Exercise sheet 2

Exercise 1

<u>Recall</u>: R domain, $\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\varphi} & R \\ 1 & \longmapsto & 1 \end{array}$ ~~the~~ canonical Ring hom.

$\operatorname{Ker}\varphi = \underbrace{(p)}_{\text{char } R = p}$ with p prime or $\underbrace{(0)}_{\text{char } R = 0}$

(Algebra 1 Bemerkung Seit 35): $R \subset S$ domains $\Rightarrow$ char R = char S

It follows from Bem. that char A = char F

A is of equal char $\overset{\text{def}}{\Longleftrightarrow}$ A contains a field K

<u>Claim</u>: A is of equal char $\Leftrightarrow$ $\varphi(\mathbb{Z})^{\setminus\{0\}} \subset A^*$, where

$\varphi : \mathbb{Z} \longrightarrow A$
$1 \longmapsto 1$

"$\Rightarrow$" $\varphi(\mathbb{Z})^{\setminus\{0\}} \subseteq K^* \subseteq A^*$

field of fractions

"$\Leftarrow$" $\underset{\text{Ring + domain}}{\varphi(\mathbb{Z})^{\setminus\{0\}}} \subset A^* \Rightarrow Q(\varphi(\mathbb{Z})) \subset A^*$

Consider

$$\begin{tikzcd} \mathbb{Z} \arrow[d] \arrow[dr, "\varphi"] & & \\ \mathbb{Z}/(\operatorname{Ker}\varphi) \arrow[r, "\tilde{\varphi}"] & A \arrow[r, "\psi"] & A/\mathfrak{m} \end{tikzcd}$$

Then $\varphi(\mathbb{Z}) \subset A^* \Leftrightarrow \varphi(\mathbb{Z}) \cap \mathfrak{m} = \emptyset \Leftrightarrow \psi|_{\varphi(\mathbb{Z}) = \tilde{\varphi}(\mathbb{Z})}$ is injective

$\Leftrightarrow \psi \circ \tilde{\varphi}$ is injective $\Leftrightarrow \operatorname{Ker}\varphi = \operatorname{Ker}\psi\circ\varphi \Leftrightarrow$

$\Leftrightarrow$ char A = char $A/\mathfrak{m}$

It follows that

A is of mixed char $\iff$ $\operatorname{char} A \neq \operatorname{char} A/m$

$$\left(\text{in this case } \underset{\substack{\| \\ (0)}}{\operatorname{Ker}\varphi} \subsetneq \underset{\substack{\| \\ (p)}}{\operatorname{Ker}\psi\circ\varphi}\right)$$

$\operatorname{char} A = 0$ & $\operatorname{char} A/m = p$

Exercise 2

$$\begin{tikzcd}
A[[X]] \arrow[r] & A[[X]]/(X^n) \\
A[X] \arrow[u] \arrow[ur, "\varphi"] \arrow[r] & A[X]/(X^n) \arrow[u, "\wr\ \tilde{\varphi}"']
\end{tikzcd}$$

surjective and injective

Note that $(X^n) \subset \operatorname{Ker}\varphi$, so $\exists\ \tilde{\varphi}$ as above.

So we get $A[[X]] \longrightarrow A[[X]]/(X^n) \xrightarrow[\sim]{\tilde{\varphi}^{-1}} A[X]/(X^n) \longrightarrow A[X]/(X) \cong A$

Remark: $\varphi : A \to B$ Ring hom., then $\varphi(A^*) \subset B^*$

(1) $A[[X]] \xrightarrow{\varphi} A \simeq A[[X]]/(X)$

$f = a_0 + a_1 X + a_2 X^2 + \dots \longmapsto a_0$

$f \in A[[X]]^* \underset{\text{Remark}}{\Longrightarrow} \varphi(f) = a_0 \in A^*$

Let now $f = a_0 + a_1 X + a_2 X^2 + \dots \in A[[X]]$ with $a_0 \in A^*$

We construct the inverse $g = b_0 + b_1 X + b_2 X^2 + \dots$ as follows:

$$1 = (a_0 + a_1 X + \dots)(b_0 + b_1 X + \dots)$$

$X^0 \quad \cdot \quad a_0 b_0 = 1 \iff b_0 := a_0^{-1} \in A$ (since $a_0 \in A^*$)

Then recursively: (assume $b_0, b_1, \dots, b_{n-1}$ are defined)

$$X^n \quad 0 = \sum_{k=0}^{n} a_k b_{n-k} = \underbrace{\sum_{k=1}^{n} a_k b_{n-k}}_{\text{defined}} + a_0 b_n$$

We set $b_n := -a_0^{-1}\left(\sum_{k=1}^{n} a_k b_{n-1}\right)$

By construction $f \cdot g = 1$ in $A[[X]]$

Similar proof works for $A[X]/(X^n)$

Alternatively:

$$A[[X]] \xrightarrow{\psi} A[[X]]/(X^n) \simeq A[X]/(X^n)$$

$$h = a_0 + a_1X + \dots + a_{n-1}X^{n-1} + (X^n)$$

If $a_0 \in A^*$, then $h = \psi(\underbrace{a_0 + a_1X + \dots + a_{n-1}X^{n-1}}_{\text{invertible in } A[[X]]^*})$ invert. in $A[X]/(X^n)$ by Remark

(2) Claim:

$A[[X]]$ is local $\iff$ A is local

"$\Rightarrow$" If A has two different max. ideals m_1 and m_2

Then $(m_1, X) \neq (m_2, X)$ are two different max ideals in $A[[X]]$.

