§ 8 Artinian modules and Rings

<u>Definition 8.1</u> We say an R-module M is Artinian if every descending chain of submodules stabilizes.
That is $\forall$ $M_1 \supseteq M_2 \supseteq \dots \supseteq M_k \supseteq \dots$ chain of submodules
$\exists\, n \in \mathbb{N}$, such that $M_k = M_n$ for all $k \geq n$

$$(M_n = M_{n+1} = M_{n+2} = \dots).$$

<u>Remark</u> 8.2 M Artinian module $\Leftrightarrow$ every non empty set of submodules has a minimal element (with respect to inclusion)

<u>Definition 8.3</u> ~~A ring R~~
a ring R is Artinian, if it is Artinian as R-module

Examples: 8.4

- $\mathbb{Z}$ is not ~~artthi~~ Artinian.

  For $a \neq 0$, a non-unit we have

$$(a) \supsetneq (a^2) \supsetneq (a^3) \supsetneq (a^4) \supsetneq \cdots (a^n)$$

descending chain of ideals with strict inclusions

- Every finite R-module is Arthinian.

  In particular, every finite abelian group (as $\mathbb{Z}$-module) is Arthinian.

Example: $\mathbb{Z}/(n)$

K field

- R is a finite dimension K-algebra

  R is Arthinian.

  Example: $K[X,Y]/(X^2, Y^2)$

- (Exercise): ~~$M = \mathbb{Z}$~~ Consider $\mathbb{Z}_p = \left\{ \frac{a}{p^n} \mid a \in \mathbb{Z}, n \in \mathbb{N} \right\}$ p prime

Then $\mathbb{Z}_p/\mathbb{Z}$ is an Arthinian $\mathbb{Z}$-module

but not Noetherian.

However for rings we will see that every Arthinian is Noetherian.

Proposition 8.5

Let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be an exact seq. of R-modules

Then

M is Artinian $\Leftrightarrow$ M' & M'' are Artinian.

"$\Rightarrow$" Take a descending sequence of submodules in M' (resp M'') and apply f (resp g^{-1}) $\rightsquigarrow$ descending sequence of submodules in M $\Longrightarrow$ it stabilizes in M

$\Rightarrow$ the initial sequence in M' (or M'') also stabilizes.

"$\Leftarrow$" Let $M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$ in M

By Remark 6.69 for every $k \in \mathbb{N}$ ~~then~~ $0 \to f^{-1}(M_k) \xrightarrow{f} M_k \xrightarrow{g} g(M_k) \to 0$ is exact

$$\begin{tikzcd} 0 \arrow[r] & f^{-1}(M_{k+1}) \arrow[r, "f"] \arrow[d, equal] & M_{k+1} \arrow[r, "g"] \arrow[d, hook] & g(M_{k+1}) \arrow[r] \arrow[d, equal] & 0 & \text{exact} \\ 0 \arrow[r] & f^{-1}(M_n) \arrow[r] & M_k \arrow[r] & g(M_n) \arrow[r] & 0 & \text{exact} \end{tikzcd}$$

Since M' and M'' Artinian, the descending sequences $\{f^{-1}(M_k)\}_{k\in\mathbb{N}}$ & $\{g(M_k)\}_{k\in\mathbb{N}}$ stabilizes at $k=n$ for some $n \in \mathbb{N}$

$\Rightarrow$ $\{M_k\}_{k\in\mathbb{N}}$ also stabilizes at $n=k$

(By Snake-lemma or 5-lemma)

It follows that M is Artinian.

Corollary 8.6

a) M_1, M_2 is Artinian $\iff$ $M_1 \oplus M_2$ is Artinian

b) $N \subset M$, then M Artinian $\iff$ N, M/N Artinian

c) R - Artinian ring.
Then every finitely generated R-module M is Artinian.
($\exists$ surjection $R^n \xrightarrow{\varphi} M$, R^n Artinian by a), and by b) $\underbrace{M \simeq R^n/\ker\varphi}_{\text{Artinian}}$)

d) R Artinian ring $\Longrightarrow$ R/I Artinian ring.

Next goal: to show that (R is Artinian $\iff$ R is Noetherian and $\underbrace{\text{Spec } R = \text{Max}(R)}$)

we say $\dim R = 0$

Proposition 8.7

R Artinian ring

Then every prime ideal in R is maximal

Proof Let $\rho \in \text{Spec } R$.

Then $A := R/\rho$ is a $\underbrace{\text{Artinian}}$ integral domain.

by Corollary 8.6 d)

To show: A is a field.

