Continuity in Type Theory

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Brouwer's continuity principle

The value of $f(\alpha)$ of a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ depends only on a finite prefix of the sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$.

In *e.g.* higher-type Heyting arithmetic (HA^{ω}),

$$\forall (f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\ \forall (\alpha : \mathbb{N}^{\mathbb{N}}).\ \exists (n : \mathbb{N}).\ \forall (\beta : \mathbb{N}^{\mathbb{N}}).\ \alpha =_n \beta \Rightarrow f(\alpha) = f(\beta)$$

is not provable (or disprovable).

But it's validated by *e.g.* Johnstone's topological topos, among other well-known models.

Continuity in type theory

How should it be formulated in intuitionistic type theory?

- We of course don't expect it to be provable.
- But much less we expect it to be disprovable.

Its Curry-Howard interpretation ($\mathrm{CH\text{-}Cont}$)

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}).\ \Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\ \Sigma(n : \mathbb{N}).\ \Pi(\beta : \mathbb{N}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta)$$

is provably false in intensional Martin-Löf type theory.

What does it mean?

What is the the correct formulation of the continuity principle in type theory?

What about uniform continuity of functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$?

Failure of the CH interpretation of the continuity principle

A theorem in intensional Martin-Löf type theory (with $\mathbb{N}, \Pi, \Sigma, \mathrm{Id}$):

$$\Big(\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\alpha : \mathbb{N}^{\mathbb{N}}).\ \Sigma(n : \mathbb{N}).\ \Pi(\beta : \mathbb{N}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta)\Big) \to 0 = 1$$

by adaptation of an old argument due to Kreisel, originally relying on choice and extensionality.

By projection, $\mathrm{CH\text{-}Cont}$ gives a modulus-of-continuity functional

$$M : (\mathbb{N}^{\mathbb{N}} \to \mathbb{N}) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$$

assigning a modulus $n = M(f, \alpha)$ to f at the point α.

Trouble: While all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ may be continuous, there can't be any continuous modulus-of-continuity functional.

Proof of $\mathrm{CH\text{-}Cont} \to 0 = 1$

- Assuming $\mathrm{CH\text{-}Cont}$, we get M and write $M(f) = M(f, 0^\omega)$, where
  - 0^ω is the infinite sequence of zeros, and
  - $0^n k^\omega$ consists of n zeros followed by infinitely many k's.
  - Two facts: $0^\omega =_n 0^n k^\omega$ and $(0^n k^\omega)(n) = k$ for any n, k.
- Let $m = M(\lambda\alpha.0)$.
  Define $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ to be $f(\beta) = M(\lambda\alpha.\beta(\alpha m))$.
- By expanding the definitions (which involves the ξ-rule), we get
  $$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha m)) = M(\lambda\alpha.0) = m.$$
- By the definition of M, we have
  $$\Pi(\beta : \mathbb{N}^{\mathbb{N}}).\ 0^\omega =_{M(f)} \beta \to m = f\beta.$$
- Choosing $\beta = 0^{M(f)+1}1^\omega$, we have $0^\omega =_{M(f)} \beta$ and hence $f(\beta) = m$.
- By the continuity of $\lambda\alpha.\beta(\alpha m)$, we get
  $$\Pi(\alpha : \mathbb{N}^{\mathbb{N}}).\ 0^\omega =_m \alpha \to \beta 0 = \beta(\alpha m).$$
- Choosing $\alpha = 0^m(M(f)+1)^\omega$, we have $0^\omega =_m \alpha$ and hence
  $$0 = \beta 0 = \beta(\alpha m) = \beta(M(f)+1) = 1.$$

Formalisation in Agda

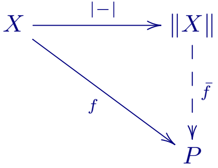
Discussion

1. No continuous/extensional modulus-of-continuity functional M:
   We used our hypothetical M to define a non-continuous function f and hence prove M wrong.
2. And this is exactly what is happening in the topological topos:
   - All functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are continuous.
   - But there is no continuous way of finding moduli of continuity.
3. The conversion
   $$f(0^\omega) = M(\lambda\alpha.0^\omega(\alpha m)) = M(\lambda\alpha.0) = m$$
   in the proof relies on the ξ-rule (reduction under λ).
4. In HA^ω, the ξ-rule, the axiom of choice, and the continuity of all functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ are together impossible.
5. Since ξ-rule holds in categories, any locally cartesian closed category with a natural numbers object (*e.g.* any topos) disproves $\mathrm{CH\text{-}Cont}$.

