

§ 9 Integral extensions

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Def 9.1 Let B be a ring and $A \subseteq B$ a subring.

An element $\alpha \in B$ is said to be integral over A if α is a root of a monic polynomial with coefficients in A , that is if α satisfies an equation of the form

$$\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_0 = 0 \quad \text{in } B, \text{ where } a_i \in A$$

Example 9.2

$$\mathbb{Z} \subseteq \mathbb{R}$$

- $\alpha = \sqrt{3}$ is integral over \mathbb{Z} , since $\alpha^2 - 3 = 0$ in \mathbb{R}
- $\alpha = \frac{1+\sqrt{5}}{2}$ is integral over \mathbb{Z} , since $\alpha^2 - \alpha - 1 = 0$ in \mathbb{R} .
- A ring, every element from A is integral over A

Proposition 9.3

$A \subseteq B$ rings.

Then the following conditions are equiv.

- 1) $\alpha \in B$ is integral over A
- 2) $A[\alpha]$ is finitely generated A -module a subring in B
 $\left(A[\alpha] = \{ f(\alpha) \in B \mid f \in A[X] \} \right)$
" $\text{Im} (A[X] \rightarrow B)$
 $X \mapsto \alpha$
- 3) $A[\alpha] \subseteq C \subseteq B$, where C is a finitely generated A -module.
 \uparrow
subring

Proof

(1) \Rightarrow (2) (*) $d^n + a_{n-1}d^{n-1} + \dots + a_0 = 0$ in B

$$d^n = -a_{n-1}d^{n-1} - a_{n-2}d^{n-2} - \dots - a_0 \in \langle 1, d, \dots, d^{n-1} \rangle$$

↑
 A -module

By induction on n one can show that $d^m \in \langle 1, d, \dots, d^{n-1} \rangle$ for every m .

Clear for $m=1, \dots, n-1$ and for $m=n$ (follows from *)

For $m > n$ just multiply (*) by d^{m-n} .

It follows $A[d] \subseteq \langle 1, d, \dots, d^{n-1} \rangle \Rightarrow A[d] = \langle 1, d, \dots, d^{n-1} \rangle$
↑
finitely generated A -module.

(2) \Rightarrow (3) Take $C = A[d]$.

(3) \Rightarrow (1) $d \in C$. Consider $\varphi_d: C \rightarrow C$ well-defined since C is a ring.
 $x \mapsto dx$

φ_d is an A -linear map. $\varphi_x(M) \subseteq dM$

By Proposition 6.47 (Take $M = C$ and $d = A$) (Cayley-Hamilton Theorem) ↖ finitely generated

we have $P(\varphi_d) = \varphi_d^k + a_{n-1}\varphi_d^{k-1} + \dots + a_1\varphi_d + a_0 = 0 \in \text{End}_R(C)$

for some $a_1, \dots, a_n \in A$. ($P(x) \in A[x]$)

Then $0 = P(\varphi_d)(1) = \underbrace{d^k}_{\in C} + a_{n-1}d^{k-1} + \dots + a_1d + a_0 \in B$

$\Rightarrow d$ is integral over A .

Corollary 9.4 $A \subseteq B$ rings.

Let d_1, \dots, d_n be elements of B , d_i ~~exact~~ integral over A for $i=1, \dots, n$.

Then the subring $A[d_1, \dots, d_n] \subseteq B$ is a finitely generated A -module.

Proof: Induction on n .

$n=1$ follows from Prop. 9.3

~~$A[d_1, \dots, d_n] = A[d_n]$~~ Denote $A_k = A[d_1, \dots, d_k]$

Then $A_n = A_{n-1}[d_n]$ (Assume A_{n-1} f.g. over A)

d_n integral over $A \Rightarrow$ integral over A_{n-1} (since $A \subseteq A_{n-1}$)

By Prop 9.3 $A_{n-1}[d_n]$ is finitely generated A_{n-1} -module

(x_1, \dots, x_k) as A_{n-1} -module

By induction hypothesis A_{n-1} is finitely generated

over A ; $A_{n-1} = Ay_1 + \dots + Ay_m$ for some $y_i \in A_{n-1}$.

