Algebra 2

Exercise Sheet 11

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Let R be a ring and M an R-module. The *support* of M is the set

$$\mathrm{Supp}(M) = \{\rho \in \mathrm{Spec}\,(R) \mid M_\rho \neq 0\}\,.$$

Exercise 1. Let R be a ring and M an R-module of finite lenght. Let

$$M = M_0 \supset M_1 \supset \cdots \supset M_0 = 0$$

be a composition series of M.
(1) Show that

$$\mathrm{Supp}(M) = \{\mathfrak{m} \in \mathrm{Max}(R) \mid M_{i-1}/M_i \simeq R/\mathfrak{m} \text{ for some } i\}\,.$$

Hint: Localize the above composition series at $\rho \in \mathrm{Spec}\,(R)$.
(2) Prove the Jordan-Hölder Theorem for modules of finite lenght (see Remark 8.20).

Exercise 2. Let R be a ring and M an R-module of finite lenght. Show that the canonical map

$$M \longrightarrow \bigoplus_{\mathfrak{m}\in\mathrm{Supp}(M)} M_{\mathfrak{m}}$$

is an isomorphism of R-modules.

Exercise 3. Let A be a factorial ring and $Q(A)$ its quotient field. Show that A is integrally closed in $Q(A)$.

Exercise 4. Let K be a field.
(1) Show that $X^3 - Y^2$ is irreducible in $K[X,Y]$.
(2) Denote by A the ring $K[X,Y]/(X^3 - Y^2)$ and by $Q(A)$ the quotient field of A. Show that there is a unique ring homomorphism: $\varphi : A \to K[t]$, such that $\varphi(X) = t^2$, $\varphi(Y) = t^3$. Show that φ is injective. Describe $\varphi(A)$ and show that φ is not surjective.
(3) Show that the field of fractions of $\varphi(A)$ is $K(t)$. Find the integral closure of $\varphi(A)$ in $K(t)$.
(4) Using (3) find the integral closure of A in $Q(A)$.

Exercise 1

Lemma 1 Let $\mathfrak{m} \in \mathrm{Max}(R)$

~~Then Supp(~~

Let $\mathfrak{p} \in \operatorname{Spec} R$, Then $\left(R/\mathfrak{m}\right)_{\mathfrak{p}} = \begin{cases} 0, & \text{if } \mathfrak{p} \neq \mathfrak{m} \\ \neq 0, & \text{if } \mathfrak{p} = \mathfrak{m} \end{cases}$

That is $\operatorname{Supp}(R/\mathfrak{m}) = \mathfrak{m}$

~~foll~~ follows from Tutorium 11. Ex 1.6

$$\operatorname{Supp}(R/\mathfrak{m}) = V(\cancel{R}\operatorname{Ann}(R/\mathfrak{m})) = V(\mathfrak{m}) = \{\mathfrak{m}\}.$$

Lemma 2 M R-module, $S \subseteq R$ mult. system.

$$\mu_s : M \longrightarrow M$$
$$m \longmapsto sm$$

Then $M \xrightarrow{\varphi} S^{-1}M$ is an isomorphism of R-modules
$$m \longmapsto \frac{m}{1}$$

$\Leftrightarrow$ μ_s is iso $\forall s \in S$

"$\Leftarrow$" Injectivity of φ: $\varphi(m) = \frac{m}{1} = 0 \iff sm = 0$ for some $s \in S$

$$\Updownarrow$$

$m = 0$ (μ_s iso)

Surjectivity of φ: $\frac{m}{s} \in S^{-1}M$, then $m = sm'$ for some $m' \in M$

$$\frac{m}{s} = \frac{sm'}{s} = \frac{m'}{1} = \varphi(m')$$

In particular, $R/\mathfrak{m} \xrightarrow{\sim} \left(R/\mathfrak{m}\right)_{\mathfrak{m}}$ ~~$R/\mathfrak{m}$~~ $= S^{-1}R/S^{-1}\mathfrak{m} = R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$

Every element s in $R \setminus \mathfrak{m}$ is invertible in $R/\mathfrak{m}$

$\Rightarrow$ μ_s an iso.

