DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2017

LIE GROUPS

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Problems 01

1. A topological space X is a *Hausdorff* space if any two distinct points $x \neq y \in X$ have disjoint neighbourhoods. Show:

i) For a topological space X the following two statements are equivalent:

- X is a Hausdorff space.
- The diagonal
$$\Delta := \{(x,x) \in X \times X : x \in X\}$$
is a closed subset of the Cartesian product $X \times X$.

ii) A compact subset of a Hausdorff space is closed.

iii) A closed subset of a compact space is compact.

2. Consider a continuous and bijective map $f : X \to Y$ between two Hausdorff spaces. Show:

i) If X is compact then f is a homeomorphism.

ii) One cannot omit the assumption of compactness in part i).

3. Consider a continuous surjective map $p : X \to Y$ between topological spaces. Show the equivalence of the following two properties:

i) The space Y carries the quotient topology with respect to p.

ii) Any map $g : Y \to Z$ with a topological space Z is continuous if the composition $g \circ p$ is continuous.

4. Consider a group G and a topology $\mathscr{T}$ on G. Show the equivalence of the following two statements:

1. $(G, \mathscr{T})$ is a topological group.

2. The map
$$G \times G \to G, (x,y) \mapsto x \cdot y^{-1},$$
is continuous with respect to $\mathscr{T}$.

Discussion: Tuesday, 2.5.2017, 12.15 p.m.

Problems 02

5. Consider a topological group G and a subgroup H.

- Determine a topological space X - canonically attached to G and H - and a group operation
$$\phi : G \times X \to X$$
with isotropy group $G_x = H$ at a distinguished point $x \in X$.

- Show: The operation ϕ is transitive and any two isotropy groups
$$G_{x_1}, G_{x_2}, x_1, x_2 \in X,$$
are conjugate subgroups of G.

- Show: The G-space (G, X) is homogeneous.

6. i) Consider a topological group and a subgroup $H \subset G$. Show the equivalence of the following two properties:

- The subgroup $H \subset G$ is open.

- The quotient G/H is a discrete topological space.

ii) Find a topological group G, carrying a non-discrete topology, with an open subgroup $H \subset G$ of index $[G : H] = 2$.

7. i) *Exotic subgroup*: Consider the topological group $G := \mathbb{R}/\mathbb{Z}$. Find a subgroup
$$H :=< \alpha > \subsetneq G,$$
generated by a single element α, with closure
$$\overline{H} = G.$$

ii) *Exotic quotient group*: Consider the indiscrete topology $\mathscr{T}_{ind}$ on $\mathbb{R}$. Determine an isomorphism of topological groups

$$\overline{f} : \mathbb{R}/\mathbb{Q} \xrightarrow{\simeq} (\mathbb{R}, \mathscr{T}_{ind}).$$

Hint: Extend the element $v_0 = 1 \in \mathbb{R}$ to a Hamel basis $(v_i)_{i \in \mathbb{N} \dot\cup I}$ of the $\mathbb{Q}$-vector space $\mathbb{R}$. Set $v_{-1} := 0$ and consider the $\mathbb{Q}$-linear map

$$f : \mathbb{R} \to \mathbb{R}, v_i \mapsto \begin{cases} v_{i-1} & i \in \mathbb{N} \\ v_i & i \in I \end{cases}$$

8. Consider a topological group G and a subgroup H which is locally closed at the neutral element $e \in H$, i.e. a neighbourhood V of e in G exists such that $V \cap H$ is closed in V.

Show: $H \subset G$ is closed.

Discussion: Tuesday, 9.5.2017, 12.15 p.m.

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Problems 03

9. Consider a connected topological group G and a normal subgroup $H \subset G$, which considered as a subspace carries the discrete topology.
Show: H is contained in the center of G

$$Z(G) := \{g \in G : g \cdot x \cdot g^{-1} = x \text{ for all } x \in G\}.$$

10. For the base field $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ the n-dimensional projective spaces are defined as

$$\mathbb{P}^n(\mathbb{K}) := (\mathbb{K}^{n+1} \setminus \{0\}) / \sim$$

with respect to the equivalence relation

$$x \sim y := \exists \lambda \in \mathbb{K}^* : \lambda \cdot x = y.$$

Show: For any $n \geq 2$ exist embeddings

$$O(n-1, \mathbb{R}) \hookrightarrow SO(n, \mathbb{R}) \text{ and } U(n-1) \hookrightarrow SU(n)$$

of closed subgroups which induce homeomorphisms

$$SO(n, \mathbb{R})/O(n-1, \mathbb{R}) \simeq \mathbb{P}^{n-1}(\mathbb{R}) \text{ and } SU(n)/U(n-1) \simeq \mathbb{P}^{n-1}(\mathbb{C}).$$

