§ 9 Integral extensions

Def 9.1 Let B be a ring and $A \subseteq B$ a subring.
An element α ~~β~~ $\in B$ is said to be integral over A if α is a root of a monic polynomial with coefficients in A, that is if α satisfies an equation of the form

$$\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_0 = 0 \quad \text{in } B, \text{ where } a_i \in A$$

Example 9.2 $\mathbb{Z} \subseteq \mathbb{R}$

- $\alpha = \sqrt{3}$ is integral over $\mathbb{Z}$, since $\alpha^2 - 3 = 0$ in $\mathbb{R}$
- $\alpha = \frac{1+\sqrt{5}}{2}$ is integral over $\mathbb{Z}$, since $\alpha^2 - \alpha - 1 = 0$ in $\mathbb{R}$.
- A ring, every element from A is integral over A

Proposition 9.3 $A \subseteq B$ rings.
Then the following conditions are equiv.

1) $\alpha \in B$ is integral over A

2) $A[\alpha]$ is finitely generated A-module a subring in B $\left(A[\alpha] = \{ f(\alpha) \in B \mid f \in A[X] \} \right)$

$$\text{Im}\left(A[X] \longrightarrow B \right), \quad X \longmapsto \alpha$$

3) $A[\alpha] \subseteq C \subseteq B$, where C is a finitely generated A-module. (C: subring)

Proof

(1) $\Rightarrow$ (2) $\quad (*) \quad \alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_0 = 0$ in B

$$\alpha^n = -a_{n-1}\alpha^{n-1} - a_{n-2}\alpha^{n-2} - \dots - a_0 \in \underbrace{\langle 1, \alpha, \dots, \alpha^{n-1}\rangle}_{A\text{-module}}$$

By induction on m one can show that $\alpha^m \in \langle 1, \alpha, \dots, \alpha^{n-1}\rangle$ for every m.

Clear for $m = 1, \dots, n-1$ and for $m = n$ (follows from $*$)

For $m > n$ just multiply $(*)$ by α^{m-n}.

It follows $A[\alpha] \subseteq \langle 1, \alpha, \dots, \alpha^{n-1}\rangle \Rightarrow A[\alpha] = \underbrace{\langle 1, \alpha, \dots, \alpha^{n-1}\rangle}_{\text{finitely generated } A\text{-module}}$.

(2) $\Rightarrow$ (3) $\quad$ Take $C = A[\alpha]$.

(3) $\Rightarrow$ (1) $\quad \alpha \in C$. Consider

$$\varphi_\alpha : C \longrightarrow C, \quad x \longmapsto \alpha x$$

well-defined since C is a ring.

φ_α is an A-linear map. $\quad \varphi_\alpha(M) \subseteq \mathfrak{a} M$

By Proposition 6.47 (Take $M = C$ finitely generated and $\mathfrak{a} = A$)
(Cayley-Hamilton Theorem)
we have $P(\varphi_\alpha) = \varphi_\alpha^k + a_{n-1}\varphi_\alpha^{k-1} + \dots + a_1 \varphi_\alpha + a_0 = 0 \in \mathrm{End}_R(C)$
for some $a_1, \dots, a_n \in A$. ($P(X) \in A[X]$)

Then $0 = P(\varphi_\alpha)(\underset{\substack{\cap \\ C}}{1}) = \alpha^k + a_{n-1}\alpha^{k-1} + \dots + a_1 \alpha + a_0 \in B$

$\Rightarrow \alpha$ is integral over A.

Corollary 9.4 $A \subseteq B$ rings.

Let $\alpha_1, \dots, \alpha_n$ be elements of B, ~~each~~ α_i integral over A. for $i = 1, \dots, n$.

Then the subring $A[\alpha_1, \dots, \alpha_n] \subseteq B$ is a finitely generated A-module

Proof: Induction on n.

$n = 1$ follows from Prop. 9.3

~~$A[\alpha_1, \dots, \alpha_n] = A'[\alpha_n]$~~ Denote $A_k = A[\alpha_1, \dots, \alpha_k]$

Then $A_n = A_{n-1}[\alpha_n]$ (Assume A_{n-1} f.g. over A)

α_n integral over $A \Rightarrow$ integral over A_{n-1} (since $A \subseteq A_{n-1}$)

By Prop 9.3 $A_{n-1}.[\alpha_n]$ is finitely generated A_{n-1}-module

$(x_1, \dots, x_k)$ as A_{n-1}-module

By induction hypothesis A_{n-1} is finitely generated over A: $A_{n-1} = Ay_1 + \dots + Ay_m$ for some $y_i \in A_{n-1}$.

