4.23)

~~Spec_{RAB}~~ $\quad \mathrm{Spec}_{rab}(A) = \{ \mathfrak{m} \cap A \in \mathrm{Spec}\, A \mid \mathfrak{m} \subseteq A[X] \text{ max.} \}$.

I ideal in A.

By Lemma 4.8 (3) we have $\quad \sqrt{I} = \bigcap\limits_{\substack{\rho \in \mathrm{Spec}\, A, \\ I \subseteq \rho}} \rho$

To show: $\quad \sqrt{I} \overset{\text{"}\subset\text{"}}{\underset{\text{"}\supset\text{"}}{=}} \bigcap\limits_{\substack{\rho \in \mathrm{Spec}_{rab}(A) \\ I \subseteq \rho}} \rho$

Since $\mathrm{Spec}_{rab}(A) \subseteq \mathrm{Spec}(A)$, the inclusion "$\subset$" holds.

Let now $a \in \bigcap\limits_{\substack{\rho \in \mathrm{Spec}_{rab}(A) \\ I \subseteq \rho}} \rho$

Consider the ideal $\quad J := \langle I, ax-1 \rangle \subseteq R[X]$

ideal generated by elements from I and $ax-1$

Assume $J \subsetneq R[X] \underset{\text{lecture}}{\Longrightarrow} J \subseteq \mathfrak{m}$ for some max ideal $\mathfrak{m} \subseteq R[X]$.

We have $\quad I \subseteq R \cap J \subseteq \underset{\underset{\rho}{\parallel}}{R \cap \mathfrak{m}} \in \mathrm{Spec}_{rab}(R)$

It follows $a \in R \cap \mathfrak{m} \Rightarrow a \in \mathfrak{m}$ & $\underset{\underset{ax-1}{\ni}}{J} \subseteq \mathfrak{m}$

Hence, $a \in \mathfrak{m}$ & $ax-1 \in \mathfrak{m} \Rightarrow 1 \in \mathfrak{m}$ ↯ It follows $J = R[X]$.

We get in $R[X]$: $\quad (*) \sum\limits_{j=1}^{n} f_j b_j + g(ax-1) = 1 \quad$ for some $g, f_j \in R[X]$ and $b_j \in I$

~~Set~~ Replacing x by $\frac{1}{x}$ in $(*)$ we get:

$$(**) \quad \sum_{j=1}^{n} f_j\left(\frac{1}{x}\right) b_j + g\left(\frac{1}{x}\right)\left(a\frac{1}{x} - 1\right) = 1$$

holds in $R[x, \frac{1}{x^{-1}}]$

Ring of Laurent polynomials.

Let $k := \max\{\deg f_j,\ j = 1, \dots, n,\ \deg g + 1\}$

Multiplying $(**)$ by x^k we get the following polynomial equality in $R[x]$

$$\sum_{j=1}^{n} \tilde{f}_j(x) b_j + \hat{g}(x)(a - x) = x^k,$$

where

$\tilde{f}_j(x) := x^k f\left(\frac{1}{x}\right)$

$\hat{g}(x) := x^k g\left(\frac{1}{x}\right)$

polynomials in $R[x]$.

Take $x = a$:

$$a^k = \sum_{j=1}^{n} \tilde{f}_j(a)\, b_j \in I \quad (\text{since } b_j \in I)$$

Therefore, $a \in \sqrt{I}$.