11. Consider a locally connected space B, two covering projections

$$p_i : E_i \to B, i = 1, 2,$$

and a continous map $f : E_1 \to E_2$ such that the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & B & \end{array}$$

Show:

i) If f is surjective then f is a covering projection.

ii) If B is connected and locally path-connected, and E_2 is connected, then f is surjective.

12. i) Show: The fundamental group $\pi_1(G,e)$ of a path-connected topological group G is Abelian.

ii) Find a connected topological space X with a non-Abelian fundamental group $\pi_1(X,*)$.

Hint: ad i) Consider on $\pi_1(G,e)$ both the multiplication $*$ from the catenation of homotopy classes and the multiplication $\cdot$ from the topological group G. Prove: On the unit square a continous map $\Phi : I^2 \to I^2$ exists with

- $\Phi(-,0)$ is a parametrization of the diagonal and
- $\Phi(-,1)$ is a suitable parametrization of the path along the bottom edge of I^2 followed by the path along the right edge of I^2.

For two paths α_1, α_2 define $\alpha_{12}(t_1,t_2) := \alpha_1(t_1) \cdot \alpha_2(t_2)$. Then $\alpha_{12} \circ \Phi$ is a homotopy from $\alpha_1 \cdot \alpha_2$ to $\alpha_1 * \alpha_2$. By using the other two edges of I^2 show analogously $\alpha_1 \cdot \alpha_2 \simeq \alpha_2 * \alpha_1$.

Discussion: Tuesday, 16.5.2017, 12.15 p.m.

Problems 04

13. Consider the defining projection

$$p : S^1 \to \mathbb{P}^1(\mathbb{R}).$$

i) Determine the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{R}), *)$.

Hint: Consider the restriction of p to the closed upper hemicircle $D^1 \subset S^1$.

ii) How does the induced map

$$\pi_1(p) : \pi_1(S^1, *) \to \pi_1(\mathbb{P}^1(\mathbb{R}), *)$$

relate to the exact homotopy sequence of the continuous fibre bundle p?

14. Consider the continuous left $SO(3,\mathbb{R})$-operation

$$\phi : SO(3,\mathbb{R}) \times S^2 \to S^2, (A,z) \mapsto A \cdot z,$$

and the orbit map of $e_1 \in S^2$, the first canonical basis vector of $\mathbb{R}^3$,

$$\psi := \phi_{e_1} : SO(3,\mathbb{R}) \to S^2.$$

Prove:

i) A continuous section against ψ, i.e a continuous map

$$\sigma : S^2 \to SO(3,\mathbb{R})$$

with $\psi \circ \sigma = id_{S^2}$, induces a homeomorphism

$$SO(3,\mathbb{R}) \simeq S^2 \times SO(2,\mathbb{R}).$$

ii) The existence of a section σ in part i) induces an isomorphism of groups

$$\pi_1(SO(3,\mathbb{R}), *) \simeq \pi_1(SO(2,\mathbb{R}), *).$$

15. It is impossible to "comb a hedgehog" (German: Satz vom Igel). More precisely: Consider a continuous map

$$f : S^2 \to \mathbb{R}^3$$

such that the scalar product in $\mathbb{R}^3$ satisfies

$$< x, f(x) >= 0$$

for all $x \in S^2$. Then at least one point $x_0 \in S^2$ exists with

$$f(x_0) = 0.$$

For an indirect proof of the claim assume $f(x) \neq 0$ for all $x \in S^2$ and show:

i) The orbit map ψ of the point $e_1 \in S^2$ belonging to the canonical left $SO(3,\mathbb{R})$-operation on S^2 has a section.

ii) The result of part i) implies a contradiction.