Exercise 1 (1) To show: $\mathrm{Supp}(M) = \{ m \in \mathrm{Max}(R) \mid M_{i-1}/M_i \simeq R/m \text{ for some } i \} = \{m_1, \dots, m_r\}$

Let (*) $M = M_0 \supset M_1 \supset \dots \supset M_n = 0$ be a comp. series of M

M_{i-1}/M_i simple. $\forall i = 1, \dots, n$.

Let $\rho \in$ ~~Supp(M)~~ $\mathrm{Spec}(R)$

We localize (*) at ρ, we get

$$M_\rho = (M_0)_\rho \supseteq (M_1)_\rho \supseteq \dots \supseteq (M_n)_\rho = 0$$

~~Since~~ $(M_\rho) \neq 0 \overset{\Longleftrightarrow}{\cancel{\text{then}}} \exists\, i \in \{1, \dots, n\}$, s.t.

$$(M_{i-1})_\rho \underset{\text{strict}}{\supsetneq} (M_i)_\rho \Longleftrightarrow 0 \neq (M_{i-1})_\rho / (M_i)_\rho =$$

$$= \Big(\underbrace{M_{i-1}/M_i}_{R/m_i}\Big)_\rho \underset{\text{Lemma 1}}{\Longleftrightarrow} \rho = m_i$$

R/m_i for some ~~$m \in \mathrm{Max}(R)$~~ $i \in [1, r]$

Exercise 2.3
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(2) Let (**) $M = M'_0 \supset M'_1 \supset \dots \supset M'_n = 0$

be another composition series

For (*) and (respectively (**)) ~~to~~ let $M_{i-1}/M_i \simeq R/m_i$ (resp R/m'_i)

As sets we have equality

$$\mathrm{Supp}\, M = \underbrace{\{m_1, \dots, m_n\}}_{(a)} = \underbrace{\{m'_1, \dots, m'_n\}}_{(b)}.$$

We need to show that $m \in \mathrm{Supp}(M)$ appears in (a) and (b) the same number of times.

Localize (*) and (**) at m

Then (after cancelling the same submodules $(M_i)_p : (M_{i-1})_p = (M_i)_p$

we get two comp. series for M_m

and the number of m in (a) and in (b)

is equal to $\ell(M_m)$

Exercise 2 :

Note that by Exercise 1 $\operatorname{Supp} M_{\mathfrak{m}} = \{\mathfrak{m}\}$. (~~that is $(M_{\mathfrak{m}})_{\mathfrak{p}} = \{ 0, \text{if } \mathfrak{p} \neq \mathfrak{m}$~~

~~Let $\varphi : M \longrightarrow \bigoplus$~~

That is for $\mathfrak{p} \in \operatorname{Spec}(R)$: $(M_{\mathfrak{m}})_{\mathfrak{p}} = \begin{cases} 0, & \text{if } \mathfrak{p} \neq \mathfrak{m} \\ \neq 0, & \text{if } \mathfrak{p} = \mathfrak{m} \end{cases}$

By Lemma 2 one can show that $M_{\mathfrak{m}} \simeq (M_{\mathfrak{m}})_{\mathfrak{m}}$

Let
$$\varphi : M \longrightarrow \bigoplus_{\mathfrak{m} \in \operatorname{Supp}(M)} M_{\mathfrak{m}}$$
$$x \longmapsto \left(\frac{x}{1}\right)_{\mathfrak{m} \in \operatorname{Supp}(M)}$$

$\forall\ \mathfrak{m}' \in \operatorname{Max}(R)$ we have

$$\varphi_{\mathfrak{m}'} : M_{\mathfrak{m}'} \xrightarrow{\sim} \bigoplus_{\mathfrak{m} \in \operatorname{Supp}(M)} (M_{\mathfrak{m}})_{\mathfrak{m}'} = M_{\mathfrak{m}'}$$

By Prop 7.25 (Remark 7.26) φ is an isomorphism.

Exercise 3

Assume $y = \frac{a}{b} \in Q(A)$, $a \in A$, $b \in A$, $b \neq 0$

is integral over A. and $\frac{a}{b} \notin A$

Without loss of generality we can assume that a and b do not have common irreducible factors

Then $\frac{a}{b} \notin A \Rightarrow \exists$ irreduciple p, $p \mid b$ and $p \nmid a$.