Let $x \in A$, $x \neq 0$. To show: $x \in A^*$.

Consider the descending chain of ideals: $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \ldots \supseteq (x^n) \supseteq \ldots$

Since A is Artinian, we have $(x^n) = (x^{n+1})$ for some $n \in \mathbb{N}$.

Hence, ~~That is~~ $x^n = x^{n+1} y$ for some $y \in A$.

$\Downarrow$

$x^n(1 - xy) = 0 \xrightarrow[\text{A integral domain}]{} xy = 1 \Longrightarrow x \in A^* \Longrightarrow A$ is a field

$\Longrightarrow \rho$ is maximal. It follows that $\text{Spec}(R) = \text{Max}(R)$ $\square$

Remark If $\operatorname{Spec} R = \operatorname{Max} R$, we say $\dim R = 0$.

Proposition 8.8 R Artinian, then $\operatorname{Max}(R)$ is finite.

Proof Assume $\operatorname{Max}(R)$ is infinite and

$\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_n, \dots$ are pairwise different max. ideals $\mathfrak{m}_i \neq \mathfrak{m}_j$ for $i \neq j$.

Consider $(*)$ $\mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \dots \supseteq \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_n \supseteq \dots$

Claim: $\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{n-1} \supsetneq \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{n-1} \cdot \mathfrak{m}_n$

Assume "$=$"

$\forall\, i = 1, \dots, n-1$ $\exists\, a_i \in \mathfrak{m}_i \setminus \mathfrak{m}_n$

Then $a_1 \cdot \dots \cdot a_{n-1} \in \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{n-1} = \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{n-1} \cdot \mathfrak{m}_n \subseteq \underset{\text{prime}}{\mathfrak{m}_n}$

But $a_i \notin \mathfrak{m}_n$ $\lightning$

It follows that Claim is true $\Longrightarrow$ the sequence $(*)$ does not stabilize $\Longrightarrow$ contradiction, since R Artinian.

Therefore, $\operatorname{Max}(R)$ is finite. $\square$

Proposition 8.9 R Artinian, then the nilradical $\mathcal{N}_R$ is nilpotent (that is $\mathcal{N}_R^k = 0$ for some $k \in \mathbb{N}$).

Proof R Artinian $\Longrightarrow$ $\mathcal{N}_R^k = \mathcal{N}_R^{k+1} = \dots$ (denote this ideal by I)

Claim: $I = 0$.

Assume $I \neq 0$ and consider the set of ideals

$\Sigma = \{J \text{ ideal in } R \mid JI \neq 0\}$ $\Sigma \neq \emptyset$, since $I \in \Sigma$.

R Artinian $\Longrightarrow$ Σ has a minimal element J', $J'I \neq 0$

$\exists\ x \in J'$, such that $xI \neq 0$ $\Longrightarrow$ $(x)I \neq 0$ $\Longrightarrow$ $(x) \in \Sigma$ &
$(x) \subseteq J'$

$\xrightarrow[\text{by minimality of } J' \text{ in } \Sigma]{}$ $(x) = J'$

We also have $((x)I)\cdot I = (x)I^2 = (x)I \neq 0$

$\Downarrow$

$(x)I \in \Sigma$

But $(x)I \subseteq (x)$

$\Downarrow$ minimality of $(x) = J$ in Σ

$(x)I = (x)$

It follows that $x = xy$ for some $y \in I = N_R^k \subseteq N_R$

that is y is nilpotent,
$y^n = 0$ for some $n \in \mathbb{N}$.

We get $x = xy = (xy)y = xy^2 = xy^3 = \dots = xy^n = 0$

$\Rightarrow \underset{(x)=}{J'} = 0$, ~~and~~ but $IJ' \neq 0$ Contradiction $\Longrightarrow$ $I = 0$ ∎

Recall:

A Artinian Ring $\overset{def}{\iff}$ descending chain of ideals in A stabilizes. $\forall\ I_1 \supseteq I_2 \supseteq \dots \supseteq I_k \supseteq \dots$

Properties: R Artinian Ring

- $\mathrm{Max}(R) = \mathrm{Spec}(R)$ (Prop 8.7) (we say $\dim R = 0$)
- $\mathrm{Max}(R)$ is finite (Prop 8.8)
- $N_R^k = 0$ for some $k \in \mathbb{N}$ (Prop 8.9)

Definition 8.10: R comm. Ring, M R-module.