$(y_1, \dots, y_m)$

Take $f \in A_{n-1}[\alpha_n] \Rightarrow f = a_1' x_1 + \dots + a_k' x_k$ $a_k' \in A_{n-1}$

and every a_i' can be written as $a_{1i} y_1 + \dots + a_{mi} y_m$

$a_{ji} \in A$

$\Rightarrow f = \sum_{i,j} a_{ji} y_j x_i \Rightarrow \{y_j x_i\}_{\substack{1 \le i \le k \\ 1 \le j \le m}}$ generate A_n over A.

Remark 9.4' $A \subseteq B$ Rings, such that B is fin. generated as ~~A-mod~~ A-module. Let M be a fin. generated B-module, then M is fin. generated as A-module.

Same argument as in the Proof of Corollary 9.4.

~~Cor.~~ Proposition 9.5 $A \subseteq B$ rings

~~The set C~~

Let $C = \{ \alpha \in B \mid \alpha \text{ is integral over } A \}$.

Then C is a subring of B, $A \subseteq C \subseteq B$.

Proof: Let $\alpha, \beta \in C$

Then by Corollary 9.4

$A[\alpha, \beta]$ is finitely generated A-module.

By Proposition 9.3, $\alpha \pm \beta$ and $\alpha\beta$ are integral over A.

Definition 9.6 • C is called the integral closure of A in B

$A \subseteq C \subseteq B$

- if $A = C$ then A is called integrally closed in B
- if $C = B$, then B is integral over A.
-

Proposition 9.7 $A \subseteq B \subseteq C$ are rings

and $\left.\begin{array}{l} B \text{ is integral over } A \\ C \text{ is integral over } B \end{array}\right\} \Rightarrow C$ is integral over A

Proof

Let $\alpha \in C$, then we have in C

$$\alpha^n + b_{n-1}\alpha^{n-1} + \dots + b_1 \alpha + b_0 = 0 \quad \text{for some } \underbrace{b_0, \dots, b_{n-1}}_{\text{integral over } A} \in B$$

~~Thus~~ The ring $B' = A[b_0, \dots, b_{n-1}]$ is finitely generated A-module by Corollary 9.4

By Prop 9.3 $B'[\alpha]$ is fin. gen. over B'

By Remark 9.4' $B'[\alpha]$ is fin. gen. over A $\rightarrow$ α integral over A

Remarks/
Example 9.6':

- A ~~factorial~~ integral domain, $A \subseteq \overset{B}{\overset{\shortparallel}{Q(A)}}$

Ex If A is factorial, then
A is integrally closed in $Q(A)$ (Exercise) (such rings a called normal?)
In particular, $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$.

- $$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\text{alg. extension}} & K \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & \mathcal{O}_K \end{array}$$ integral closure of $\mathbb{Z}$ in K.
(studied in alg. number theory)

- $K = \mathbb{Q}(i)$, $\mathcal{O}_K = \mathbb{Z}[i]$

- $K = \mathbb{Q}(\sqrt{5})$, $\mathcal{O}_K = \mathbb{Z}[\alpha]$, where $\alpha = \frac{1+\sqrt{5}}{2}$

- $A \subseteq C \subseteq B$
C: integral closure of A in B.
Then C is integrally closed in B.
If $x \in B$ is integral over C $\Rightarrow$ then x is integral over A (by Proposition 9.6)
$\Longrightarrow x \in C$.

- $K \subset L$ algebraic extension of field
Then L is integral over K.

~~Proposition~~ Remark 9.8 : $A \overset{\varphi}{\subseteq} B$ rings,

& $J \subset B$ ideal $I = \varphi^{-1}(J) = A \cap J$

Then ~~we have~~ φ induces $A/_I \hookrightarrow B/_J$

an injection $a+I \longmapsto a+J$

and $B/_J$ is integral over $A/_I$.