16. Show:

i) Any continous map $f : S^2 \to S^2$ has a point $x_0 \in S^2$ with

- $f(x_0) = x_0$ (fixed point)
- or $f(x_0) = -x_0$ (antipodal point)

Hint: Define a continous map $S^2 \to \mathbb{R}^3$ with the properties from Problem 15.

ii) On the topological space S^2 no topological group structure exists.

Hint: For an indirect proof consider a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \neq e$ and $\lim_{n\to\infty} x_n = e$. Consider the maps

$$h_n : S^2 \to S^2, y \mapsto x_n \cdot y.$$

Discussion: Tuesday, 23.5.2017, 12.15 p.m.

Problems 05

17. Consider a surjective morphism

$$f : G \to H$$

of topological groups. Assume G is σ-compact and H is locally compact.

Show: f is an open map.

Hint: Consider a suitable group operation.

18. Consider an analytic $\mathbb{K}$-manifold. For any open subset $U \subset X$ denote by $\mathscr{O}(U)$ the ring of $\mathbb{K}$-analytic functions defined on U.

For $U \subset X$ open a *derivation* on U is a $\mathbb{K}$-linear map

$$D : \mathscr{O}(U) \to \mathscr{O}(U)$$

satisfying the product rule

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$$

for all $f, g \in \mathscr{O}(U)$. An *analytic vector field* on X is a family

$$D = (D_U)_{U \subset X \ open}$$

of derivations

$$D_U : \mathscr{O}(U) \to \mathscr{O}(U), U \subset X \ open,$$

such that for any pair of open sets $V \subset U \subset X$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{O}(U) & \xrightarrow{D_U} & \mathscr{O}(U) \\ \downarrow \rho^U_V & & \downarrow \rho^U_V \\ \mathscr{O}(V) & \xrightarrow{D_V} & \mathscr{O}(V) \end{array}$$

Here

$$\rho_V^U : \mathscr{O}(U) \to \mathscr{O}(V)$$

denotes the restriction of analytic functions.

Show: An analytic vector field D on X defines for each point $x \in X$ a tangent vector

$$D(x) \in T_x X$$

according to the diagram

$$\begin{array}{ccc}
\mathscr{O}(U) & \xrightarrow{D_U} & \mathscr{O}(U) \\
\downarrow & & \downarrow \\
\mathscr{O}_{X,x} & \xrightarrow{D_x} & \mathscr{O}_{X,x} \\
& \overset{D(x)}{\searrow} & \downarrow \varepsilon_x \\
& & \mathscr{O}_{X,x}/\mathfrak{m}_x
\end{array}$$

valid for any open neighbourhood U of x. Here

$$\mathscr{O}(U) \to \mathscr{O}_{X,x}, f \mapsto [f],$$

denotes the canonical map to the local ring. And

$$\varepsilon_x : \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathfrak{m}_x \simeq \mathbb{K}, [f] \mapsto f(x),$$

denotes the evaluation at the point $x \in X$.

Hint. Verify the following steps: i) The map D_x does not depend on the choice of the neighbourhood U. ii) The map D_x satisfies the product rule. iii) The map $D(x)$ satisfies the product rule.

Note. The following problems 19 and 20 illustrate the broad scope of the concept of local rings. The problems do not relate to Lie group theory.

19. *Spectrum*: Consider the set

$$P := \{\mathfrak{p} : \mathfrak{p} \subset \mathbb{Z} \text{ prime ideal}\}$$

of all prime ideals of the ring $\mathbb{Z}$ and the subsets

$$V(n) := \{\mathfrak{p} \in P : (n) \subset \mathfrak{p}\}, n \in \mathbb{Z}.$$

i) Show: The elements $V(n), n \in \mathbb{Z}$, are the closed sets of a topology $\mathscr{T}$ on P (*Zariski topology*).

The topological space

$$Spec\ \mathbb{Z} := (P, \mathscr{T})$$

is named the *spectrum* of the ring $\mathbb{Z}$.

ii) Which points $\mathfrak{p} \in Spec\ \mathbb{Z}$ determine a closed singleton $\{\mathfrak{p}\} \subset Spec\ \mathbb{Z}$?

iii) Determine the closure $\overline{\{\eta\}} \subset Spec\ \mathbb{Z}$ of the singleton determined by the *generic* point $\eta := (0) \in Spec\ \mathbb{Z}$.

