Algebra 2

Exercise Sheet 11

Prof. Markus Land	Summer Semester 2023
Dr. Maksim Zhykhovich	12.07.2023

Let R be a ring and M an R-module. The *support* of M is the set

$$\mathrm{Supp}(M) = \{\rho \in \mathrm{Spec}\,(R) \mid M_\rho \neq 0\}\,.$$

Exercise 1. Let R be a ring and M an R-module of finite lenght. Let

$$M = M_0 \supset M_1 \supset \cdots \supset M_0 = 0$$

be a composition series of M.
(1) Show that

$$\mathrm{Supp}(M) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid M_{i-1}/M_i \simeq R/\mathfrak{m} \text{ for some } i\}\,.$$

Hint: Localize the above composition series at $\rho \in \mathrm{Spec}\,(R)$.
(2) Prove the Jordan-Hölder Theorem for modules of finite lenght (see Remark 8.20).

Exercise 2. Let R be a ring and M an R-module of finite lenght. Show that the canonical map

$$M \longrightarrow \bigoplus_{\mathfrak{m}\in \mathrm{Supp}(M)} M_{\mathfrak{m}}$$

is an isomorphism of R-modules.

Exercise 3. Let A be a factorial ring and $Q(A)$ its quotient field. Show that A is integrally closed in $Q(A)$.

Exercise 4. Let K be a field.
(1) Show that $X^3 - Y^2$ is irreducible in $K[X, Y]$.
(2) Denote by A the ring $K[X, Y]/(X^3 - Y^2)$ and by $Q(A)$ the quotient field of A. Show that there is a unique ring homomorphism: $\varphi : A \to K[t]$, such that $\varphi(X) = t^2$, $\varphi(Y) = t^3$. Show that φ is injective. Describe $\varphi(A)$ and show that φ is not surjective.
(3) Show that the field of fractions of $\varphi(A)$ is $K(t)$. Find the integral closure of $\varphi(A)$ in $K(t)$.
(4) Using (3) find the integral closure of A in $Q(A)$.