# § 7 Localization

**Recall:** $R$ integral ring

$R$ integral ring $\rightsquigarrow$ Quotient field $Q(R) := \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\}$

was defined as $R \times (R \setminus (0)) / \sim$

where $(a,b) \sim (a',b')$ if $ab' = a'b$

eq. class denoted by $\frac{a}{b}$

$R$ comm. ring

**Definition 7.1** Multiplicatively closed subset $S$ of $R$ is a subset $S \subseteq R$ such that

- $1 \in S$
- $s \in S$ & $s' \in S \Rightarrow s \cdot s' \in S$

**Example:**
- $R$ integral domain, $S = R \setminus \{0\}$
- $R$ arbitrary, $S = R \setminus \mathfrak{p}$ where $\mathfrak{p} \in \operatorname{Spec} R$.

**Definition 7.2** Let $S \subseteq R$ mult. closed subset

(Localization of $R$ along $S$)

$$S^{-1}R := R \times S / \sim$$

where $(a, s_1) \sim (b, s_2)$ if $\exists\, s \in S$ such that

$a s_2 s = b s_1 s$ for some $s \in S$

$\Updownarrow$

$(a s_2 - b s_1) s = 0$

$\sim$ is an equivalence relation

- reflexive (take $s = 1 \in S$)
- clearly symmetric
- transitive

$$(a, s_1) \sim (b, s_2) \sim (c, s_3) \overset{?}{\Longrightarrow} (a, s_1) \sim (c, s_3)$$

$$0 = (as_2 - bs_1)s \qquad 0 = (bs_3 - cs_3)s'$$

It follows $(as_3 - cs_1)\underbrace{s_2 s s'}_{\in S} = 0 \Rightarrow (a, s_1) \sim (c, s_3)$

We denote by $\frac{a}{s}$ the equiv. class of $(a,s)$.

Define :

- $\frac{a}{s_1} \cdot \frac{b}{s_2} = \frac{ab}{s_1 s_2}$ (well-defined if $\frac{b'}{s_2'} = \frac{b}{s_2}$, check that $\frac{ab'}{s_1 s_2'} = \frac{ab}{s_1 s_2}$)

- $\frac{a}{s_1} + \frac{b}{s_2} = \frac{as_2 + bs_1}{s_1 s_2}$

$S^{-1}R$ is a ring with respect to $+, \cdot$

$\frac{0}{1} =$ zero element

$\frac{1}{1} =$ 1-element.

We also have a natural ring hom.

$$\begin{aligned} R &\longrightarrow S^{-1}R \\ r &\longmapsto \frac{r}{1} \end{aligned}$$ (not always injective!)

## Remarks / examples 7.3

1) In $S^{-1}R$ holds: $\frac{a}{s} = \frac{as'}{ss'}$ $\forall a \in R,\ s, s' \in S$

2) $S^{-1}R = 0 \iff 0 \in S$ (Tutorium 9)

3) $\varphi: R \longrightarrow S^{-1}R$ canonical ring hom.

Elements from $S$ become units in $S^{-1}R$

4) $R$ integral

Then $\varphi$ is injective

$$R \hookrightarrow S^{-1}R = \{ \tfrac{a}{s} \mid a \in R,\ s \in S \} \subseteq \overbrace{Q(R)}^{\text{quotient ring of } R}$$

$$a \longmapsto \tfrac{a}{1}$$

can be seen as a subring in $Q(R)$.

5) $f \in R$, $S = \{1, f, f^2, \dots\} = \{f^n\}_{n \in \mathbb{N}}$ mult. closed in $R$

$R_f := S^{-1}R$

$f = 3$, $R = \mathbb{Z}$ $\quad \mathbb{Z}_3 = \{ \frac{a}{3^n} \mid a \in \mathbb{Z},\ n \in \mathbb{N} \} \subseteq \mathbb{Q}$

6) $R$ integral, $S = R \setminus \{0\}$ Then $S^{-1}R = Q(R)$

7) ~~More generaly,~~ $R$ arbitrary, let $\mathfrak{p} \in \operatorname{Spec} R$. Then $S = R \setminus \mathfrak{p} \subseteq R$ mult. closed

$R_\mathfrak{p}$ ~~denoted as~~ $:= S^{-1}R$ ("localization at $\mathfrak{p}$")

Example: $R = \mathbb{Z}$, $p$ prime number $\mathbb{Z}_{(p)} = \{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ p \nmid b \}$

subring of $\mathbb{Q}$.