Take $\bar{b} \in B/_J$ (b is a root of a monic polynomial $\in A[X]$

$$b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0 = 0$$

$\downarrow$ $B \to B/_J$ (take mod J)

$$\bar{b}^n + \bar{a}_{n-1} \bar{b}^{n-1} + \dots + \bar{a}_1 \bar{b} + \bar{a}_0 = 0 \Rightarrow \bar{b} \text{ integral over } A/_I$$

Proposition 9.9 : $A \leq B$ rings, $S \subseteq A$ mult. subset in A (also in B)

C the integral closure of A in B (denote by C_A)

$$S^{-1} C_A = C_{S^{-1}A}$$

$$\begin{array}{ccc} S^{-1}A & \leq & S^{-1}B \\ \uparrow & & \uparrow \\ A & \leq & B \end{array}$$

$$\frac{\overbrace{b^n + a_{n-1} b^{n-1} + \dots + a_1 b + a_0}^{\text{monic polynomial eq. for } b \in B} = 0}{s^n}$$

Proof "$\subset$" Let $\frac{b}{s} \in S^{-1}C_A$, with $b \in C_A$, $s \in S$

Then we have in $S^{-1}B$: $\left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s}\left(\frac{b}{s}\right)^{n-1} + \dots + \frac{a_1}{s^{n-1}}\left(\frac{b}{s}\right) + \frac{a_0}{s^n}$

coeff. $\frac{a_{n-i}}{s^i} \in S^{-1}A$.

$\Rightarrow \frac{b}{s}$ integral over $S^{-1}A$.

"$\supset$" Let $\frac{b}{s} \in C_{S^{-1}A}$, then $\frac{b}{s}$ satisfies

$$(*) \left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s_{n-1}}\left(\frac{b}{s}\right)^{n-1} + \dots + \frac{a_1}{s_1}\left(\frac{b}{s}\right) + \frac{a_0}{s_0} = 0 \qquad \frac{b}{s} \in \overline{S^{-1}A}_{S^{-1}B}$$

where $a_i \in A$, $s_i \in S$ (so $\frac{a_i}{s_i} \in S^{-1}A$)

~~$s' = s_0 s_1 \dots s_{n-1} s$~~ ~~Multiply $(*)$ by $(s')^n$ we get~~ ~~$s's b \in \bar{A}_B$~~

Damn $\frac{b}{\ } = \frac{\ }{s'}$

Take $s' = s_0 \cdot s_1 \cdot \ldots \cdot s_{n-1} \in S$

Multiplying $(*)$ by $(s')^n s^n$, we get in $S^{-1}B$

$$\underbrace{\overbrace{(s'b)^n + c_{n-1}(s'b)^{n-1} + \ldots + c_1(s'b) + c_0}^{f}}_{1} \overset{?}{=} 0$$

for some $c_i \in A$.

Hence, $s'' f = 0$ for some $s'' \in S$ $\Rightarrow$ $(s'')^n f = 0$

$\Rightarrow (s''s'b)^n + s''c_{n-1}(s''s'b)^{n-1} + \ldots = 0$

It follows $s''s'b \in C_A$

Then $\frac{b}{s} = \frac{s''s'b}{s''s's}$ and $s''s'b \in C_A$

$\Rightarrow \frac{b}{s} \in S^{-1}C_A$.

Remark 9.9' In particular, if B is integral over A, then $S^{-1}B$ is integral over $S^{-1}A$.

Lemma 9.10 (Exercise, Tutorium)

$A \subseteq B$ rings, B integral over A

Then A is a field $\iff$ B is a field.

Proposition 9.11 $A \overset{\varphi}{\subseteq} B$ rings, B integral over A.

Let $q \in \operatorname{Spec} B$ and $p = \varphi^{-1}(q) = A \cap p \in \operatorname{Spec} A$

(we say q lies over p)

Then (1) $p \in \operatorname{Max}(A) \iff q \in \operatorname{Max}(B)$

(2) If $q \subseteq q'$, where $q' \in \operatorname{Spec}(B)$ and $p = \varphi^{-1}(q) = \varphi^{-1}(q')$, then $q = q'$.

Proof

(1) $A/p \longrightarrow B/q$ (integral ring extension by Remark ~~9.8~~ 9.8)

By Lemma 9.10: A/p field $\iff$ B/q field

Hence, $p \in \operatorname{Max}(A) \iff q \in \operatorname{Max}(B)$.