Since y is integral over A, we have

$$y^n + a_{n-1} y^{n-1} + \dots + a_1 y + a_0 = 0$$ for some $a_i \in A$ and $n \in \mathbb{N}$

Multiplying this equality by b^n we get

$$a^n + \underbrace{b a_{n-1} a^{n-1} + \dots + b^{n-1} a_1 a + b^n a_0}_{\text{divisible by } b \;\Rightarrow\; \text{divisible by } p} = 0$$

$\Rightarrow p \mid a^n \Rightarrow p \mid a$ ↯

Exercise 4

(1) $X^3 - Y^2$ irreducible $\iff$ $\cancel{X}\ Y^2 - X^3 \in K[X][Y]$ irreducible

$\uparrow$ monic in Y (primitive)

If $Y^2 - X^3$ is reducible, then we have

$$Y^2 - X^3 = (Y - f_1(X))(Y - f_2(X)) \Rightarrow f_1^2(X) = X^3 \text{ in } K[X]$$

$\cancel{\&}$ $2 \deg f_1 = 3$ $\lightning$

So $A = K[X,Y]/(X^3 - Y^2)$ is an integral domain

(2)

Consider the ring hom. $\tilde{\varphi}: K[X,Y] \longrightarrow K[t]$

$$X \longmapsto t^2$$
$$Y \longmapsto t^3$$

Since $\tilde{\varphi}(X^3 - Y^2) = \tilde{\varphi}(X)^3 - \tilde{\varphi}(Y)^2 = (t^2)^3 - (t^3)^2 = 0$

$\Rightarrow (X^3 - Y^2) \in \operatorname{Ker} \tilde{\varphi}$, then

$$\begin{tikzcd} K[X,Y] \arrow[r, "\tilde{\varphi}"] \arrow[d] & K[t] \\ A = K[X,Y]/(X^3-Y^2) \arrow[ur, dashed, "\exists \varphi"'] & \end{tikzcd}$$

?

$\exists$ $\varphi: A \longrightarrow F[t]$

$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

where x, y = classes of X and Y in A.

Since $y^2 = x^3$ in A, every element in A can be written as $g(x) + f(x)y$, where $g, f \in K[X]$.

Then $\varphi(g(x) + f(x)y) = g(t^2) + f(t^2)t^3 \neq 0$ if $f \neq 0$ or $g \neq 0$

monomials only with odd degrees

monomials with even degrees

It follows that φ is injective.

$$\underset{\varphi(A)}{\underset{\shortparallel}{\mathrm{Im}\,\varphi}} = \mathrm{Im}\,\tilde{\varphi} = K[t^2, t^3] \subseteq K[t]$$

$$= K \oplus Kt^2 \oplus Kt^3 \oplus Kt^4 \ldots$$

$$= K \oplus \bigoplus_{n=2}^{\infty} Kt^n \leftarrow K\text{- subalgebra in } K[t].$$

$t \notin \mathrm{Im}\,\varphi \Rightarrow \varphi$ not surjective.

(3) $Q(\varphi(A)) \subseteq Q(K[t]) = K(t)$

But $t = \dfrac{t^3}{t^2} \in Q(\varphi(A))$. Hence, $Q(\varphi(A)) = K(t)$

(*) integrally closed in $K(t)$ (by Ex 3) since $K[t]$ is factorial

We have $\varphi(A) \hookrightarrow K[t] \hookrightarrow K(t)$

integral extension, since t is a root of $Z^2 - \underset{\varphi(A)}{\underset{\cap}{t^2}}$

$\Downarrow$

t integral over $\varphi(A)$

It follows that the integral closure C of $\varphi(A)$ in $K(t)$ is equal to $K[t]$.

Indeed, $K[t] \subseteq C$ and if $f \in K(t)$ and integral over $\varphi(A)$ then also integral over $K[t]$ $\underset{\text{by } (*)}{\Longrightarrow}$ $f \in K[t]$.

(4)

$$\begin{tikzcd}
Y/X \arrow[r, mapsto] & t \\
\cancel{\mathrm{Frac}}\,Q(A) \arrow[r, "\sim"] & F(t) \\
C_A \arrow[u, hook] & F[t] \arrow[u, hook] \\
A \arrow[u, hook] \arrow[r, "\sim"] & \varphi(A) \arrow[u, hook]
\end{tikzcd}$$

integral closure of A in $Q(A)$ $\rightarrow C_A$

$\rightsquigarrow \quad C_A = A[Y/X]$.