8) $I \subset R$ ideal, $S = 1 + I$ mult closed subset

9) $S = R^*$ mult. closed in $R$, in this case $S^{-1}R \simeq R$ (Tutorium)

$S^{-1}R$ "universal" example of an $R$-algebra in which all the elements of $S$ become units.

~~Theorem~~ "Universal property" of $S^{-1}R$:

<u>Proposition 7.4</u>

$$a \longmapsto \frac{a}{1}$$

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & S^{-1}R \arrow[d, dashed, "f"] \\ & R' \end{tikzcd}$$

~~Given a ring hom. $\psi: R \to R'$,~~

Let $\psi: R \to R'$ be a ring hom., such that $\psi(S) \subset R'^{*}$. Then $\exists!\ f: S^{-1}R \to R'$ such that the above diagram commutes: (that is $\psi = f \circ \varphi$)

<u>Proof</u>: Assume that such $f$ exists and we show that then $f$ is is uniquely determined by $\psi$.

1) <u>Uniqueness</u> $f\left(\frac{a}{1}\right) = \psi(a) \quad \forall a \in R$ (Indeed, $f\left(\frac{a}{1}\right) = f(\varphi(a)) = \psi(a)$)

$1 = f\left(\frac{s}{1}\right) f\left(\frac{1}{s}\right)$ in $R'$

if $s \in S$, $f\left(\frac{1}{s}\right) = f\left(\frac{s}{1}\right)^{-1} = \psi(s)^{-1}$

$$f\left(\frac{a}{s}\right) = f\left(\frac{a}{1}\right) \cdot f\left(\frac{1}{s}\right) = \psi(a)\psi(s)^{-1}$$

2) <u>Existence</u>: Define $f: S^{-1}R \to R'$

$$\frac{a}{s} \longmapsto \psi(a)\psi(s)^{-1}$$

~~Well defined~~

$$\cancel{\frac{a}{s} + \frac{a'}{s'} \longmapsto \psi(a)\psi(s)^{-1} + \psi(a')\psi(s')^{-1}}$$

We need to check that $f$ is well-defined & $f$ is a ring homomorphism

(Well-defined:)

Suppose $\frac{a}{s} = \frac{a'}{s'}$ $\quad (as' - a's)t = 0$ for some $t \in S$

$\swarrow$ apply $\psi$

$(\psi(a)\psi(s') - \psi(a')\psi(s))\underbrace{\psi(t)}_{\in R'^*} = 0 \Rightarrow \psi(a)\psi(s') - \psi(a')\psi(s) = 0$

$\Rightarrow \psi(a)\psi(s)^{-1} = \psi(a')\psi(s')^{-1} \Rightarrow$ $\quad f: S^{-1}R \longrightarrow R'$, $\frac{a}{s} \longmapsto \psi(a)\psi(s)^{-1}$ is well-defined

Straightforward to check that $f$ is a ring hom.

$$f\left(\frac{a}{s}\cdot\frac{a'}{s'}\right) \overset{\text{to check}}{=} f\left(\frac{a}{s}\right) f\left(\frac{a'}{s'}\right)$$

$$f\left(\frac{a}{s} + \frac{a'}{s'}\right) = f\left(\frac{as' + a's}{ss'}\right) \overset{\text{to check}}{=} f\left(\frac{a}{s}\right) + f\left(\frac{a}{s'}\right).$$

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## Ideals in $S^{-1}R$.