20. *Local ring*: Set

$$X := Spec(\mathbb{Z}).$$

i) For each $\mathfrak{p} \in X$ define

$$\mathscr{O}_{X,\mathfrak{p}} := \mathbb{Z}_{\mathfrak{p}} := \{q \in \mathbb{Q} : q = \frac{n}{m} \textit{ with } n, m \in \mathbb{Z}, m \notin \mathfrak{p}\}.$$

Determine the unique maximal ideal $\mathfrak{m}_{\mathfrak{p}} \subset \mathbb{Z}_{\mathfrak{p}}$ and the *residue field*

$$k(\mathfrak{p}) := \mathbb{Z}_{\mathfrak{p}} / \mathfrak{m}_{\mathfrak{p}}$$

of the local ring $\mathscr{O}_{X,\mathfrak{p}}$.

ii) Set

$$K := \dot{\bigcup_{\mathfrak{p} \in Spec(\mathbb{Z})}} k(\mathfrak{p}).$$

Show: Each integer $n \in \mathbb{Z}$ defines a function, equally named,

$$n : X \to K, \mathfrak{p} \mapsto n(\mathfrak{p}) := [n] \in k(\mathfrak{p}).$$

iii) Show: For each $\mathfrak{p} \in X$ holds

$$\mathscr{O}_{X,\mathfrak{p}} = \{q \in \mathbb{Q} : q = \frac{n}{m} \textit{ with } m(\mathfrak{p}) \neq 0\}$$

and

$$\mathfrak{m}_{\mathfrak{p}} = \{q \in \mathscr{O}_{\mathbb{Z},\mathfrak{p}} : q = \frac{n}{m} \textit{ with } n(\mathfrak{p}) = 0\}$$

the set of non-units of $\mathscr{O}_{X,\mathfrak{p}}$.

Discussion: Tuesday, 30.5.2017, 12.15 p.m.

Problems 06

21. Show: Any Lie group is a Hausdorff space.

22. Show: For an analytic map
$$f : X \to Y$$
between two analytic manifolds the graph
$$\Gamma(f) := \{(x, f(x)) \in X \times Y : x \in X\} \subset X \times Y$$
is a submanifold of the product.

23. (*Partial holomorphy and Cauchy integral*) Consider a continuous function
$$f : D \to \mathbb{C}$$
on an open set $D \subset \mathbb{K}^n$, which is holomorphic in each variable separately, i.e. for any index $j = 1, .., n$ and for any arbitrary but fixed point $a = (a_1, ..., a_n) \in D$ the restriction
$$f|D_j(a) : D_j(a) \to \mathbb{C}$$
is holomorphic on the 1-dimensional open slice
$$D_j(a) := \{z \in \mathbb{C} : (a_1, ..., a_{j-1}, z, a_{j+1}, ..., a_n) \in D\}.$$
Prove:

i) The function f satisfies the Cauchy integral formula, i.e. for any $w = (w_1, ..., w_n) \in D$ a polydisc around w
$$\Delta(w; r) := \{(\zeta_1, ..., \zeta_n) \in \mathbb{C}^n : |\zeta_j - w_j| < r_j \ for \ j = 1, ..., n\}$$
with a suitable polyradius
$$r = (r_1, ..., r_n), r_j > 0 \ for \ j = 1, ..., n,$$
exists, such that for all $z \in \Delta(w; r)$

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \cdot \int_{\partial\Delta(w;r)} \frac{f(\zeta)\, d\zeta_1 \cdot \ldots \cdot d\zeta_n}{(\zeta_1 - z_1) \cdot \ldots \cdot (\zeta_n - z_n)}$$

ii) f is holomorphic.

24. (*Maximum principle*) i) Consider a holomorphic function

$$f : G \to \mathbb{C}$$

defined on a domain $G \subset \mathbb{K}^n$. Assume that $|f|$ attends a local maximum, i.e. a point $z_0 \in G$ and a neighbourhood $U \subset G$ of z_0 exist such that

$$|f(z)| \leq |f(z_0)|$$

for all $z \in U$.

Show: The function f is constant, i.e. $f(z) = f(z_0)$ for all $z \in G$.