(2) Consider,

$$\begin{tikzcd} A_{\mathfrak{p}} \arrow[r, hook] & B_{\mathfrak{p}} = S^{-1}B \\ A \arrow[u] \arrow[r, hook] & B \arrow[u] \end{tikzcd}, \text{ where } S = A \setminus \mathfrak{p}.$$

$A_{\mathfrak{p}}$ is local (Corollary 7.11) with the unique max. ideal $\mathfrak{p} A_{\mathfrak{p}}$.

Note that $\mathfrak{q} \cap S = \mathfrak{q}' \cap S = \emptyset$ in B

$\Rightarrow$ $S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{q}'$ are two prime ideals in $S^{-1}B$ (see Corollary 7.10)

If we know that $(*)$ $\mathfrak{p} A_{\mathfrak{p}} = S^{-1}\mathfrak{q} \cap A_{\mathfrak{p}} = S^{-1}\mathfrak{q}' \cap A_{\mathfrak{p}}$,

then by (1) both $S^{-1}\mathfrak{q}$ and $S^{-1}\mathfrak{q}'$ are maximal ideals in $S^{-1}B$ (Note that $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ is integral by Remark 9.9')

But, since $S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{q}'$, we get $S^{-1}\mathfrak{q} = S^{-1}\mathfrak{q}'$ and by Corollary 7.10 $\mathfrak{q} = \mathfrak{q}'$.

So it is enough to prove $(*)$

Let us prove $\mathfrak{p} A_{\mathfrak{p}} = S^{-1}\mathfrak{q} \cap A_{\mathfrak{p}}$ (same prove for $\mathfrak{p} A_{\mathfrak{p}} = S^{-1}\mathfrak{q}' \cap A_{\mathfrak{p}}$)

"$\subset$" clear (since $\mathfrak{p} \subseteq \mathfrak{q}$)

"$\supset$" Let $\frac{b}{s} \in S^{-1}\mathfrak{q} \cap A_{\mathfrak{p}}$ (recall that $A_{\mathfrak{p}} = S^{-1}A$)

We can assume $b \in \mathfrak{q}$ (since $\frac{b}{s} \in S^{-1}\mathfrak{q}$)

$\frac{b}{s} \in S^{-1}A \Rightarrow \frac{b}{s} = \frac{a}{s'}$ for some $a \in A$, $s' \in S$

$\Rightarrow s''(bs' - as) = 0$ for some $s'' \in S$

$\Rightarrow bs''s' = as''s \in A$ and $\in \mathfrak{q}$ $\Rightarrow$ $\in \mathfrak{p} = A \cap \mathfrak{q}$

Then $\frac{b}{s} = \frac{s''s'b}{s''s's} \in S^{-1}\mathfrak{p} = \mathfrak{p} A_{\mathfrak{p}}$. $\blacksquare$

<u>Proposition 9.12</u> $A \subseteq B$ Rings, B integral over A

Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $\exists\, \mathfrak{q} \in \operatorname{Spec} B$, such that $\mathfrak{q} \cap A = \mathfrak{p}$.

<u>Proof</u> Consider the comm. diagramm:

$$\begin{tikzcd}
\tilde{f}^{-1}(\mathfrak{m}) = \mathfrak{p} A_{\mathfrak{p}} \arrow[d] & A_{\mathfrak{p}} \arrow[r, hook, "\tilde{f}", "\text{integral}"'] & B_{\mathfrak{p}} & \mathfrak{m} \arrow[lll, bend right] \arrow[d, bend left, "\varphi_B^{-1}"] \\
\mathfrak{p} = \varphi_A^{-1}(\mathfrak{p} A_{\mathfrak{p}}) & A \arrow[u, "\varphi_A"] \arrow[r, hook, "f"] \arrow[ur, phantom, "\circlearrowright"] & B \arrow[u, "\varphi_B"'] & \mathfrak{q} \arrow[lll, bend left]
\end{tikzcd}$$

Note that $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ integral (by Remark 9.9')

and $A_{\mathfrak{p}}$ local with the unique max. ideal $= \mathfrak{p} A_{\mathfrak{p}}$

Take a max. ideal $\mathfrak{m} \in B_{\mathfrak{p}}$ $(B_{\mathfrak{p}} \neq 0)$

By Prop 9.11 (1): $A_{\mathfrak{p}} \cap \mathfrak{m} = \mathfrak{p} A_{\mathfrak{p}}$, then $\varphi_A^{-1}(\mathfrak{p} A_{\mathfrak{p}}) = \mathfrak{p}$.