~~Remark~~ Remark-Definition 7.5
~~Definition~~ $R$ comm. ring, $S$ multiplicative set. $I \subseteq R$ ideal. Then

- $I^S := \{ a \in R \mid \exists s \in S \text{ with } a \cdot s \in I \}$ (saturation of $I$ with respect to $S$)

$I^S$ is an ideal

Indeed, let $a, b \in I^S$, $r \in R$

$\Downarrow$

$as \in I$ and $bs' \in I$ for some $s, s' \in S$

Then $(ra)\cdot s \in I \Rightarrow ra \in I^S$

$(a + a')\cdot ss' \in I \Rightarrow a + a' \in I^S$

- We have the following properties: • $I \subseteq I^S$, • $(I^S)^S = I^S$, • $\mathrm{Ker}(R \longrightarrow S^{-1}R) = (0)^S$

- $S^{-1}I := \{ \frac{i}{s} \mid i \in I, s \in S \}$ an ideal in $S^{-1}R$ (Tutorium)

ideal in $S^{-1}I$
generated by elements from $\varphi(I)$, $\varphi: R \longrightarrow S^{-1}R$

- $I_1 \subseteq I_2 \implies S^{-1}I_1 \subseteq S^{-1}I_2$

**Proposition 7.6** $I \subseteq R$ ideal, $S$ mult. subset of $R$.

$S^{-1}I = S^{-1}R \iff I \cap S \neq \emptyset$

**Proof**

"$\Leftarrow$" Let $i \in I \cap S$

Then $\frac{i}{1} \in S^{-1}I \cap (S^{-1}R)^* \implies S^{-1}I = S^{-1}R$

"$\Rightarrow$" If $S^{-1}I = S^{-1}R$, then $\frac{1}{1} = \frac{i}{s}$ for some $i \in I$, $s \in S$

$\implies (i-s)s' = 0$ for some $s' \in S \implies \underset{I}{\underset{\cap}{is'}} = \underset{S}{\underset{\cap}{ss'}} \implies I \cap S \neq \emptyset$.

$\varphi: R \longrightarrow S^{-1}R,\ a \longmapsto \frac{a}{1}$

**Proposition 7.7** Let $J \subset S^{-1}R$ an ideal. Set $I = \underbrace{\varphi^{-1}(J)}_{\text{ideal in } R}$

Then (1) $J = S^{-1}I$ and (2) $I = I^S$

**Proof** (1) "$\subset$" Let $\frac{a}{s} \in J$, then $\frac{a}{s} \cdot \frac{s}{1} = \frac{a}{1} \in J$ (since $J$ is an ideal)

$\implies \varphi(a) \in J \implies a \in \varphi^{-1}(J) = I \implies \frac{a}{s} \in S^{-1}I$

"$\supset$" Let $\frac{i}{s} \in S^{-1}I$. Since $i \in I$, $\varphi(i) = \frac{i}{1} \in J$

Then $\frac{i}{s} = \underbrace{\frac{i}{1}}_{\in J} \cdot \underbrace{\frac{1}{s}}_{\in S^{-1}R} \in J$

(2) Clearly $I \subset I^S$
Let $a \in I^S$ (that is $as \in \underset{\varphi^{-1}(J)}{I}$ for some $s \in S$)

$\implies \frac{as}{1} = \varphi(as) \in J \implies \frac{as}{1} \cdot \frac{1}{s} \in J \implies \frac{as}{s} \in J \implies \frac{a}{1} \in J \implies a \in I$

$\blacksquare$

**Proposition 7.8** $\varphi : R \longrightarrow S^{-1}R$, $a \longmapsto \frac{a}{1}$ canonical map, $I \subseteq R$ ideal

Then $\varphi^{-1}(S^{-1}I) = \underset{\substack{\uparrow \\ \text{recall}}}{I^S =} \{a \in R \mid as \in I \text{ for some } s \in S\}$

**Proof**

"$\subseteq$" Let $a \in \varphi^{-1}(S^{-1}I)$, then $\varphi(a) = \frac{a}{1} \in S^{-1}I$

That is $\frac{a}{1} = \frac{i}{s}$ for some $i \in I$, $s \in S$

$\Rightarrow (as - i)\underset{\substack{\cap \\ S}}{s'} = 0 \Rightarrow \underbrace{ss'}_{\in S} a = is' \in I \Rightarrow a \in I^S$

"$\supseteq$" Let $a \in I^S$, then $as \in I$ for some $s \in S$

$\varphi(a) = \frac{a}{1} = \frac{\overbrace{as}^{\in I}}{s} \in S^{-1}I$

**Proposition 7.9** $\mathfrak{p} \in \operatorname{Spec} R$. Assume $\mathfrak{p} \cap S = \emptyset$

Then (1) $\mathfrak{p}^S = \mathfrak{p}$ (2) $S^{-1}\mathfrak{p} \in \operatorname{Spec}(S^{-1}R)$