The **annihilator** of M is the ideal

$$\mathrm{Ann}_R(M) = \{ r \in R \mid rm = 0 \ \forall\, m \in M \}$$
$$\|$$
$$\mathrm{Ker}\,(R \xrightarrow{\varphi} \mathrm{Mod}_{\mathbb{Z}}(M))$$
$$r \longmapsto \mu_r \quad , \quad \mu_r : M \to M,\ m \mapsto rm$$

Remark 8.11 If $I \subseteq \mathrm{Ann}_R(M)$, then M is canonically R/I-module

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & \mathrm{Mod}_{\mathbb{Z}}(M) \\ R/I \arrow[ur, dashed, "\exists \varphi'"'] & \end{tikzcd} \quad , \quad I \subset \mathrm{Ker}(M)$$

M

$(r+I)m := rm$ well-defined (since $I \subseteq \mathrm{Ann}_R(M)$).

$M' \subset M$ ~~abelian~~ subset

M' $\{$ ~~to be~~ ~~set of~~ R-submodules of ~~M~~ R-module M $\}$ $\iff$ $\{$ ~~to be~~ Set of R/I-submodules of R/I-module M $\}$

<u>Proposition 8.12</u> R comm. ring.

$\mathfrak{m}_1, \dots, \mathfrak{m}_n$ max. ideals in R (not necessary different)
such that $\mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \dots \cdot \mathfrak{m}_n = 0$

Show that: R ~~Noetherian~~ Artinian $\iff$ R ~~Artinian~~ Noetherian

<u>Proof</u>:

(by (*) see below)
Artinian $\iff$ Noetherian

Consider $R = \mathfrak{m}_0 \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \dots \supset \underbrace{\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{n-1}}_{M_{n-1}} \supset \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_n = 0$

$$M_i := \frac{\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{i-1}}{\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{i-1} \cdot \mathfrak{m}_i} \qquad (i = 1, \dots, n)$$

$\mathfrak{m}_i \in \mathrm{Ann}_R(M_i)$

M_i is an $\underbrace{R/\mathfrak{m}_i}_{\text{field}}$ - module (by Remark 8.11)

$\Rightarrow$ $R/\mathfrak{m}_i$ - vector space.

Then (*) M_i Artinian $\iff$ M_i Noetherian ($\iff$ M_i finite dim. $R/\mathfrak{m}_i$ - vector space)

Even If for $\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_i$ holds ~~No~~ (Artinian $\iff$ Noetherian)
Denote $N \iff A$

Then ($N \iff A$) also holds for $\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{i-1}$. Indeed,

$$0 \to \underset{\substack{(N \iff A)\\ \text{Induction hypothesis}}}{\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_i} \longrightarrow \mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{i-1} \longrightarrow \underset{(N \iff A)}{M_i} \to 0 \quad \text{exact.}$$

By Prop. 8.5 (and similar result for Noetherian modules)

for $\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_{i-1}$ we have ($N \iff A$)

$i = n-1$ follows from (*). (since $\mathfrak{m}_1 \cdot \dots \cdot \mathfrak{m}_n = 0$)

By decreasing induction on i ($N \iff A$) holds for $\mathfrak{m}_0 = R$.

Prop 8.13

R Artinian $\iff$ R Noetherian and $\dim R = 0$

Proof:

"$\Rightarrow$"

Let R be an Artinian ring.

Prop. 8.7 $\Rightarrow$ $\dim R = 0$

Prop 8.8 $\Rightarrow$ $\mathrm{Max}(R) = \{m_1, \dots, m_n\}$ finite

~~fa~~

$$m_1 \cdot \ldots \cdot m_n \subseteq m_1 \cap \ldots \cap m_n = N_R$$

$\Downarrow$ Prop 8.9

$m_1^k \cdot \ldots \cdot m_n^k = 0$ for some $k \in \mathbb{N}$.

$\Downarrow$ Prop 8.12

R Noetherian

"$\Leftarrow$" Let R be a Noetherian ring.

By Proposition 3.18, R has only fin. many minimal prime ideals $\underbrace{\{m_1, \dots, m_n\}}_{\mathrm{Max}(R)}$, since $\dim R = 0$

&

$$m_1 \cdot \ldots \cdot m_n \subseteq m_1 \cap \ldots \cap m_n = N_R$$

Claim: N_R is nilpotent. $N_R = (f_1, \dots, f_e)$, $f_i^{k_i} = 0$

Then $\forall f = a_1 f_1 + \ldots + a_e f_e$ we have $f^{k = k_1 + \ldots + k_e} = 0$

So for $k = k_1 + \ldots + k_e$, $N_R^k = 0$.