(y_1, \dots, y_m)

Take $f \in A_{n-1}[d_n] \Rightarrow f = a'_1 x_1 + \dots + a'_k x_k$ $a'_k \in A_{n-1}$

and every a'_i can be written as $a_{i1}y_1 + \dots + a_{im}y_m$

$a_{ji} \in A$

$\Rightarrow f = \sum_{i,j} a_{ji} y_j x_i \Rightarrow \{y_j x_i\}_{\substack{1 \leq i \leq k \\ 1 \leq j \leq m}}$ generate A_n over A .

Remark 9.4' $A \subseteq B$ rings, such that B is fin. generated as ~~A -module~~ A -module. Let M be a fin. generated B -module, then M is fin. generated as A -module.

Same argument as in the Proof of Corollary 9.4.

Proposition 9.5 $A \subseteq B$ rings

~~The set~~

Let $C = \{ d \in B \mid d \text{ is integral over } A \}$.

Then C is a subring of B , $A \subseteq C \subseteq B$.

Proof: Let $\alpha, \beta \in C$

Then by Corollary 9.4

$A[\alpha, \beta]$ is finitely generated A -module.

By Proposition 9.3, $\alpha \pm \beta$ and $\alpha\beta$ are integral over A .

Definition 9.6 • C is called the integral closure of A in B

$$A \subseteq C \subseteq B$$

- if $A = C$ then A is called integrally closed in B
- if $C = B$, then B is integral over A .
-

Proposition 9.7 $A \subseteq B \subseteq C$ are rings

and B is integral over A } $\Rightarrow C$ is integral over A
 C is integral over B

Proof

Let $\alpha \in C$, then we have in C

$$\alpha^n + b_{n-1}\alpha^{n-1} + \dots + b_1\alpha + b_0 = 0 \quad \text{for some } \underbrace{b_0, \dots, b_{n-1}}_{\text{integral over } A} \in B$$

The ring $B' = A[b_0, \dots, b_{n-1}]$ is finitely generated

A -module by Corollary 9.4

By Prop 9.3 $B'[\alpha]$ is fin. gen. over B'

By Remark 9.4' $B'[\alpha]$ is fin. gen. over $A \rightarrow \alpha$ integral over A

Example 9.6':

- A factorial ~~is~~ integral domain, $A \subseteq Q(A)$

~~Ex~~ If A is factorial, then

A is integrally closed in $Q(A)$ (Exercise) (such rings are called normal?)

In particular, \mathbb{Z} is integrally closed in \mathbb{Q} .

- $$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\text{alg. extension}} & K \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & \mathcal{O}_K \end{array}$$

integral closure of \mathbb{Z} in K .
(studied in alg. number theory)

- $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$

- $K = \mathbb{Q}(\sqrt{5})$, $\mathcal{O}_K = \mathbb{Z}[\alpha]$, where $\alpha = \frac{1+\sqrt{5}}{2}$

- $A \subseteq C \subseteq B$

\uparrow
integral closure of A in B .

Then C is integrally closed in B .

If $x \in B$ is integral over $C \Rightarrow$ then x is integral over A

$\Rightarrow x \in C$. \uparrow by Proposition 9.6

- $K \subset L$ algebraic extension of field
Then L is integral over K .

Remark Proposition 9.8: $A \subseteq B$ rings,

$\exists J \subset B$ ideal $I = \varphi^{-1}(J) = A \cap J$

Then we have φ induces an injection $A/I \xrightarrow{\varphi} B/J$
 $a+I \mapsto a+J$

and B/J is integral over A/I .