Proof (next lecture)

Using Propositions 7.6 – 7.9 we get

~~Proposition 7.9~~ **Corollary 7.10** There is 1-to-1 correspondence (bijection)

$$\left\{\begin{array}{c}\text{prime ideals} \\ \text{in } R \text{ such that} \\ \mathfrak{p} \cap S = \emptyset\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{prime ideals} \\ \text{in } S^{-1}R\end{array}\right\}$$

$$\mathfrak{p} \longmapsto S^{-1}\mathfrak{p}$$

$$\varphi^{-1}(\mathfrak{q}) \longleftarrow\!\shortmid \mathfrak{q}$$

Proposition 7.9 $\mathfrak{p} \in \operatorname{Spec} R$. Assume $\mathfrak{p} \cap S = \emptyset$

Then (1) $\mathfrak{p}^S = \mathfrak{p}$ (2) $S^{-1}\mathfrak{p} \in \operatorname{Spec}(S^{-1}R)$

Proof (1) $\mathfrak{p}^S = \{ a \in R \mid as \in \mathfrak{p} \text{ for some } s \in S \}$

Clearly $\mathfrak{p} \subseteq \mathfrak{p}^S$

"$\supseteq$" Let $a \in \mathfrak{p}^S \Rightarrow \exists s \in S$ s.t. $a\overset{\notin \mathfrak{p}}{s} \in \mathfrak{p}$ ← prime

$$\Downarrow$$

$$a \in \mathfrak{p}$$

(2) Assume $\frac{a}{s_1} \cdot \frac{b}{s_2} = \frac{x}{s} \in S^{-1}\mathfrak{p}$, where $x \in \mathfrak{p}$

Then in R: $ab \underbrace{ss'}_{\substack{\in S \\ \notin \mathfrak{p}}} = \underbrace{x s_1 s_2 s'}_{\in \mathfrak{p}}$ for some $s' \in S$

$\Rightarrow ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p} \Rightarrow \frac{a}{s_1} \in S^{-1}\mathfrak{p}$ or $\frac{b}{s_2} \in S^{-1}\mathfrak{p}$

It follows that $S^{-1}\mathfrak{p} \in \operatorname{Spec}(S^{-1}R)$

Using Propositions 7.6 – 7.9 we get

Corollary 7.10 ~~There~~

(1) There is a 1-to-1 correspondence (bijection) between

$$\left\{\begin{array}{l}\text{all saturated} \\ \text{ideals in } R\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{all ideals} \\ \text{in } S^{-1}R\end{array}\right\}$$

$$\varphi^{-1}(J) \xleftarrow{\ \varphi^{-1}\ } J$$

$$I \xrightarrow{\ S^{-1}\ } S^{-1}I$$

(2) There is a 1-to-1 correspondence (bijection) between

$$\left\{\begin{array}{l}\text{prime ideals} \\ \text{in } R \text{ such that} \\ \quad \mathfrak{p} \cap S = \emptyset\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{prime ideals} \\ \text{in } S^{-1}R\end{array}\right\}$$

$$\mathfrak{p} \xrightarrow{\ S^{-1}\ } S^{-1}\mathfrak{p}$$

$$\varphi^{-1}(\mathfrak{q}) \xleftarrow{\ \varphi^{-1}\ } \mathfrak{q}$$

Proof: (1) By Prop 7.7 $\varphi^{-1}$ is well-defined (that is $\varphi^{-1}(J)$ is saturated in $R$)

and $S^{-1} \circ \varphi^{-1} = id$

By Prop 7.8 $\varphi^{-1} \circ S^{-1} = id$

(2) Follows from Prop 7.9

**Corollary 7.11** Let $\mathfrak{p} \in \operatorname{Spec} R$. $S = R \setminus \mathfrak{p}$ mult subset of $R$

$R_{\mathfrak{p}} = S^{-1}R$ is local with maximal ideal $\mathfrak{p}R_{\mathfrak{p}} = S^{-1}\mathfrak{p}$.

