Mathematisches Institut der LMU
Dr. Heribert Zenk
D.T. Nguyen

Mathematische Quantenmechanik II
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Excercise Sheet 8

Übung 1

Let χ be a measurable real–valued function on $\mathbb{R}$. Prove that

a) The transformation operators $U(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(U(t)\psi)(x) := e^{it\chi(x)}\psi(x)$, for $t \in \mathbb{R}$ and $\psi \in \mathcal{S}(\mathbb{R})$ define a strongly continuous, one–parameter unitary group $\{U(t) : t \in \mathbb{R}\}$.

b) The operator A defined on

$$D(A) = \left\{ \psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\chi(x)\psi(x)|^2 \mathrm{d}x < \infty \right\}$$

as the multiplication operator with χ is the infinitesimal generator of $\{U(t) : t \in \mathbb{R}\}$.

Übung 2

a) Prove that the dilation operator $U(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by $(U(t)\psi)(x) := e^{-\frac{nt}{2}}\psi(e^{-t}x)$, for $\psi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, forms a strongly continuous one–parameter unitary group. Determine its infinitesimal generator. Here $e = \exp(1)$.

b) Consider the Hamiltonian $H = -\Delta + V$ with V symmetric, relatively bounded with respect to $-\Delta$, and $U(-t)VU(t) = e^{-t}V$ on $D(-\Delta) = H^2(\mathbb{R}^n)$. Prove that every normalized eigenfunction ψ corresponding to an eigenvalue λ satisfies

$$\lambda = -\langle u, -\Delta u\rangle = \frac{1}{2}\langle u, Vu\rangle.$$

Übung 3

Es seien $L_j : \mathcal{D}(L_j) \to L^2(\mathbb{R}^3)$ die Komponenten des Drehimpulses, also

$$(L_j[\psi])(\underline{x}) = \sum_{k,l=1}^{3} \operatorname{sign}\begin{pmatrix} 1 & 2 & 3 \\ j & k & l \end{pmatrix} x_k (p_l[\psi])(\underline{x})$$

für alle $\psi \in \mathcal{S}(\mathbb{R}^3)$, $\underline{x} = (x_1, x_2, x_3)$ und $j = 1, 2, 3$. Zeige:

a) $[L_j, L_k]\psi = i \sum_{l=1}^{3} \operatorname{sign}\begin{pmatrix} 1 & 2 & 3 \\ j & k & l \end{pmatrix} L_l\psi$ für alle $\psi \in \mathcal{S}(\mathbb{R}^3)$.

b) Für $L^2 := L_1^2 + L_2^2 + L_3^2$ gilt $[L^2, L_j]\psi = 0$ für alle $\psi \in \mathcal{S}(\mathbb{R}^3)$.