Take $\bar{b} \in B/J$ (b is a root of a monic polynomial $\in A[X]$)

$$b^n + a_{n-1}b^{n-1} + \dots + a_1b + a_0 = 0$$

$\downarrow B \rightarrow B/J$ (take mod J)

$$\bar{b}^n + \bar{a}_{n-1}\bar{b}^{n-1} + \dots + \bar{a}_1\bar{b} + \bar{a}_0 = 0 \Rightarrow \bar{b} \text{ integral over } A/I$$

Proposition 9.9: $A \subseteq B$ rings, $S \subseteq A$ mult. subset in A (also in B)
 C the integral closure of A in B (denote by \overline{A}_B , C_A)

$$S^{-1} \overline{A}_B C_A = \overline{S^{-1}A}_{S^{-1}B} C_{S^{-1}A} \quad S^{-1}A \subseteq S^{-1}B$$

$\uparrow \quad \uparrow$
 $A \subseteq B$

monic polynomial eq. for $b \in B$

$$\frac{b^n + a_{n-1}b^{n-1} + \dots + a_1b + a_0 = 0}{s^n}$$

Proof "C" Let $\frac{b}{s} \in S^{-1}C_A$, with $b \in C_A, s \in S$

Then we have in $S^{-1}B$: $\left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s} \left(\frac{b}{s}\right)^{n-1} + \dots + \frac{a_1}{s_1} \left(\frac{b}{s}\right) + \frac{a_0}{s_0} = 0$
 coeff. $\frac{a_{n-i}}{s_i} \in S^{-1}A$.

$\Rightarrow \frac{b}{s}$ integral over $S^{-1}A$.

Let $\frac{b}{s} \in C_{S^{-1}A}$, then $\frac{b}{s}$ satisfies $\frac{b}{s} \in \overline{S^{-1}A}_{S^{-1}B}$

" \supset " (*) $\left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s_{n-1}} \left(\frac{b}{s}\right)^{n-1} + \dots + \frac{a_1}{s_1} \left(\frac{b}{s}\right) + \frac{a_0}{s_0} = 0$
 where $a_i \in A, s_i \in S$ (so $\frac{a_i}{s_i} \in S^{-1}A$)

~~$B = \frac{S_0 \cdot S_1 \cdot \dots \cdot S_{n-1} \cdot s}{s}$~~ Multiply (*) by $(s')^n$ we get ~~$\frac{b}{s} \in \overline{S^{-1}A}_{S^{-1}B}$~~
 ~~$\frac{b}{s} \in A_B$~~

Take $s' = s_0 \cdot s_1 \cdot \dots \cdot s_{n-1} \in S$

Multiplying (*) by $(s')^n s^n$, we get in $S^{-1}B$

$$\underbrace{(s'b)^n + c_{n-1}(s'b)^{n-1} + \dots + c_1(s'b) + c_0}_{=0} = 0$$

for some $c_i \in A$.

Hence, $s''f = 0$ for some $s'' \in S \Rightarrow (s'')^n f = 0$

It follows $s''s'b \in C_A$

$$\Rightarrow (s''s'b)^n + s''c_{n-1}(s''s'b)^{n-1} + \dots = 0$$

Then $\frac{b}{s} = \frac{s''s'b}{s''s's}$ and $s''s'b \in C_A$

$\Rightarrow \frac{b}{s} \in S^{-1}C_A$.

Remark 9.9' In particular, if B is integral over A , then $S^{-1}B$ is integral over $S^{-1}A$.

Lemma 9.10 (Exercise, Tutorium)

$A \subseteq B$ rings, B integral over A

Then A is a field $\Leftrightarrow B$ is a field.

