Algebra 2

Exercise Sheet 1

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Summer Semester 2023
25.04.2023

Exercise 1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a commutative ring A.
(1) Show that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$.
(2) Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime (that is $\mathfrak{a} + \mathfrak{b} = A$). Show that $\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$.
(3) Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be pairwise coprime ideals in A. Show that $\prod_{i=1}^n \mathfrak{a}_i = \bigcap_{i=1}^n \mathfrak{a}_i$.

Exercise 2. Let I and J be two ideals of a commutative ring A and $\pi : A \to A/I$ the canonical projection. Show that $\pi(J)$ is an ideal in A/I and

$$(A/I)/\pi(J) \simeq A/(I + J)\,.$$

Exercise 3. Let p be a prime number.
(1) Show that -1 is not a square in $\mathbb{F}_p$ if and only if $p = 3 \mod 4$.
Remark: See Aufgabe 1, Tutoriumsblatt 3 (Algebra 1).
(2) Let $\mathbb{Z}[i]$ be the ring of Gaussian intergers. Show that the ideal (p) is prime if and only if $p = 3 \mod 4$.
Hint: Consider the quotient ring $\mathbb{Z}[i]/(p)$, observe that $\mathbb{Z}[i] \simeq \mathbb{Z}[X]/(X^2+1)$ and use Exercise 2 and question (1).

Exercise 4. Show that every prime ideal $\mathfrak{p}$ in $\mathbb{Z}[X]$ has one of the following form

(1) $\mathfrak{p} = (0)$.
(2) $\mathfrak{p} = (p)$, where p is a prime number.
(3) $\mathfrak{p} = (f)$, where f is an irreducible polynomial in $\mathbb{Z}[X]$.
(4) $\mathfrak{p} = (p, f)$, where p is a prime number and f a polynomial in $\mathbb{Z}[X]$ irreducible modulo p.

Hint: Show that $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$ and consider two cases: $\mathfrak{p} \cap \mathbb{Z} \neq (0)$ and $\mathfrak{p} \cap \mathbb{Z} = (0)$.