Then we ~~apply~~ get $m_1^k \cdot \ldots \cdot m_n^k = 0 \underset{\text{Prop 8.12}}{\Longrightarrow} R$ Artinian.

Proposition 8.14

An Artinian Ring R is isomorphic to a finite direct product, ~~where~~ $\prod_{i=1}^{n} R_i$, where R_i are Artinian local Rings.

Proof $\mathrm{Max}(R) = \{m_1, \dots, m_n\}$ (by Proposition 8.8)

$\forall i \neq j$: m_i and m_j are coprime $\xRightarrow{\text{(Tutorium 1, Exercise 1)}}$ m_i^k and m_j^k coprime $\forall i \neq j$ $\forall k \in \mathbb{N}$

$m_i + m_j = R$

as in the proof of Proposition 8.9

$\xRightarrow{\text{Ex. sheet 1}}$ $m_1^k \cap \dots \cap m_n^k = m_1^k \cdot \dots \cdot m_n^k = 0$

for some k, s.t. $N_R^k = 0$.

as

$$R/_{m_1^k \cap \dots \cap m_n^k} = R/_{\underbrace{m_1^k \cdot \dots \cdot m_n^k}_{0}} = R$$

$\cong$ ← Chinese Remainder Theorem (Theorem 2.21)

$$R/_{m_1^k} \times \dots \times R/_{m_n^k}$$

Every $R/_{m_i^k}$ is Artinian (by Corollary 8.6 d)

It remains to show that $R/_{m_i^k}$ is local.

We know from the lecture that

$\pi: R \longrightarrow R/_{m_i^k}$ canonical projection

$$\mathrm{Max}\left(R/_{m_i^k}\right) = \left\{\pi(m') \mid m' \in \mathrm{Max}(R) \text{ and } m_i^k \subseteq m'\right\} =$$

$= \{m_i\}$

only m_i satisfies this property

if $m' \neq m_i$, take $x \in m_i \setminus m'$ ~~and~~ then $x^k \notin m'$

It follow that

$R/_{m_i^k}$ has the unique max ideal $\Rightarrow R/_{m_i}$ is local

Modules of finite lenght.

Definition / Remark 8.15

- An R-Module is called *simple* if M has no submodules except 0 & M itself.

- M simple $\iff M \simeq R/\mathfrak{m}$, where $\mathfrak{m}$ is maximal in $\mathfrak{m}$.

(note that R/I simple $\iff I$ maximal

if $x \neq 0$, $x \in M$, then $(x) = M \Rightarrow M \simeq R/\mathrm{Ann}(x)$)

Definition 8.16

We say that the chain of submodules

$$(*) \qquad M = M_0 \supset M_1 \supset \dots \supset M_n = 0 \qquad \text{(strict inclusions)}$$

has length $= n$.

$(*)$ ~~A~~ is a *composition series* of M ~~is a max~~ if $\forall\, i = 1, \dots, n$: M_{i-1}/M_i is simple

(~~that is in~~ $\iff$ $(*)$ ~~one can that~~ is maximal, that is one can not insert extra submodules).

Definition 8.17

$$\ell(M) := \inf \{ m \,/\, M \text{ has a composition series of length } m \}$$

By convention,

- $\ell(M) = \infty$ if M has no comp. series
- $\ell(0) = 0$

If $\ell(M) < \infty$, we say M is a module of *finite length*

Proposition 8.18 M R-module.

Assume $\ell(M) = n$.

Then every composition series of M has length n, and every chain of submodules in M can be extended to a composition series.

Proof

Claim 1: if $N \subsetneq M$ (proper submodule), then $\ell(N) < \ell(M)$.

$\ell(M) = n \Rightarrow \exists$ a composition series of M of length $= n$.

$$M = M_0 \supset M_1 \supset M_2 \supset \dots \supset M_n = 0$$

Consider $N_i := N \cap M_i \subset M_i$

Then clearly $N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_n = 0$ $(*)$

We have

$$\underbrace{N_{i-1} \hookrightarrow M_{i-1} \longrightarrow M_{i-1}/M_i}_{\text{Ker of this composition}}$$

is $N_{i-1} \cap M_i = \cancel{(M_{i-1} \cap N)}\ (N \cap M_{i-1}) \cap M_i = N \cap M_i = N_i$.