Propositional truncation

A type is called a proposition if it has at most one element.

A propositional truncation of a type X, if it exists, is a proposition $\|X\|$ together with a map $|-|: X \to \|X\|$ such that for any proposition P and $f: X \to P$ we can find $\bar{f}: \|X\| \to P$.



Intuitively, $\|X\|$ is

- the truth value of the inhabitedness of X;
- the quotient of the type X by the chaotic equivalence relation.

The logic of propositions in HoTT and toposes

In HoTT, $\|X\|$ is defined as a higher inductive type.

The logic of propositions

$$
\begin{aligned}
\bot \quad &:\equiv \quad \mathbb{0} \\
\top \quad &:\equiv \quad \mathbb{1} \\
P \wedge Q \quad &:\equiv \quad P \times Q \\
P \vee Q \quad &:\equiv \quad \|P + Q\| \\
P \Rightarrow Q \quad &:\equiv \quad P \to Q \\
\forall (x : A). P(x) \quad &:\equiv \quad \Pi (x : A). P(x) \\
\exists (x : A). P(x) \quad &:\equiv \quad \|\Sigma (x : A). P(x)\|
\end{aligned}
$$

The correct type-theoretic formulation of continuity

$$\Pi(f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\alpha : \mathbb{N}^{\mathbb{N}}).\ \| \ \Sigma(n : \mathbb{N}).\ \Pi(\beta : \mathbb{N}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta)\ \|$$

- It's validated in *e.g.* the topological topos.
- The continuous dependency of n on inputs f and α is now broken.
- Because the axiom of choice

$$\Pi(x : X).\|\Sigma(y : Y).A(x, y)\| \to \|\Sigma(f : X \to Y).\Pi(x : X).A(x, y)\|$$

is not provable.

Uniform continuity of functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$

$$\forall (f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \exists (n : \mathbb{N}).\ \forall (\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \Rightarrow f(\alpha) = f(\beta)$$

- Not provable but consistent in HA^ω.
- Its Curry–Howard interpretation
$$\Pi (f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \Sigma (n : \mathbb{N}).\ \Pi (\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta)$$
is also consistent in MLTT.
- Moreover, it's logically equivalent to
$$\Pi (f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \|\ \Sigma (n : \mathbb{N}).\ \Pi (\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta)\ \|$$
assuming function extensionality.

Disclosing secrets from truncations

In general we don't have $\|X\| \to X$ for arbitrary X, because it gives a constructive taboo and also contradicts univalence.

However, for some types X, we can disclose a secret $\|X\|$ to X.

Lemma. For any family A of types indexed by natural numbers such that

1. $A(n)$ is a proposition for every $n : \mathbb{N}$, and
2. $A(n)$ implies that $A(m)$ is decidable for every $m < n$

we have

$$\Sigma(n\!:\!\mathbb{N}).A(n) \;\leftrightarrow\; \|\Sigma(n\!:\!\mathbb{N}).A(n)\|.$$

Proof sketch of ($\leftarrow$). Given n with $A(n)$, we can find the minimal k with $A(k)$, using the decidability of $A(m)$ for $m < n$. Since "having a minimal k with $A(k)$" is a proposition (proved using function extensionality), the elimination rule of $\| - \|$ gives the desired result.

Equivalence of the two formulations of uniform continuity

Theorem. The proposition

$$\Pi(f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \|\ \Sigma(n : \mathbb{N}).\ \Pi(\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta)\ \|$$

is logically equivalent to the type

$$\Pi(f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(n : \mathbb{N}).\ \Pi(\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta).$$

Proof sketch. Use the lemma by taking

$$A(n) \ :\equiv\ \Pi(\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \to f(\alpha) = f(\beta).$$

Adding uniform continuity to type theory

Simply adding a constant as an axiom Ax to a theory $\mathcal{T}$ may destroy its canonicity, i.e. not every closed natural number in $\mathcal{T}$ evaluates to a numeral.

Instead, one can build a constructive/computational model $\mathcal{M}$ of the theory $\mathcal{T}$

$$\mathcal{T} \xrightarrow{[\![-]\!]} \mathcal{M}$$

such that the axiom Ax has an interpretation $[\![\mathrm{Ax}]\!] \in \mathcal{M}$.

Then the evaluation of terms in $\mathcal{T} + \mathrm{Ax}$ becomes the one in the model $\mathcal{M}$.

We build such a model of type theory extended with the uniform-continuity principle, using C-spaces.

