Algebra 2

Exercise Sheet 10

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Summer Semester 2023
05.07.2023

Exercise 1. Let R be a von Neumann regular ring.
(1) Show that R_ρ is a field for every prime ideal $\rho \in \operatorname{Spec} R$.
Hint: Use the properties of von Neumann regular rings from Lemma 6.42.
(2) Deduce from (1) that every module over R is flat.

Exercise 2. (1) Let R be a commutative ring. Assume that
(a) $R_\mathfrak{m}$ is a Noetherian ring for every maximal ideal $\mathfrak{m}$ of R.
(b) Every nonzero element $x \in R$ lies only in finitely many maximal ideals.
Show that R is Noetherian.
(2) Using Exercise 1 find an example of a non-Noetherian ring R, such that $R_\mathfrak{m}$ is Noetherian for every maximal ideal $\mathfrak{m}$ of R.
Remark: It means that in general "to be a Noetherian ring" is not a local property (that is the property (a) (without (b)) from (1) does not always imply that R is Noetherian).

Exercise 3. Let p be a prime number. Consider the following $\mathbb{Z}$-module: $\mathbb{Z}_p = \{\frac{a}{p^n} \mid a \in \mathbb{Z}, n \in \mathbb{N}\} \subset \mathbb{Q}$. Show that the $\mathbb{Z}$-module $\mathbb{Z}_p/\mathbb{Z}$ is Artinian and not Noetherian.

Exercise 4. Let K be a field and A a finitely generated K-algebra. Show that the following conditions are equivalent:
(a) A is an Artinian ring.
(b) A is a finite dimensional K-vector space.
Hint: For " $\implies$ " first reduce to the case where A is an Artin local ring. Then use Lemma 5.5 to show that the residue field of A is a finite extension of K.