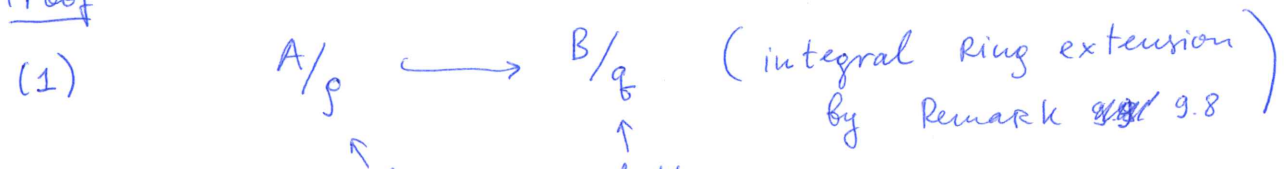
Proposition 9.11 $A \xrightarrow{\varphi} B$ rings, B integral over A .

Let $q \in \text{Spec } B$ and $\mathfrak{p} = \varphi^{-1}(q) = A \cap q \in \text{Spec } A$
 (we say q lies over \mathfrak{p})

Then (1) $\mathfrak{p} \in \text{Max}(A) \Leftrightarrow q \in \text{Max}(B)$

(2) If $q \subseteq q'$, where $q' \in \text{Spec}(B)$ and $\mathfrak{p} = \varphi^{-1}(q) = \varphi^{-1}(q')$, then $q = q'$.

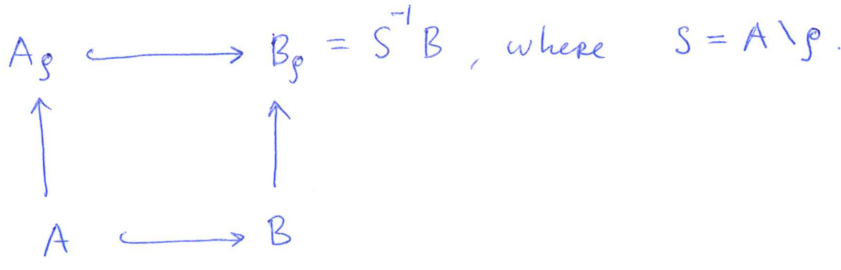
Proof



By Lemma 9.10:

Hence, $\mathfrak{p} \in \text{Max}(A) \Leftrightarrow q \in \text{Max}(B)$.

(2) Consider,



$A_{\mathfrak{p}}$ is local (Corollary 7.11) with the unique max. ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Note that $\mathfrak{q} \cap S = \mathfrak{q}' \cap S = \emptyset$ in B

$\Rightarrow \bar{S}^{-1}\mathfrak{q} \subseteq \bar{S}^{-1}\mathfrak{q}'$ are two prime ideals in $\bar{S}^{-1}B$
(see Corollary 7.10)

If we know that (*) $\mathfrak{p}A_{\mathfrak{p}} = \bar{S}^{-1}\mathfrak{q} \cap A_{\mathfrak{p}} = \bar{S}^{-1}\mathfrak{q}' \cap A_{\mathfrak{p}}$,

then by (1) both $\bar{S}^{-1}\mathfrak{q}$ and $\bar{S}^{-1}\mathfrak{q}'$ are maximal ideals in $\bar{S}^{-1}B$ (Note that $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$ is integral)

But, since $\bar{S}^{-1}\mathfrak{q} \subseteq \bar{S}^{-1}\mathfrak{q}'$, we get $\bar{S}^{-1}\mathfrak{q} = \bar{S}^{-1}\mathfrak{q}'$ and by Corollary 7.10 $\mathfrak{q} = \mathfrak{q}'$.

So it ~~is~~ is enough to prove (*)

Let us prove $\mathfrak{p}A_{\mathfrak{p}} = \bar{S}^{-1}\mathfrak{q} \cap A_{\mathfrak{p}}$ (same prove for $\mathfrak{p}A_{\mathfrak{p}} = \bar{S}^{-1}\mathfrak{q}' \cap A_{\mathfrak{p}}$)

" \subset " clear (since $\mathfrak{p} \subseteq \mathfrak{q}$)