Hint: First use the Cauchy formula from Problem 23 to generalize the mean value formula in the case of one variable

$$|f(z_0)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} |f(z_0 + r \cdot e^{2\pi i \phi})| d\phi.$$

Secondly, prove that f is constant in a neighbourhoof of z_0. Eventually apply - without proof - the identity theorem for holomorphic functions: Two holomorphic functions, defined on a common domain, are equal if they coincide on an non-empty open subset.

ii) Show: Any holomorphic function defined on a connected compact complex manifold is constant.

Discussion: Tuesday, 13.6.2017, 12.15 p.m.

Problems 07

25. Consider a Lie group H and an analytic H-principal bundle $p : X \to Y$. For two bundle charts of p

$$\phi_i : p^{-1}(U_i) \to U_i \times H \text{ and } \phi_j : p^{-1}(U_j) \to U_j \times H$$

assume the existence of a map

$$h_{ji} : U_i \cap U_j \to H$$

satisfying

$$\phi_j \circ (\phi_i | (U_i \cap U_j) \times H)^{-1} : (U_i \cap U_j) \times H \to (U_i \cap U_j) \times H, (y,h) \mapsto (y, h_{ji}(y) \cdot h).$$

Show: The map h_{ji} is analytic.

26. Consider a finite-dimensional associative $\mathbb{K}$-algebra A with product

$$m : A \times A \to A, (x,y) \mapsto x \cdot y.$$

Show: The automorphism group of A

$$Aut(A) := \{\phi : A \to A | \phi \text{ a } \mathbb{K}-\text{linear automorphism with } \phi(x \cdot y) = \phi(x) \cdot \phi(y)\}$$

with respect to the multiplication

$$Aut(A) \times Aut(A) \to Aut(A), (\phi_1, \phi_2) \mapsto \phi_1 \circ \phi_2.$$

is a $\mathbb{K}$-matrix group.

Hint: Consider the Lie group $G := GL(n, \mathbb{K})$ of automorphism of the $\mathbb{K}$-vector space underlying A. On the $\mathbb{K}$-vector space $X := Bil(A \times A, A)$ of $\mathbb{K}$-bilinear maps define a suitable G-left operation

$$\phi : G \times X \to X$$

with $G_m = Aut(A)$.

27. Consider a Lie group G, a group H, and a surjective group homomorphism

$$\pi : G \to H.$$

Show:

i) The following two properties are equivalent.

- On H exists a Lie group structure such that π is a morphism of Lie groups.

- The subgroup $ker\ \pi \subset G$ is a Lie subgroup.

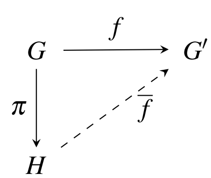
Assume that the properties of part i) are satisfied. Show:

ii) The Lie group structure of H is uniquely determined.

iii) Any morphism $f : G \to G'$ of Lie groups with $ker\ \pi \subset ker\ f$ induces a unique morphism of Lie groups

$$\overline{f} : H \to G'$$

such that the following diagram commutes



28. Consider the analytic manifold $X := M(2 \times 2, \mathbb{C})$, the Lie group $H := GL(2, \mathbb{C})$, and the analytic G-right operation

$$\phi : X \times G \to X, (A, B) \mapsto B^{-1} \cdot A \cdot B.$$

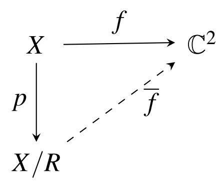
Show: i) The equivalence relation $R \subset X \times X$ induced by ϕ is not closed.

ii) No analytic structure exists on X/R such that the canonical projection $p : X \to X/R$ is analytic.

iii) The analytic map

$$f : X \to \mathbb{C}^2,\ A \mapsto (trace\ A, det\ A),$$

factorizes via p, i.e. a unique map $\overline{f} : X/H \to \mathbb{C}^2$ exists such that the following diagram commutes



Hint for part i) and ii): Consider matrices $A_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{C}$.

Discussion: Tuesday, 20.6.2017, 12.15 p.m.

Problems 08

29. Show for all $n \geq 1$:

i) The group $SU(n)$ is simply connected.

ii) The group $SL(n,\mathbb{C})$ is simply connected.

Hint: Use part i).