Set $\mathfrak{q} = \varphi_B^{-1}(\mathfrak{m}) \in \operatorname{Spec} B$

Since the above diagramm commutes, we get

$A \cap \mathfrak{q} = \mathfrak{p}$.

Proposition 9.13 ("Going up" theorem)

$A \subseteq B$ rings, B integral over A.

Let ~~$\mathfrak{p}_1 \in \operatorname{Spec} A$ and $\mathfrak{q}_1 \in \operatorname{Spec} B$, s.t.~~

$\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ be prime ideals in A.

And let $\mathfrak{q}_1$ be a prime ideal in B lying over $\mathfrak{p}_1$.

Then $\exists\, \mathfrak{q}_2$, s.t. $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\mathfrak{q}_2$ lies over $\mathfrak{p}_2$

$$\begin{array}{ccc} \mathfrak{q}_1 & \subseteq & \exists\,\mathfrak{q}_2 \\ \downarrow & & \downarrow \\ \mathfrak{p}_1 & \subseteq & \mathfrak{p}_2 \end{array} \quad \text{in} \quad \begin{array}{cc} \operatorname{Spec} B & \mathfrak{q} \\ \downarrow & \downarrow \\ \operatorname{Spec} A & \mathfrak{p} \cap A. \end{array}$$

Proof: $A/\mathfrak{p}_1 \subseteq B/\mathfrak{q}_1$ integral by Remark 9.8

$\mathfrak{p}_2/\mathfrak{p}_1$ ~~is a prim~~ $\in \operatorname{Spec}(A/\mathfrak{p}_1)$ (since $\mathfrak{p}_2 \in \operatorname{Spec} A$ and $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$)

By Proposition 9.2 $\exists\, \mathfrak{q}' \in \operatorname{Spec}(B/\mathfrak{q}_1)$ lying over $\mathfrak{p}_2/\mathfrak{p}_1$

has form $\mathfrak{q}_2/\mathfrak{q}_1$ for some prime ideal $\mathfrak{q}_2 \in \operatorname{Spec} B$, $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$

The diagramm commutes

$$\begin{tikzcd} \mathfrak{p}_2 & & \mathfrak{q}_2 \arrow[ll] \\ & A \arrow[r, hook] \arrow[d] & B \arrow[d] & \\ & A/\mathfrak{p}_1 \arrow[r] & B/\mathfrak{q}_1 & \\ \mathfrak{p}_2/\mathfrak{p}_1 \arrow[uuu, bend left] & & \mathfrak{q}_2/\mathfrak{q}_1 \arrow[ll] \arrow[uuu, bend right] \end{tikzcd}$$

Let φ be the composition $A \to A/\mathfrak{p}_1 \to B/\mathfrak{q}_1$

then $\varphi^{-1}(\mathfrak{q}_2/\mathfrak{q}_1) = \mathfrak{p}_2$. It follows that $\mathfrak{q}_2$ lies over $\mathfrak{p}_2$

(since the above diagramm commutes)

Remark 9.14

The following "Going down theorem" also holds.

$A \subseteq B$ integral domains ~~integral~~, A integrally closed, B integral over A

~~$\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ prime ideals in Spec B~~

Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ two prime ideals in Spec A and $\mathfrak{q}_2 \in \operatorname{Spec} B$, lies over $\mathfrak{p}_2$

$$\begin{array}{cccc} \exists\ \mathfrak{q}_1 & \subseteq & \mathfrak{q}_2 \text{ (lies over } \mathfrak{p}_2) & \qquad \operatorname{Spec} B \\ \downarrow & & \downarrow & \qquad \downarrow \\ \mathfrak{p}_1 & \subseteq & \mathfrak{p}_2 \text{ prime ideals in} & \qquad \operatorname{Spec} A \end{array}$$

Then $\exists\ \mathfrak{q}_1 \in \operatorname{Spec} B$, s.t. $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ & $\mathfrak{q}_1$ lies over $\mathfrak{p}_1$.

Definition 9.15 A a comm. ring.

The (Krull)-dimension ~~dim(A)~~ of A is defined as follows:

$$\dim(A) := \sup\{n \mid \exists \text{ a chain of primes } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n \text{ in } A \text{ with strict inclusions}\}$$

Example 9.16

- A Artinian, $\dim A = 0$ ($\dim K = 0$, K field)
- ~~dim~~ A PID, $(0) \in \operatorname{Spec} A$
  Every prime ideal $\neq (0)$ is maximal $\Rightarrow \dim A = 1$

  In particular, $\dim \mathbb{Z} = 1$

  $\dim K[X] = 1$
- (Later) $\dim K[X_1, \dots, X_n] = n$.

