

Exercise sheet 2

Exercise 1

Recall: R domain, $\mathbb{Z} \xrightarrow{\varphi} R$ ~~the~~ canonical ring hom.
 $1 \mapsto 1$

$$\text{Ker } \varphi = \begin{matrix} (p) \text{ with } p \text{ prime} & \text{or} & (0) \\ \downarrow & & \downarrow \\ \text{char } R = p & & \text{char } R = 0 \end{matrix}$$

(Algebra 1 Bemerkung Seite 35): $R \subset S$ domains $\Rightarrow \text{char } R = \text{char } S$

It follows from Bem. that $\text{char } A = \text{char } F$

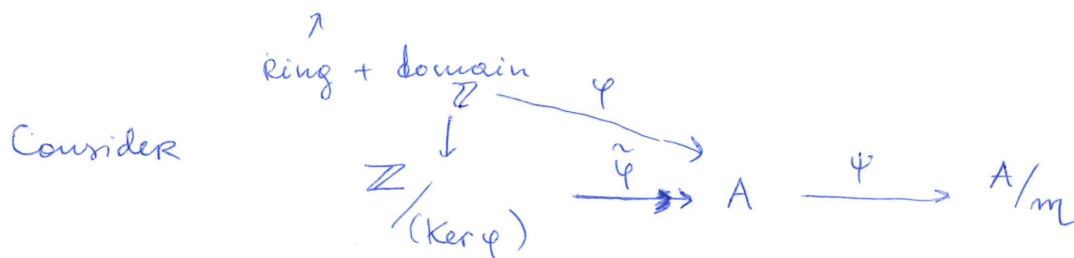
A is of equal char $\stackrel{\text{def}}{\iff} A$ contains a field K

Claim: A is of equal char $\iff \varphi(\mathbb{Z}) \stackrel{\{\cdot\}}{\subset} A^*$, where

$$\varphi: \mathbb{Z} \longrightarrow A \\ 1 \longmapsto 1$$

" \Rightarrow " $\varphi(\mathbb{Z}) \stackrel{\{\cdot\}}{\subset} K^* \subseteq A^*$

" \Leftarrow " $\varphi(\mathbb{Z}) \stackrel{\{\cdot\}}{\subset} A^* \Rightarrow \mathbb{Q}(\varphi(\mathbb{Z})) \subset A^*$
↙ field of fractions

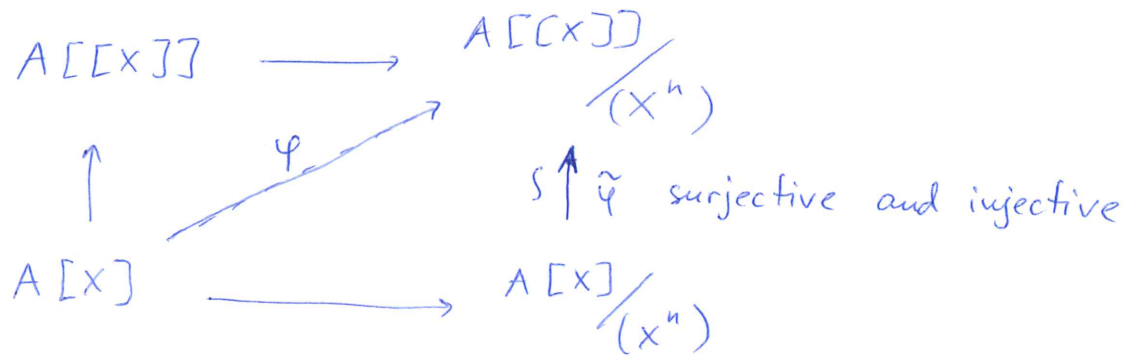


Then $\varphi(\mathbb{Z}) \subset A^* \iff \varphi(\mathbb{Z}) \cap \mathfrak{m} = \emptyset \iff \psi|_{\varphi(\mathbb{Z})} = \tilde{\varphi}(\mathbb{Z})$ is injective

$\iff \psi \circ \tilde{\varphi}$ is injective $\iff \text{Ker } \varphi = \text{Ker } \psi \circ \tilde{\varphi} \iff$

$\iff \text{char } A = \text{char } A/\mathfrak{m}$

Exercise 2



Note that $(x^n) \subset \text{Ker } \varphi$, so $\exists \tilde{\varphi}$ as above.

So we get $A[[x]] \xrightarrow{\quad} A[x] / (x^n) \xrightarrow{\tilde{\varphi}^{-1}} A[x] / (x^n) \xrightarrow{\cong} A[x] / (x)$
|
A

Remark: $\varphi: A \rightarrow B$ Ring hom., then $\varphi(A^*) \subset B^*$

(1) $A[[x]] \xrightarrow{\varphi} A^* \cong A[[x]] / (x)$

$f = a_0 + a_1x + a_2x^2 + \dots \mapsto a_0$

$f \in A[[x]]^* \Rightarrow \varphi(f) = a_0 \in A^*$
 Remark

Let now $f = a_0 + a_1x + a_2x^2 + \dots \in A[[x]]$ with $a_0 \in A^*$

We construct the inverse $g = b_0 + b_1x + b_2x^2 + \dots$
 as follows:

$1 = (a_0 + a_1x + \dots)(b_0 + b_1x + \dots)$

$x^0 \cdot a_0 b_0 = 1 \Leftrightarrow b_0 := a_0^{-1} \in A$ (since $a_0 \in A^*$)

Then recursively: (assume b_0, b_1, \dots, b_{n-1} are defined)

$x^n \cdot 0 = \sum_{k=0}^n a_k b_{n-k} = \underbrace{\sum_{k=1}^n a_k b_{n-k}}_{\text{defined}} + a_0 b_n$

We set $b_n := -a_0^{-1} \left(\sum_{k=1}^n a_k b_{n-k} \right)$

By construction $f \cdot g = 1$ in $A[[x]]$

Similar proof works for $A[x]/(x^n)$

Alternatively:

$$A[[x]] \xrightarrow{\psi} A[[x]]/(x^n) \cong A[x]/(x^n)$$

$$h = a_0 + a_1x + \dots + a_{n-1}x^{n-1} + (x^n)$$

If $a_0 \in A^*$, then $h = \underbrace{\psi(a_0 + a_1x + \dots + a_{n-1}x^{n-1})}_{\text{invertible in } A[[x]]^*}$ invert. in $A[x]/(x^n)$ by Remark

(2) Claim:

$$A[[x]] \text{ is local } \Leftrightarrow A \text{ is local}$$

" \Rightarrow " If A has two different max. ideals m_1 and m_2
 Then $(m_1, x) \neq (m_2, x)$ are two different max ideals in $A[[x]]$.

" \Leftarrow " Assume A local with unique max ideal m .
 Then $A \setminus A^* = m$

And by question (1) : $A[[x]] \setminus A[[x]]^* = (m, x)$
 \uparrow
 max. in $A[[x]]$

By lemma 2.37 $A[[x]]$ is local

Same proof works for $A[x]/(x^n)$.

Exercise 3

(5)

Recall

T is reducible $\Leftrightarrow \exists T_1$ and T_2 closed in T ($T_i \neq T$),
 s.t. $T = T_1 \cup T_2$

Assume $\bar{T} = Y_1 \cup Y_2$ (where $Y_i \neq \bar{T}$)
 $\uparrow \quad \uparrow$
 closed in \bar{T} (hence closed in X)

Then $T = (Y_1 \cap T) \cup (Y_2 \cap T)$
 $\uparrow \quad \uparrow$
 closed in T

It remains to show that $Y_i \cap T \neq T$

Assume $Y_i \cap T = T$, then $\exists_{\mathbb{B}} T \subseteq Y_i$
 \uparrow
 closed in X

$\Rightarrow \bar{T} \subseteq Y_i \subseteq \bar{T} \Rightarrow Y_i = \bar{T} \downarrow$

The converse is also true:

$T = T_1 \cup T_2$ then
 reducible
 ($T_i \neq T$)

Exercise
 \downarrow
 $\bar{T} = \overline{T_1 \cup T_2} = \bar{T}_1 \cup \bar{T}_2$

(if $\bar{T}_i = \bar{T}$,
 then \forall closed Z :
 $T_i \subset Z \Rightarrow T \subset Z$)
 But $T_i = Z \cap T$ for some
 closed $Z \Rightarrow T_i \subset Z \Rightarrow T \subset Z$
 $\Rightarrow T_i = T \downarrow$

Exercise 4

(6)

If α is finitely generated, then $\alpha \in S$ is maximal.

Assume S has a maximal element I .

Then $\forall a \in A$, $I \subseteq (I, a) \xRightarrow{I \text{ maximal}} I = (I, a) \Rightarrow a \in I$

It follows $I = \alpha$ is finitely generated.

Exercise 5

(1) Remark: X is quasi-compact if the following holds:
 $\forall \{X_i\}_{i \in I}$, X_i closed

$$\bigcap_{i \in I} X_i = \emptyset \Rightarrow \bigcap_{i \in J} X_i = \emptyset \quad \text{for some finite subset } J \subset I$$

$$X = \text{Spec } A$$

$$X_i = V(\alpha_i) \quad \text{for some ideal } \alpha_i$$

$$\bigcap_{i \in I} X_i = \bigcap_{i \in I} V(\alpha_i) = V\left(\sum_{i \in I} \alpha_i\right)$$

see Lemma 2.42 (1)

$$\text{If } \bigcap_{i \in I} X_i = \emptyset, \text{ then } \sum_{i \in I} \alpha_i = A$$

and one can write $1 = \sum_{i \in J} a_i$, where $a_i \in \alpha_i$ and $J \subset I$ is finite

It follows

$$\emptyset = V\left(\sum_{i \in I} \alpha_i\right) = V\left(\sum_{i \in J} \alpha_i\right) = \bigcap_{i \in J} X_i$$

(3) If I is infinite, then (2) is wrong:

$$\coprod_{i \in I} \text{Spec } A_i$$

not a quasi-compact space

$$\text{Spec} \left(\prod_{i \in I} A_i \right)$$

quasi-compact by question (1).