" \supset " Let $\frac{b}{s} \in \bar{S}^{-1}\mathfrak{q} \cap A_{\mathfrak{p}}$ (recall that $A_{\mathfrak{p}} = \bar{S}^{-1}A$)

We can assume $b \in \mathfrak{q}$ (since $\frac{b}{s} \in \bar{S}^{-1}\mathfrak{q}$)

$\frac{b}{s} \in \bar{S}^{-1}A \Rightarrow \frac{b}{s} = \frac{a}{s'}$ for some $a \in A, s' \in S$

$\Rightarrow s''(bs' - as) = 0$ for some $s'' \in S$

$\Rightarrow bs''s' = as''s \in A$ and $\in \mathfrak{q} \Rightarrow \in \mathfrak{p} = A \cap \mathfrak{q}$

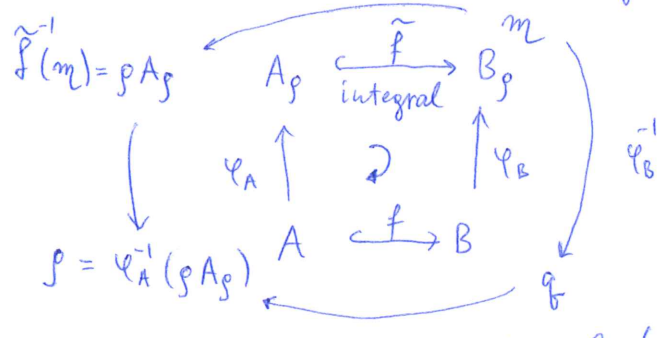
Then $\frac{b}{s} = \frac{s''s'b}{s''s} \in \bar{S}^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$.

Proposition 9.12

$A \subseteq B$ rings, B integral over A

Let $\mathfrak{p} \in \text{Spec } A$. Then $\exists \mathfrak{q} \in \text{Spec } B$, such that $\mathfrak{q} \cap A = \mathfrak{p}$.

Proof Consider the comm. diagramm:



Note that $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ integral (by Remark 9.9')

and $A_{\mathfrak{p}}$ local with the unique max. ideal $= \mathfrak{p}A_{\mathfrak{p}}$

Take a max. ideal $\mathfrak{m} \in B_{\mathfrak{p}}$ ($B_{\mathfrak{p}} \neq 0$)

By Prop 9.11 (1): $A_{\mathfrak{p}} \cap \mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$, then $\psi_A^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}$.

Set $\mathfrak{q} = \psi_B^{-1}(\mathfrak{m}) \in \text{Spec } B$

Since the above diagramm commutes, we get

$A \cap \mathfrak{q} = \mathfrak{p}$.

Proposition 9.13 ("Going up" theorem)

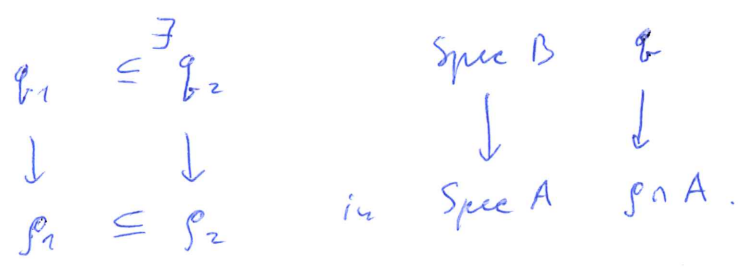
$A \subseteq B$ rings, B integral over A .

Let ~~$\mathfrak{p}_1 \in \text{Spec } A$ and $\mathfrak{q}_1 \in \text{Spec } B$, s.t.~~

$\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ be prime ideals in A .

And let \mathfrak{q}_1 be a prime ideal in B lying over \mathfrak{p}_1 .