C-spaces and continuous maps

Def. A C-topology on a set X is a collection P of probes $\mathbf{2}^{\mathbb{N}} \to X$ subject to the following probe axioms:

1. All constant maps are in P.
2. If $t \colon \mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is uniformly continuous and $p \in P$, then $p \circ t \in P$.
   (Presheaf condition)
3. For any two maps $p_0, p_1 \in P$, the unique map $p \colon \mathbf{2}^{\mathbb{N}} \to X$ defined by $p(i * \alpha) = p_i(\alpha)$ is in P.
   (Sheaf condition)

A C-space is a set X equipped with C-topology.

A function $f \colon X \to Y$ of C-spaces is continuous if $f \circ p \in P_Y$ whenever $p \in P_X$. (Naturality condition)

Examples of $\mathbb{C}$-spaces

All continuous maps from $\mathbf{2}^{\mathbb{N}}$ (with the usual topology) to any topological space X form a $\mathbb{C}$-topology on X:

- Any constant map $\mathbf{2}^{\mathbb{N}} \to X$ is continuous.
- The composite $\mathbf{2}^{\mathbb{N}} \xrightarrow{t} \mathbf{2}^{\mathbb{N}} \xrightarrow{p} X$ of two continuous maps is continuous.
- The sheaf condition is satisfied because $\mathbf{2}^{\mathbb{N}}$ is compact Hausdorff.

Any continuous map of topological spaces is continuous w.r.t. the above $\mathbb{C}$-topology, as composition preserves continuity.

Discrete C-spaces

Def. A map $p \colon \mathbf{2}^{\mathbb{N}} \to X$ into a set X is called locally constant iff $\exists(n : \mathbb{N}).\ \forall(\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \Rightarrow f(\alpha) = f(\beta)$.

Lemma

The locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ form a C-topology which has the smallest amount of probes on X.

Def. A C-space X is discrete if all functions $X \to Y$ into any C-space Y are continuous.

Lemma

A C-space is discrete iff its probes are precisely the locally constant functions.

Def. We thus refer to the collection of all locally constant maps $\mathbf{2}^{\mathbb{N}} \to X$ as the discrete C-topology on X.

- The discrete C-topology on $\mathbf{2}$ or $\mathbb{N}$ is the set of uniformly continuous maps.
- The discrete space $\mathbf{2}$ is the coproduct of two copies of the terminal space.
- The discrete space $\mathbb{N}$ is the natural numbers object.

Yoneda Lemma and Fan functional

The monoid C of uniformly continuous $\mathbf{2}^{\mathbb{N}} \to \mathbf{2}^{\mathbb{N}}$ is a C-topology on $\mathbf{2}^{\mathbb{N}}$.

$$(\mathbf{2}^{\mathbb{N}}, \mathrm{C}) \;=\; \text{the exponential of the two discrete } \mathrm{C}\text{-spaces}$$

The Yoneda Lemma says that a map $\mathbf{2}^{\mathbb{N}} \to X$ into a C-space X is a probe iff it is continuous in the sense of the category of C-spaces.

Lemma
The exponential $\mathbb{N}^{\mathbf{2}^{\mathbb{N}}}$ is a discrete C-space.

Theorem
There is a continuous functional $\mathrm{fan} \colon \mathbb{N}^{\mathbf{2}^{\mathbb{N}}} \to \mathbb{N}$ that calculates (minimal) moduli of uniform continuity.

Modelling uniform continuity

$\mathbb{C}$-spaces provide a model of system T and dependent types:

1. Cartesian closed structure — simply typed λ-calculus.
2. Locally cartesian closed structure — dependent types.
3. Natural numbers object — base type and primitive recursion principle.

Theorem
The uniform continuity axiom

$$\forall (f : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \exists (n : \mathbb{N}).\ \forall (\alpha, \beta : \mathbf{2}^{\mathbb{N}}).\ \alpha =_n \beta \Rightarrow f(\alpha) = f(\beta)$$

is validated by the fan functional.

Computing moduli of uniform continuity

A Gödel's T term $f : (\mathsf{N} \to 2) \to \mathsf{N}$ (or a term in MLTT)

↓ $[\![-]\!]$

A continuous map $[\![f]\!] : \mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ in C-**Space**

↓ Yoneda Lemma

$[\![f]\!]$ is uniformly continuous (as Σ)

↓ $\mathrm{pr}_1, \mathrm{pr}_2$

The least modulus of uniform continuity of f

An Agda program