**Proof**

By Corollary 7.10 $\operatorname{Spec} R_{\mathfrak{p}} = \{ S^{-1}\mathfrak{q} \mid \mathfrak{q} \in \operatorname{Spec} R, \underbrace{\mathfrak{q} \cap S = \emptyset}_{\Updownarrow \; \mathfrak{q} \subseteq \mathfrak{p}} \}$

It follows that $S^{-1}\mathfrak{p} = \mathfrak{p}R_{\mathfrak{p}}$ is the maximal ideal and the unique one.

**Example 7.12**

- $\mathbb{Z}_{(5)}$, $\operatorname{Spec} \mathbb{Z}_{(5)} = \{ 0, (5) \cdot \mathbb{Z}_{(5)} \}$.

- $\mathbb{Z}_6 = \{ \frac{a}{6^n} \mid a \in \mathbb{Z}, n \in \mathbb{N} \} \subseteq \mathbb{Q}$, $S = \{6^n\}_{n \in \mathbb{N}}$

  $\mathfrak{q} \in \operatorname{Spec} \mathbb{Z}, \ \mathfrak{q} \cap S = \emptyset \iff \mathfrak{q} \neq (2), (3)$.

  $\operatorname{Spec} \mathbb{Z}_6 = \cancel{\{ (\mathfrak{q}) \mathbb{Z}_6 \mid (\mathfrak{q}) \in \operatorname{Spec} \mathbb{Z}, (\mathfrak{q}) \neq (2),(3) \}}$

  $\{ \mathfrak{q} \mathbb{Z}_6 \mid \mathfrak{q} \in \operatorname{Spec} \mathbb{Z}, \ \mathfrak{q} \neq (2), (3) \}$.

# Localization of modules

**Definition 7.13** $R$ comm. Ring, $S \subseteq R$ mult. closed.

Let $M$ be an $R$-module

Define on $M \times S$ the following relation:

$$(m, s) \sim (m', s') \iff \exists\, t \in S \text{ such that } t(sm' - s'm) = 0$$

This is an equivalence relation.

$\frac{m}{s} :=$ equivalence class of $(m, s)$ in $M \times S/_\sim$

$S^{-1}M := M \times S/_\sim$ is an $S^{-1}A$ - Module

$$\frac{a}{s'} \cdot \frac{m}{s} := \frac{am}{s's}$$

$$\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'}$$

**Notation:**
- $M_\mathfrak{p} := S^{-1}M$, where $M = R \setminus \mathfrak{p}$ and $\mathfrak{p} \in \operatorname{Spec} R$
- $f \in R$ $M_f := S^{-1}M$, where $S = \{f^n\}_{n \in \mathbb{N}}$

**Proposition 7.14** Let $f : M \to N$ an $R$-linear map ($M, N$ $R$-modules)

Then
$$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N \qquad \text{is well-defined, } S^{-1}R\text{-linear}$$
$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$

Moreover, if $g : N \to L$ $R$ linear, then $S^{-1}(g \circ f) = S^{-1}g \circ S^{-1}f$

**Proof**

Well-defined $f : M \to N$

$\frac{m}{s} = \frac{m'}{s'}$ in $S^{-1}M \implies t(s'm - sm') = 0$ in $M \xRightarrow{\text{apply } f}$ $t(s'f(m) - s f(m')) = 0$ in $N$

$\implies \frac{f(m)}{s} = \frac{f(m')}{s'}$ in $S^{-1}N$.

R-linear

**Remark 7.15** $S \subseteq R$ mult. closed

Localization defines a functor

$$S^{-1}: \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(S^{-1}R)$$

$$M \longmapsto S^{-1}M$$

$$(f: M \to N) \longmapsto (S^{-1}f: S^{-1}M \to S^{-1}N)$$

**Proposition 7.16** The functor $S^{-1}$ is exact

(That is

$\forall \; M' \xrightarrow{f} M \xrightarrow{g} M''$ exact $\Rightarrow$ $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ exact

**Proof:** $g \circ f = 0 \Rightarrow S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = S^{-1}0 = 0 \Rightarrow$

$\Rightarrow \mathrm{Im}\, S^{-1}f \subseteq \mathrm{Ker}\, S^{-1}g$

"$\supseteq$"