30. Show: $\pi_n(S^1) = 0$ for all $n \geq 2$.

Hint: Use the universal covering projection of S^1.

31. Consider the continuous map

$$p : S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\} \to \mathbb{P}^1(\mathbb{C}), p(z_0, z_1) := (z_0 : z_1).$$

Here the notation

$$(z_0 : z_1) := [(z_0, z_1)] \text{ (\textit{Homogeneous coordinates})}$$

denotes the equivalence class from $\mathbb{P}^1(\mathbb{C}) := (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$.

Show: The map p is a continuous principal bundle $S^1 \hookrightarrow S^3 \xrightarrow{p} \mathbb{P}^1(\mathbb{C})$ (*Hopf bundle*).

32. Show:

i) $\pi_2(S^2) \simeq \mathbb{Z}$.

Hint: Use $\mathbb{P}^1(\mathbb{C}) \simeq S^2$.

ii) The Hopf bundle is not a product bundle, in particular there is no continuous section $s : \mathbb{P}^1(\mathbb{C}) \to S^3$ against p.

Discussion: Tuesday, 27.6.2017, 12.15 p.m.

Problems 09

33. i) Show: For $n \in \mathbb{N}^*$ the element $F := (F_1, ..., F_n)^\top \in \mathfrak{m} < X, Y >^{\oplus n}$ with variables $X = (X_1, ..., X_n), Y = (Y_1, ..., Y_n)$, defined by

$$F_j(X,Y) := X_j + Y_j,\ j = 1, ..., n,$$

is a formal group structure on $\mathfrak{m} < X >^{\oplus n}$.

ii) Show: The element

$$F(X,Y) := X + Y + XY \in \mathfrak{m} < X, Y >$$

is a formal group structure on $\mathfrak{m} < X >$.

iii) Compute the inverse of the element

$$X \in \mathfrak{m} < X >^{\oplus n}$$

in case i), and the inverse of

$$X \in \mathfrak{m} < X >$$

in case ii).

iv) Compute

$$[-,-]_F : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$$

in case i), and

$$[-,-]_F : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$$

in case ii).

34. For an n-dimensional $\mathbb{K}$-Lie group G denote by $F \in \mathbb{K} < X, Y >^{\oplus n}$ its formal group law with respect to a chart around $e \in G$, and by $[-,-]_F$ the corresponding Lie bracket.

i) Find a Lie group G with the formal group structure F from Exercise 33, part i).

ii) Find a connected Lie group G_1 with the formal group structure F from Exercise 33, part ii).

iii) Find a connected Lie group G_2, which is not homeormorphic to the Lie group G_1 from part ii), but has the same Lie bracket $[-,-]_F$. How are G_1 and G_2 related?

35. Consider a formal group structure F on $\mathfrak{m} < X >^{\oplus n}$ and denote by

$$[-,-]_F : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$$

the derived $\mathbb{K}$-bilinear map.

i) Show: The *Hall identity* implies

$$0 = [X,[Y,Z]_F]_F + [Y,[Z,X]_F]_F + [Z,[X,Y]_F]_F + O(4).$$

ii) Prove the *Jacobi identity*: $0 = [X,[Y,Z]_F]_F + [Y,[Z,X]_F]_F + [Z,[X,Y]_F]_F$.

36. Show:

i) For any cyclic group $H := \mathbb{Z}/n\mathbb{Z},\ n \in \mathbb{N}^*$, exists a Lie group G with $\pi_1(G,e) = H$.

Hint: Consider a suitable quotient by the group of n-th roots of unity.

ii) For any finitely-generated Abelian group H exists a Lie group G with $\pi_1(G,e) = H$.

Discussion: Tuesday, 4.7.2017, 12.15 p.m.

Problems 10

37. *Vector fields on open subsets of* $\mathbb{K}^n$: Consider an open subset $X \subset \mathbb{K}^n$. Show:

i) For any vector field $A = (A_U)_{U \subset X \ open}$ on X the derivation A_X has the form

$$A_X = \sum_{j=1}^{n} A_X(z_j) \cdot \frac{\partial}{\partial z_j}$$

Here $A_X(z_j)$ denotes the result of applying the derivation A_X to the analytic function $z_j \in \mathscr{O}_X(X)$.