Then $\exists \mathfrak{q}_2$, s.t. $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and \mathfrak{q}_2 lies over \mathfrak{p}_2



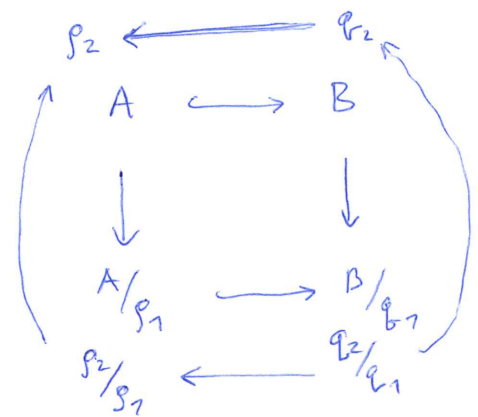
Proof: $A/\mathfrak{p}_1 \subseteq B/\mathfrak{q}_1$ integral by Remark 9.8

$\mathfrak{p}_2/\mathfrak{p}_1$ is a prime $\in \text{Spec}(A/\mathfrak{p}_1)$ (since $\mathfrak{p}_2 \in \text{Spec } A$ and $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$)

By Proposition 9.2 $\exists \mathfrak{q}' \in \text{Spec}(B/\mathfrak{q}_1)$ lying over $\mathfrak{p}_2/\mathfrak{p}_1$

↑
has form $\mathfrak{q}_2/\mathfrak{q}_1$ for some prime ideal $\mathfrak{q}_2 \in \text{Spec } B$
 $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$

The diagram commutes



Let φ be the composition $A \rightarrow A/\mathfrak{p}_1 \rightarrow B/\mathfrak{q}_1$
then $\varphi^{-1}(\mathfrak{q}_2/\mathfrak{q}_1) = \mathfrak{p}_2$. It follows that \mathfrak{q}_2 lies over \mathfrak{p}_2
(since the above diagram commutes)

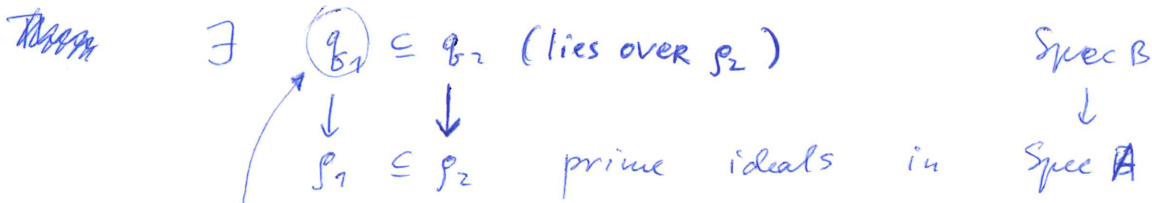
Remark 9.14

The following "Going down theorem" also holds.

$A \subseteq B$ ~~integral~~ ^{integral domains}, A integrally closed, B integral over A

~~$\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ prime ideals in $\text{Spec } B$~~

Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ two prime ideals in $\text{Spec } A$ and $\mathfrak{q}_2 \in \text{Spec } B$, lies over \mathfrak{p}_2



Then $\exists \mathfrak{q}_1 \in \text{Spec } B$, s.t. $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ & \mathfrak{q}_1 lies over \mathfrak{p}_1 .

Definition 9.15 A a comm. ring.

The (Krull) - dimension $\dim(A)$ of A is defined as follows:

$$\dim(A) := \sup \left\{ n \mid \exists \text{ a chain of primes } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n \text{ in } A \right\}$$

with strict inclusions

Example 9.16

• A Artinian, $\dim A = 0$ ($\dim K = 0$)
 K field

• $\dim A$ PID, $(0) \in \text{Spec } A$

Every prime ideal $\neq (0)$ is maximal $\Rightarrow \dim A = 1$

In particular, $\dim \mathbb{Z} = 1$

~~\dim~~ $\dim K[X] = 1$

• (Later) $\dim K[X_1, \dots, X_n] = n$.

Proposition 9.17 $A \subseteq B$ rings, B integral over A .

(70)

Then $\dim A = \dim B$.

