The following type is called the Curry–Howard interpretation of Brouwer's continuity principle

$$\text{(CH - Cont)} \qquad \Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\alpha : \mathbb{N}^{\mathbb{N}}).\ \Sigma(n : \mathbb{N}).\ \Pi(\beta : \mathbb{N}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta).$$

Theorem. In intensional Martin-Löf type theory (with Π, Σ, Id),

$$\text{CH - Cont} \to 0 = 1.$$

Proof.

1. Assuming CH - Cont, we get a *modulus-of-continuity functional*

   $$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$$

   assigning a modulus $M(f)$ to f at the point 0^ω, where

   - 0^ω is the infinite sequence of zeros (`0^w`), and
   - $0^n k^\omega$ consists of n zeros followed by infinitely many k's (`n zeros-and-then k`).
   - Fact (i): $(0^n k^\omega)(n) = k$ ($\texttt{zeros-and-then-spec}_0$).
   - Fact (ii): $0^\omega =_n 0^n k^\omega$ ($\texttt{zeros-and-then-spec}_1$).

2. Let $m = M(\lambda\alpha.0)$. Define $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ by $f(\beta) = M(\lambda\alpha.\beta(\alpha m))$.
3. By expanding the definitions (which involves the ξ-rule), we get

   $$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha m)) = M(\lambda\alpha.0) = m \quad (\texttt{claim}_0).$$

4. By the definition of M, we have

   $$\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\ 0^\omega =_{M(f)} \beta \to m = f\beta \quad (\texttt{claim}_1).$$

5. Choosing $\beta = 0^{M(f)+1}1^\omega$, we have

   $$0^\omega =_{M(f)} \beta \quad (\texttt{claim}_2)$$

   and hence

   $$m = f(\beta) \quad (\texttt{claim}_3).$$

6. By the continuity of $\lambda\alpha.\beta(\alpha m)$, we get

   $$\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\ 0^\omega =_m \alpha \to \beta 0 = \beta(\alpha m) \quad (\texttt{claim}_4).$$

7. Choosing $\alpha = 0^m(M(f)+1)^\omega$, we have

   $$0^\omega =_m \alpha \quad (\texttt{claim}_5)$$

   and hence

   $$0 = \beta 0 = \beta(\alpha m) = \beta(M(f)+1) = 1 \quad (\texttt{goal}).$$