"$\Leftarrow$" Assume A local with unique max ideal m.

Then $A \setminus A^* = m$

And by question (1): $A[[X]] \setminus A[[X]]^* = (m, X)$

$\nearrow$ max. in $A[[X]]$

By lemma 2.37 $A[[X]]$ is local

Same proof works for $A[X]/(X^n)$.

Exercise 3

Recall

T is reducible $\iff$ $\exists$ T_1 and T_2 closed in T $(T_i \neq T)$,

s.t. $T = T_1 \cup T_2$

Assume $\overline{T} = Y_1 \cup Y_2$ (where $Y_i \neq \overline{T}$)

closed in $\overline{T}$ (hence closed in X)

Then $T = (Y_1 \cap T) \cup (Y_2 \cap T)$

closed in T

It remains to show that $Y_i \cap T \neq T$

Assume $Y_i \cap T = T$, then $T \subseteq Y_i$

closed in X

$\Rightarrow$ $\overline{T} \subseteq Y_i \subseteq \overline{T}$ $\Rightarrow$ $Y_i = \overline{T}$ $\lightning$.

The converse is also true:

$T = T_1 \cup T_2$ reducible $(T_i \neq T)$ then $\overline{T} = \overline{T_1 \cup T_2} \overset{\text{Exercise}}{=} \overline{T_1} \cup \overline{T_2}$

(if $\overline{T_i} = \overline{T}$,

then $\forall$ closed Z:

$T_i \subset Z \Rightarrow T \subset Z$)

But $T_i = Z \cap T$ for some closed Z $\Rightarrow$ $T_i \subset Z$ $\Rightarrow T \subset Z$

$\Rightarrow T_i = T$ $\lightning$

Exercise 4

If α is finitely generated, then $\alpha \in S$ is maximal.

Assume S has a maximal element $I \in S$.

Then $\forall a \in A$, $I \subseteq (I, a) \underset{I \text{ maximal}}{\Longrightarrow} I = (I,a) \Rightarrow a \in I$

It follows $I = \alpha$ is finitely generated.

Exercise 5

(1) Remark: X is quasi-compact if the following holds:
$\forall \{X_i\}_{i \in I}$, X_i closed

$$\bigcap_{i \in I} X_i = \emptyset \implies \bigcap_{i \in J} X_i = \emptyset \quad \text{for some finite subset } J \subset I$$

$X = \operatorname{Spec} A$

$X_i = V(\alpha_i)$ for some ideal α_i

$$\bigcap_{i \in I} X_i = \bigcap_{i \in I} V(\alpha_i) = V\Big(\sum_{i \in I} \alpha_i\Big)$$

see Lemma 2.42 (1)

If $\bigcap_{i \in I} X_i = \emptyset$, then $\sum_{i \in I} \alpha_i = A$

and one can write $1 = \sum_{i \in J} a_i$, where $a_i \in \alpha_i$ and $J \subset I$ is finite

It follows

$$\emptyset = V\Big(\sum_{i \in I} \alpha_i\Big) = V\Big(\sum_{i \in J} \alpha_i\Big) = \bigcap_{i \in J} X_i .$$

(2) $I = \{1, 2, \dots, n\}$

~~Note~~

$$\operatorname{Spec} A_i \longrightarrow \operatorname{Spec}(A_1 \times A_2 \times \dots \times A_n)$$
$$\wp_i \longmapsto A_1 \times \dots \times A_{i-1} \times \wp \times A_{i+1} \times \dots \times A_n \quad (*)$$

is continious by Lemma 2.47

Hence, the ~~co~~ map φ from the exercise is also continious

Clearly, φ is injective.

It remains to show that φ is surjective.

That is every prime ideal in $\prod_{i=1}^{n} A_i$ is of the form $(*)$.

Let $e_i = (0, \dots, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, \dots 0) \in A_1 \times \dots \times A_i \times \dots \times A_n$

Then $1 = e_1 + \dots + e_n$ and $e_i e_j = 0 \quad \forall i \neq j$.

~~If~~ Let $I \in \operatorname{Spec}(A_1 \times \dots \times A_n)$

If all $e_i \in I$, then $1 \in I \Rightarrow I = \prod_{i=1}^{n} A_i$ ↯

Assume $e_i \notin I$ for some $i \in \{1, 2, \dots, n\}$.

Then $0 = e_i e_j \Rightarrow e_j \in I \quad \forall j \neq i. \Rightarrow A_j e_j \in I$

It follows that $I = A_1 \times \dots \times \underset{\substack{\cap \\ A_i}}{\wp} \times \dots \times A_n$

where $\wp$ is ~~see~~ a prime ideal in A_i.

$$\left(\begin{array}{ccc} A_1 \times \dots \times A_n & \xrightarrow{\varphi_i \ i\text{-th projection}} & A_i \\ & & \cup \\ I & \longmapsto & \wp \end{array} \right)$$

prime ideal in A_i

(see Lemma 2.26)

(3) If I is infinite, then (2) is wrong:

$\coprod_{i \in I} \operatorname{Spec} A_i$ not a quasi-compact space

$\operatorname{Spec}\left(\prod_{i \in I} A_i\right)$ quasi-compact by question (1).