Proof follows from Proposition 9.13 ("Going up" theorem)
and Proposition 9.12 & Proposition 9.11.

Example 9.18: Let K be an ~~finite~~ algebraic extension
of \mathbb{Q} . And $\mathcal{O}_K \subseteq K$ an integral closure of \mathbb{Z} in K .

Then ^{by Prop. 9.17} $\dim \mathcal{O}_K = \dim \mathbb{Z} = 1$

$$\dim \mathbb{Z}[i] = 1.$$

$$\begin{array}{c} \text{"} \\ \mathcal{O}_{\mathbb{Q}(i)} \end{array}$$

Theorem 9.19 (Noether Normalisation)

Let K be a field and $B \neq 0$ be a finitely generated K -algebra.

$$B = K[x_1, \dots, x_n] \quad (x_i \in B)$$

Then \exists algebraically independent over K elements

$$t_1, \dots, t_r \in B \quad \left(\begin{array}{l} \text{That is the map} \\ K[X_1, \dots, X_r] \longrightarrow B \\ X_i \longmapsto t_i \end{array} \right. \text{ is injective}$$

such that B is integral over $K[t_1, \dots, t_r]$

Proof: Induction on n .

$$n=0 \quad B=K \quad (\text{take } r=0).$$

$$n > 0$$

If x_1, \dots, x_n are alg. independent over K

then take $r=n$ and $y_i = x_i$.

Assume x_1, \dots, x_n are alg. dependent, that is

$$f(x_1, \dots, x_n) = 0 \quad \text{in } B \quad \text{for some } f(X_1, \dots, X_n) \in K[X_1, \dots, X_n]$$

We need the following lemma:

Lemma: $\exists d \in K$ and $a_1, \dots, a_{n-1} \in \mathbb{N}$ such that

$$d f(Y_1 + Y_n^{a_1}, \dots, Y_{n-1} + Y_n^{a_{n-1}}, Y_n) \in K[Y_1, \dots, Y_n]$$

is monic on Y_n .

Proof: exercise

Remark If K is infinite one can take

$$Y_1 + d_1 Y_n, \dots, Y_{n-1} + d_{n-1} Y_n, Y_n \quad \text{where } d_i \in K.$$

Set $y_1 := x_1 - x_n^{a_1}, \dots, y_{n-1} := x_{n-1} - x_n^{a_{n-1}}, y_n := x_n$

with a_1, \dots, a_{n-1} as in the Lemma.

Note that $x_1 = y_1 + y_n^{a_1}, \dots, x_{n-1} = y_{n-1} + y_n^{a_{n-1}}, x_n = y_n \in B$

$\Rightarrow x_i \in K[y_1, \dots, y_n] \subseteq B \quad \forall i=1, \dots, n.$

$\Rightarrow K[y_1, \dots, y_n] = B$ since x_1, \dots, x_n generate B as K -algebra.

Since by Lemma f is monic on Y_n

we get that y_n is integral over $K[y_1, \dots, y_{n-1}]$

$\Rightarrow B = K[y_1, \dots, y_{n-1}, y_n]$ is integral over $K[y_1, \dots, y_{n-1}]$

~~And \exists alg. independ.~~

By induction hypothesis \exists alg. independant t_1, \dots, t_r over K ,

such that $K[y_1, \dots, y_{n-1}]$ is integral over $K[t_1, \dots, t_r]$

$K[X_1, \dots, X_r]$
polynomial
ring in
 r -variables
with coeff.
in K .

Then by Proposition 9.7

~~$K[y_1, \dots, y_{n-1}]$~~ B is integral over $K[t_1, \dots, t_r]$

Corollary 9.20

Exercise

$\dim B = \dim K[X_1, \dots, X_r] = r$

Moreover, if B is integral, then

$\dim B = \underset{\substack{\uparrow \\ \text{transcendence degree}}}{\text{tr. deg}_K} B \subseteq Q(B)$