Let $\frac{m}{s} \in \mathrm{Ker}(S^{-1}g)$

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

$$\frac{m}{s} \longmapsto \frac{g(m)}{s} = 0$$

Then $\frac{g(m)}{s} = 0$ in $S^{-1}M''$

$\Rightarrow \exists\, s' \in S$, such that $0 = s'g(m) \underset{g \text{ is } R\text{-linear}}{=} g(s'm)$ (in $M''$) $\Rightarrow s'm \in \mathrm{Ker}\, g = \mathrm{Im}\, f$

$s'm = f(m')$ for some $m' \in M'$

Then $\frac{m}{s} = \frac{ms'}{ss'} = \frac{f(m')}{ss'} = S^{-1}f\left(\frac{m'}{ss'}\right) \in \mathrm{Im}\, S^{-1}f$ $\blacksquare$.

## Remark 7.17

- In particular, $S^{-1}$ preserves injections, so for an $R$-submodule $M' \overset{i}{\hookrightarrow} M$ we can consider $S^{-1}M'$ as a $S^{-1}R$ - submodule of $S^{-1}M$. ( $S^{-1}M' \simeq \operatorname{Im} S^{-1}i$ )

  $S^{-1}M' = \{ \frac{m}{s} \mid m \in M', s \in S \} \subseteq S^{-1}M$
- Let $I \overset{i}{\hookrightarrow} R$ be an ideal

  Then $\operatorname{Im} S^{-1}i = S^{-1}I$

  which we constructed before.

## Proposition 7.18 (Exercise)

- $N, P \leq M$ submodules. Then

1) $S^{-1}(N+P) = S^{-1}N + S^{-1}P \leq S^{-1}M$

2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$

3) $S^{-1}(M/N) \simeq S^{-1}M / S^{-1}M$ as $S^{-1}R$ - modules.

## Proposition 7.19

Let $M$ be an $R$-module, $S \subseteq R$ mult. subset.

Then the map

$$f : S^{-1}R \otimes_R M \longrightarrow S^{-1}M$$
$$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$$

is well-defined an is an isomorphism of $S^{-1}R$-modules

**Proof**:

$$S^{-1}R \times M \longrightarrow S^{-1}M \quad \text{is } R\text{-bilinear}$$
$$\left(\frac{a}{s}, m\right) \longmapsto \frac{am}{s}$$

So this map factors through

$$S^{-1}R \times M \longrightarrow \overbrace{S^{-1}R \otimes_R M \longrightarrow S^{-1}M}^{\text{denote by } f}$$
$$\left(\frac{a}{s}, m\right) \longmapsto \underbrace{\frac{a}{s} \otimes m \longmapsto \frac{am}{s}}_{f}$$

Clearly $f$ is surjective.
It remains to show that $f$ is injective.

Any element from $S^{-1}R \otimes_R M$ we can write

as $\sum_i \frac{a_i}{s_i} \otimes m_i$

Set $s = \prod_i s_i \in S$ and $t_i = \prod_{j \neq i} s_j$

$$\sum_i \frac{a_i}{s_i} \otimes m_i = \sum_i \frac{a t_i}{s} \otimes m = \sum \frac{1}{s} \otimes a_i t_i m = \frac{1}{s} \otimes \sum_i a_i t_i m$$

Hence, every element in $S^{-1}R \otimes_R M$ ~~has~~ is of the form $\frac{1}{s} \otimes m$

Assume $0 = f\left(\frac{1}{s} \otimes m\right) = \frac{m}{s} \Rightarrow s'm = 0$ for some $s' \in S$

and we have $\frac{1}{s} \otimes m = \frac{s'}{ss'} \otimes m = \frac{1}{ss'} \otimes \underbrace{s'm}_{0} = 0$ ∎

<u>Corollary 7.20</u> $S^{-1}R$ is a flat $R$-module

<u>Proof</u>: Proposition 7.19 provides an equivalence of functors $S^{-1}$ and $S^{-1}R \otimes_R -$.

exact by Prop 7.16

## Remark 7.19'

By Prop 7.19 for every $R$-module $M$ we have an isomorphism of $S^{-1}R$-modules

$$f_M : S^{-1}R \otimes_R M \longrightarrow S^{-1}M$$
$$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$$

These isomorphisms ~~define~~ induce the equivalence of functors $S^{-1}R \otimes_R -$ and $S^{-1} : \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(S^{-1}R)$