$\rightsquigarrow$ ~~then~~ we have an injection

$$N_{i-1}/N_i \longrightarrow M_{i-1}/M_i \qquad \forall\, i = 1, \dots, n.$$

simple

Since M_{i-1}/M_i simple, we have either

$$N_{i-1}/N_i = 0 \quad \text{or} \quad N_{i-1}/N_i \simeq M_{i-1}/M_i$$

$\Downarrow$

$N_{i-1} = N_i$

By removing from (*) submodules N_i, s.t. $N_i = N_{i+1}$ we get a composition series of length $\le n$.

Moreover, length $= n \iff N_{i-1}/N_i \simeq M_{i-1}/M_i \quad \forall\, i = 1, \dots, n.$

Note that $N_n = M_n = 0 \implies N_{n-1} = N_{n-1}/N_n \simeq M_{n-1}/M_n = M_n$

By decreasing induction on i we can show that $N_i = M_i \quad \forall\, i = 0, 1, \dots, n.$ (True for $n-1$)

~~(To illustrate let~~

Assume $N_i = M_i$, to show $N_{i-1} = M_{i-1}$

We have

$$\cancel{0 \to N_{i-1} \to N_i \to N_{i-1}/N}$$

$$\begin{tikzcd}
0 \arrow[r] & N_i \arrow[r] \arrow[d, "(1)"', "\parallel"] & N_{i-1} \arrow[r] \arrow[d, "(2)"'] & N_{i-1}/N_i \arrow[r] \arrow[d, "(3)"', "\wr"] & 0 & \text{exact} \\
0 \arrow[r] & M_i \arrow[r] & M_{i-1} \arrow[r] & M_{i-1}/M_i \arrow[r] & 0 & \text{exact}
\end{tikzcd}$$

Since (1) and (3) are isomorphisms, it follows from Snake Lemma, that $N_{i-1} = M_{i-1}$ (that is "(2) is iso").

Let (*) $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$ be a chain of submodules in M. (of length $= k$)

<u>Claim 2</u>: $k \leq n = \ell(M)$

By Claim 1: $n = \ell(M) > \underbrace{\ell(M_1) > \dots > \ell(M_k) = 0}_{\substack{k \text{ different integers} \\ \in [0, n-1]}} \Rightarrow k \leq n.$

- If (*) is a comp. series, then

$$\underset{\substack{\uparrow \\ \text{Claim 2}}}{k \leq} \ \ell(M) \underset{\substack{\uparrow \\ \text{by def. of } \ell(M)}}{\leq k} \qquad \Rightarrow \quad k = n$$

- If (*) has length $= \ell(M)$, then (*) is a **maximal chain** (by Claim 2). **Hence**, (*) is a composition series.

- If the length of (*) $< \ell(M)$, then it is not a composition series and we can extend it till the length $= \ell(M)$.
  Then it becomes a comp. series.

~~and we can extend it till the length ℓ(M))~~ .
~~Then it becomes a comp series~~

Prop 8.19: M is of finite length $\Leftrightarrow$ M is Artinian & Noetherian

Proof: "$\Rightarrow$" In any descending (resp. ascending) chain of submodules there are only finitely many strict inclusion $\Rightarrow$ & every such chain stabilizes

"$\Leftarrow$" $M_0 = M$ Noetherian $\Rightarrow \exists$ maximal submodule $M_1 \subset M_0$ (M_0/M_1 simple)

& Similarly, M_1 has a max. submodule M_2 ... ~~and so on~~.

$$\rightsquigarrow \quad M = M_0 \supset M_1 \supset M_2 \supset \dots \supset \overset{=0}{M_n} \supset M_{n+1} \supset \quad (*)$$

with M_{i-1}/M_i simple for $i = 1, 2, \dots$

Since M Artinian, $(*)$ stabilizes ~~and~~ by $M_n = 0$

$\Rightarrow$ $(*)$ is a comp. series

Remark 8.20 • $0 \to M' \xrightarrow{f} M \to M'' \to 0$ exact

Then $\ell(M) = \ell(M') + \ell(M'')$.

• ∮ (Jordan – Hölder theorem): unique up to permutation and iso

$M = M_0 \supset M_1 \supset \dots \supset M_n = 0$

$M' = M_0' \supset M_0 \supset \dots \supset M_n' = 0$

two comp. series $\Rightarrow \{M_{i-1}/M_i\}_{1 \le i \le n} \overset{1\text{-to-}1}{\longleftrightarrow} \{M'_{i-1}/M'_i\}_{1 \le i \le n}$.

up to permutation $M_{i-1}/M_i \simeq M'_{i-1}/M'_i$