Algebra 2

Exercise Sheet 2

Prof. Markus Land | Summer Semester 2023
Dr. Maksim Zhykhovich | 02.05.2023

Exercise 1. Let $(A, \mathfrak{m})$ be a local domain, F its field of fractions and $\kappa = A/\mathfrak{m}$ its residue field. Show that A is of equal characteristic if and only if $\mathrm{char}(F) = \mathrm{char}(\kappa)$ and of mixed characteristic if and only if $\mathrm{char}(F) = 0$ and $\mathrm{char}(\kappa) > 0$.

Exercise 2. Let A be a commutative ring. Let $A[[X]]$ be a ring of formal power series with coefficients in A, that is

$$A[[X]] = \{a_0 + a_1X + a_2X^2 + \ldots \mid a_i \in A\}.$$

Show that for each $n \geq 1$, there are ring maps

$$A[[X]] \longrightarrow A[X]/(X^n) \longrightarrow A$$

(1) Show that an element of $A[[X]]$ or of $A[X]/(X^n)$ is invertible if and only if its image under the above maps in A is invertible.
(2) Show that $A[[X]]$ and $A[X]/(X^n)$ are local rings if and only if A is a field.

Exercise 3. Let X be a topological space and T a subset. Show that if T is irreducible then so is its closure $\overline{T}$. Prove or disprove that the converse is also true, that is: if $\overline{T}$ is irreducible, then so is T.

Exercise 4. Let $\mathfrak{a} \subseteq A$ be an ideal in a commutative ring. Show that $\mathfrak{a}$ is finitely generated if and only if the set

$$S = \{(T) \subseteq A \mid T \subseteq \mathfrak{a} \text{ finite subset }\}$$

of finitely generated ideals contained in $\mathfrak{a}$ has a maximal element (which is then necessarily $\mathfrak{a}$).

Exercise 5. Let X be a topological space. An open cover of X consists of a set I and a map $I \to \mathcal{U}(X)$, written $i \mapsto U_i$, such that $\bigcup_{i\in I} U_i = X$. We write $\{U_i\}_{i\in I}$ for such an open cover. A topological space X is called *quasi-compact* if for every open cover $\{U_i\}_{i\in I}$, there exists a finite subset $J \subseteq I$ such that $X = \bigcup_{j\in J} U_j$. Now let A be a commutative ring.

(1) Show that $\mathrm{Spec}(A)$ is quasi-compact.
(2) Let I be a finite set and let A_i be a commutative ring for each $i \in I$. Show that the canonical map

$$\coprod_{i\in I} \mathrm{Spec}(A_i) \longrightarrow \mathrm{Spec}(\prod_{i\in I} A_i)$$

induced by the projections is a homeomorphism.

(3) Let I be an infinite set. Prove or disprove that the conclusion of (2) still holds.