Übungsblatt 4 zu Mathematische Quantenmechanik II

Aufgabe 1:
Consider the N-fold Hilbert space tensor product $\mathcal{H}^{\widehat{\otimes} N}$ of a Hilbert space $\mathcal{H}$ with itself. Let $\sigma \in \mathcal{S}_N$ be a permutation of $\{1, 2, \ldots, N\}$, and $\widehat{p}_\sigma : \mathcal{H}^{\widehat{\otimes} N} \to \mathcal{H}^{\widehat{\otimes} N}$ defined by

$$\widehat{p}_\sigma[u_1 \otimes \ldots \otimes u_N] := u_{\sigma^{-1}(1)} \otimes \ldots \otimes u_{\sigma^{-1}(N)}.$$

a) Show that $\widehat{p}_\sigma$ defines a unitary operator.

b) Show that $\widehat{p}_\sigma \widehat{p}_\tau = \widehat{p}_{\tau\sigma}$ for any two permutations $\sigma, \tau \in \mathcal{S}_N$.

Define the two following operators,

$$\widehat{S}_N = (N!)^{-1} \sum_{\sigma \in \mathcal{S}_N} \widehat{p}_\sigma, \quad \text{and} \quad \widehat{A}_N = (N!)^{-1} \sum_{\sigma \in \mathcal{S}_N} \operatorname{sign}(\sigma) \widehat{p}_\sigma.$$

c) Show that $\widehat{S}_N$ and $\widehat{A}_N$ are orthogonal projections satisfying $\widehat{S}_N \widehat{A}_N = 0$ if $N \geq 2$.

d) Show that $\widehat{p}_\sigma \circ \widehat{A}_N = \operatorname{sign}(\sigma) \widehat{A}_N$ for all permutation $\sigma \in \mathcal{S}_N$.

Aufgabe 2: Show $\widehat{S}_N \left[\mathcal{H}^{\widehat{\otimes} N}\right]$ and $\widehat{A}_N \left[\mathcal{H}^{\widehat{\otimes} N}\right]$ are Hilbert spaces.

Aufgabe 3: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Prove that for any sequence $(A_n)_{n \in \mathbb{N}}$ in $L(\mathcal{H}_1)$ and for $A \in L(\mathcal{H}_1)$, such that, $\|A_n[\psi] - A[\psi]\| \overset{n\to\infty}{\longrightarrow} 0$ for all $\psi \in \mathcal{H}_1$ then

$$\|(A_n \widehat{\otimes} \mathrm{id}_{\mathcal{H}_2})[\phi] - (A \widehat{\otimes} \mathrm{id}_{\mathcal{H}_2})[\phi]\| \to 0$$

for all $\phi \in \mathcal{H}_1 \widehat{\otimes} \mathcal{H}_2$, where $A \widehat{\otimes} \mathrm{id}_{\mathcal{H}_2}$ is the closure of $A \otimes \mathrm{id}_{\mathcal{H}_2}$.