Indeed, if $M \xrightarrow{\varphi} M'$ $R$-linear, the the following diagram commutes

$$\begin{tikzcd}
S^{-1}R \otimes_R M \arrow[r, "f_M"] \arrow[d, "\mathrm{id}\otimes\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & S^{-1}M \arrow[d, "S^{-1}\varphi"] \\
S^{-1}R \otimes_R M' \arrow[r, "f_{M'}"'] & S^{-1}M
\end{tikzcd}$$

## Remark 7.21

$$S^{-1}M \otimes_{S^{-1}R} S^{-1}N \longrightarrow S^{-1}(M \otimes_R N) \quad \text{is an isomorphism}$$
$$\frac{m}{s} \otimes \frac{n}{s'} \longmapsto \frac{m \otimes n}{ss'}$$

Follows from Lemma 6.20 & Prop 7.19

$\uparrow$

$$(S^{-1}R \otimes_R M) \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \xrightarrow{\sim} S^{-1}R \otimes_R (M \otimes_R N)$$

is an iso.

<u>Definition 7.22</u> R comm. ring, M R-module

A property $\Pi$ of rings (resp. R-modules) is called local property if

$R$ has $\Pi$ $\Leftrightarrow$ $R_{\mathfrak{p}}$ has $\Pi$ for all $\mathfrak{p} \in \operatorname{Spec} R$

(resp. M) (resp. $M_{\mathfrak{p}}$)

<u>Remark</u>:
- to be a reduced ring is a local property (Exercise)
- to be an integral ring is not a local property (Exercise)

(Hint: Consider $R = K \times K$, where $K$ is a field)

<u>Proposition 7.24</u> M R-module. Then the following conditions are equivalent:

(a) $M = 0$

(b) $M_{\mathfrak{p}} = 0$ $\forall \mathfrak{p} \in \operatorname{Spec} R$

(c) $M_{\mathfrak{m}} = 0$ $\forall \mathfrak{m} \in \operatorname{Max}(R) :=$ set of all max ideals in R

<u>Proof</u>: a) $\Rightarrow$ b) $\Rightarrow$ c) clear

Assume c) holds and assume $M \neq 0$.

Then $\exists m \in M$, $m \neq 0$. Consider $R \supseteq \operatorname{Ann}(m) := \{ r \in R \mid rm = 0 \}$

$$\operatorname{Ann}(m) = \operatorname{Ker}\begin{pmatrix} R \longrightarrow M \\ r \longmapsto rm \end{pmatrix} \text{ ideal in } R$$

$\exists\, \mathfrak{m}$ max. ideal of $R$ containg $\mathrm{Ann}(m)$.

$\mathrm{Ann}(m) \subseteq \mathfrak{m}$

Then since $M_{\mathfrak{m}} = 0$, $\frac{m}{1} = 0$ in $M_{\mathfrak{m}}$

$\Rightarrow$ $sm = 0$ for some $s \notin \mathfrak{m}$

$\Downarrow$

$s \in \mathrm{Ann}(m) \subset \mathfrak{m}$ $\quad$ ↯

<u>Proposition 7.25</u> $M, N$ $R$-modules

Let $\varphi : M \longrightarrow N$ be an $R$-linear map

Then the following conditions are equivalent

a) $\varphi$ is injective

b) $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}}$ injective $\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$

c) $\varphi_{\mathfrak{m}} : M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$ $\forall\, \mathfrak{m} \in \mathrm{Max}(R)$ ← set of max. ideals
is injective

<u>Proof:</u> "a) $\Rightarrow$ b)"

$0 \longrightarrow M \xrightarrow{\varphi} N$ exact $\xRightarrow[\text{Prop 7.16}]{}$ $0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} N_{\mathfrak{p}}$ exact

$\Rightarrow$ $\varphi_{\mathfrak{p}}$ injective

"b) $\Rightarrow$ c)" clear (every prime is maximal)

"c) $\Rightarrow$ a)" Sei $M' := \mathrm{Ker}\,\varphi$

$0 \longrightarrow M' \longrightarrow M \xrightarrow{\varphi} N$ exact $\xRightarrow[\text{Prop 7.16}]{}$ $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \xrightarrow{\varphi_{\mathfrak{m}}} N_{\mathfrak{m}}$ exact