Proposition 9.17 $A \subseteq B$ rings, B integral over A.

Then $\dim A = \dim B$.

Proof follows from Proposition 9.13 ("Going up" theorem) and Proposition 9.12 & Proposition 9.11.

Example 9.18: Let K be an ~~finite~~ algebraic extension of $\mathbb{Q}$. And $\mathcal{O}_K \subseteq K$ an integral closure of $\mathbb{Z}$ in K.

Then by Prop. 9.17 $\dim \mathcal{O}_K = \dim \mathbb{Z} = 1$

$$\underset{\mathcal{O}_{\mathbb{Q}(i)}}{\underset{\shortparallel}{\dim \mathbb{Z}[i]}} = 1.$$

Theorem 9.19 (Noether Normalisation)

Let K be a field and $B \neq 0$ be a finitely generated K-algebra.

$$B = K[x_1, \dots, x_n] \qquad (x_i \in B)$$

Then $\exists$ algebraically independent over K elements

$t_1, \dots, t_r \in B$ (That is the map

$$\begin{aligned} K[X_1, \dots, X_r] &\longrightarrow B \\ X_i &\longmapsto t_i \end{aligned}$$

is injective)

such that B is integral over $K[t_1, \dots, t_r]$

Proof: Induction on n.

$n = 0$ $\quad B = K$ (take $r = 0$).

$n > 0$

If $x_1, \dots, x_n$ are alg. independent over K

then take $r = n$ and $y_i = x_i$.

Assume $x_1, \dots, x_n$ are alg. dependent, that is

$f(x_1, \dots, x_n) = 0$ in B for some $f(X_1, \dots, X_n) \in K[X_1, \dots, X_n]$

We need the following lemma:

Lemma: $\exists$ $\cancel{\alpha \in}$ $\overset{\neq 0}{\alpha} \in K$ and $a_1, \dots, a_{n-1} \in \mathbb{N}$ such that

$$\alpha\, f(Y_1 + Y_n^{a_1}, \dots, Y_{n-1} + Y_n^{a_{n-1}}, Y_n) \in K[Y_1, \dots, Y_n]$$

is monic on Y_n.

Proof: exercise

Remark If K is infinite one can take

$\cancel{y_i}$ $Y_1 + \alpha_1 Y_n, \dots, Y_{n-1} + \alpha_{n-1} Y_n, Y_n$ where $\alpha_i \in K$.

Set $y_1 := x_1 - x_n^{a_1}, \dots, y_{n-1} := x_{n-1} - x_n^{a_{n-1}}, y_n := x_n$

with $a_1, \dots, a_{n-1}$ as in the Lemma.

Note that $x_1 = y_1 + y_n^{a_1}, \dots, x_{n-1} = y_{n-1} + y_n^{a_{n-1}}, x_n = y_n$

$\Rightarrow \quad x_i \in K[y_1, \dots, y_n] \subseteq B \quad \forall i = 1, \dots, n.$

$\Rightarrow \quad K[y_1, \dots, y_n] = B$ since $x_1, \dots, x_n$ generate B as K-algebra.

Since by Lemma f is monic on Y_n

we get that y_n is integral over $K[y_1, \dots, y_{n-1}]$

$\Rightarrow \quad B = K[y_1, \dots, y_{n-1}, y_n]$ is integral over $K[y_1, \dots, y_{n-1}]$

~~And $\exists$ alg. independ.~~

By induction hypothesis $\exists$ alg. independant $t_1, \dots, t_r$ over K,

such that $K[y_1, \dots, y_{n-1}]$ is integral over $K[t_1, \dots, t_r]$

$K[t_1, \dots, t_r] \cong K[X_1, \dots, X_r]$ polynomial ring in r-variables with coeff. in K.

Then by Proposition 9.7

~~$K[y]$~~ B is integral over $K[t_1, \dots, t_r]$.

<u>Corollary 9.20</u>

$\dim B = \dim K[X_1, \dots, X_r] = r$ Exercise

Moreover, if B is integral, then

$\dim B = \operatorname{tr.deg}_K Q(B)$

transcendence degree