$\Rightarrow$ $M'_{\mathfrak{m}} \simeq \mathrm{Ker}\,\varphi_{\mathfrak{m}} = 0$ $\xRightarrow[\text{Prop 7.24}]{}$ $\mathrm{Ker}\,\varphi = M' = 0 \Rightarrow \varphi$ injective

**Remark 7.26**

Same statement as in Proposition 7.25 is true for surjectivity

To consider $\operatorname{Coker} \varphi$ (instead of $\operatorname{Ker}$)

**Remark 7.26'** $R$ comm. ring

$\bigoplus_{m \in \operatorname{Max}(R)} R_m$ is faithfully flat (Exercise)

**Proposition 7.27** $M$ - $R$-module

The following is equivalent:

1) $M$ is flat $/R$

2) $M_p$ is flat $/R_p$ $\forall\, p \in \operatorname{Spec} R$

3) $M_m$ is flat $/R_m$ $\forall\, m \in \operatorname{Max}(R)$

**Proof**

**Lemma** (Exercise): $M$ $A$-module, $f: A \to B$ ring homomorphism

If $M$ flat over $A$, then $B \otimes_A M$ is flat over $B$.

1) $\Rightarrow$ 2) follows from Lemma

2) $\Rightarrow$ 3) clear

3) $\Rightarrow$ 1) $\quad M_{\mathfrak{m}}$ flat over $R_{\mathfrak{m}}$ $\forall \mathfrak{m} \in \mathrm{Max}(R)$

Let $N \xrightarrow{\varphi} N'$ injective $\underset{\text{Prop 7.25}}{\Longrightarrow}$ $N_{\mathfrak{m}} \xrightarrow{\varphi_{\mathfrak{m}}} N'_{\mathfrak{m}}$ injective $\forall \mathfrak{m} \in \mathrm{Max}(R)$

$$\underset{M_{\mathfrak{m}} \text{ flat}}{\Longrightarrow} \quad N_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \xrightarrow{\varphi_{\mathfrak{m}} \otimes id} N'_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \quad \text{injective } \forall \mathfrak{m} \in \mathrm{Max}(R)$$

$$\underset{\text{Remark 7.21}}{\Longrightarrow} \quad (N \otimes_R M)_{\mathfrak{m}} \xrightarrow{(\varphi \otimes id)_{\mathfrak{m}}} (N \otimes_R M)_{\mathfrak{m}} \quad \text{injective } \forall \mathfrak{m} \in \mathrm{Max}(R)$$

$$\underset{\text{Prop 7.25}}{\Longrightarrow} \quad N \otimes_R M \xrightarrow{\varphi \otimes id} N \otimes_R M \quad \text{injective} \Rightarrow M \text{ is flat.}$$

**Remark 7.28**

to be a free R-module is not a local property in general.

By Corollary 6.53 any finitely generated projective module over a local ring is free.

But we have seen examples of finitely generated but not free modules.

**Proposition 7.29** Let $P$ be a finitely presented module over $R$.

Then (1) $P$ is flat over $R$

(2) $P$ is projective over $R$

(3) $P_{\mathfrak{p}}$ (resp. $P_{\mathfrak{m}}$) is free over $R_{\mathfrak{p}}$ (resp. $R_{\mathfrak{m}}$) for $\forall\, \mathfrak{p} \in \operatorname{Spec} R$ (resp. $\forall\, \mathfrak{m} \in \operatorname{Max}(R)$)

**Proof**

$$(1) \xRightarrow[\text{Proposition 6.38}]{} (2) \xRightarrow[\text{Corollary 6.53}]{} (3) \xRightarrow{\text{Proposition 7.27}} (1)$$

finitely presented then flat $\Rightarrow$ projective

finitely generated over a local ring $\Downarrow$ free

**Corollary 7.30** Let $R$ be a von Neumann regular ring and $M$ an $R$-module. Then $M$ is flat

**Remark:** If every module over a ring $A$ is flat, then one says $A$ is *absolutely flat*.

**Proof of Corollary 7.30 (Exercise)** By Proposition 7.27 it is enough to show that $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module.

Hint: show that $R_{\mathfrak{p}}$ is a field (use properties from Lemma 6.42)