Hint: Use the fact that an analytic vector field A is determined by all its tangent vectors $A(x) \in T_x X, x \in X$.

ii) The map

$$\Theta(X) \to \mathscr{O}_X(X)^{\oplus n}, A \mapsto (A_X(z_1), ..., A_X(z_n))$$

is an isomorphism of $\mathbb{K}$-vector spaces. In particular, the analytic vector fields on X correspond bijectively to the derivations

$$\sum_{j=1}^{n} a_j \cdot \frac{\partial}{\partial z_j},\ a_j \in \mathscr{O}_X(X), j = 1, ..., n.$$

38. *Vector fields on manifolds*: Consider the complex projective space $\mathbb{P}^1$. Show:

i) $\mathbb{P}^1$ is a compact analytic manifold.

ii) The complex vector space of analytic vector fields $\Theta(\mathbb{P}^1)$ is isomorphic to the vector space $Pol^2 \subset \mathbb{C}[Z_0, Z_1]$ of homogenous polynomials of degree 2 in two variables:

$$Pol^2 \simeq \Theta(\mathbb{P}^1).$$

Hint: Consider the covering $\mathscr{U} = \{U_0, U_1\}$ of $\mathbb{P}^1$ by the two open sets

$$U_i := \{(z_0 : z_1) \in \mathbb{P}^1 : z_i \neq 0\}, i = 0, 1.$$

For $i = 0,1$ choose two charts $\phi_i : U_i \to \mathbb{C}$ with $\phi_1 = 1/\phi_0$ on $U_0 \cap U_1$. According to Exercise 37 an analytic vector field $\theta = (\theta_U)_{U \subset \mathbb{P}^1 \ open} \in \Theta(\mathbb{P}^1)$ is determined by a pair (θ_0, θ_1) with two derivations

$$\theta_i = \theta_{U_i} = f_i \cdot \frac{d}{d\phi_i},\ i = 0,1.$$

How do the analytic functions f_0 *and* f_1 relate to each other on $U_0 \cap U_1$?
Conversely, for a homogeneous polonomial $P(Z_0, Z_1) \in Pol^2$ set

$$f_0(z_0 : z_1) := P(1, z_1/z_0) \text{ and } f_1(z_0 : z_1) := -P(z_0/z_1, 1).$$

iii) Determine the dimension of the vector space $\Theta(\mathbb{P}^1)$.

39. The Lemma on *Low order approximation* from the lecture states:

A formal group structure

$$F \in \mathfrak{m} < X, Y >^{\oplus n}$$

satisfies the following approximations:

1. If $U, V, U', V' \in \mathfrak{m} < X >^{\oplus n}$ and $U' = U + O(2), V' = V + O(2)$ then

$$B(U', V') - B(V', U') = B(U, V) - B(V, U) + O(3)$$

2. If $U, V \in \mathfrak{m} < X >^{\oplus n}$ then

$$F(U, V) - F(V, U) = B(U, V) - B(V, U) + O(3)$$

3. If $U, U' \in \mathfrak{m} < X >^{\oplus n}$ and $U' = U + O(2)$ then

$$U'(F(X, Y)) - U'(F(Y, X)) = U(F(X, Y)) - U(F(Y, X)) + O(3).$$

Prove part 1) and part 2).

40. The Lemma on the *Independence of the Lie algebra structure* from the lecture states:

Consider two Lie groups G and G', a morphism $f : G \to G'$ of Lie groups, and two charts $\phi : U \to \mathbb{K}^n$ and $\phi' : U' \to \mathbb{K}^{n'}$ around the neutral elements with $f^{-1}(U') \subset U$. Denote by

$$F \in \mathfrak{m} < X, Y >^{\oplus n}, F' \in \mathfrak{m} < X', Y' >^{\oplus n'}$$

the induced formal group structures and by

$$[-,-]_{(F)} \text{ and } [-,-]_{(F')}$$

the induced Lie brackets on the tangent spaces respectively T_eG and $T_{e'}G'$. Then

$$T_ef : (T_eG, [-,-]_{(F)}) \to (T_{e'}G', [-,-]_{(F')})$$

is a morphism of Lie algebras.

Justify each of the following eight steps from the first part of the proof of the Lemma: If g_1 denotes the linear part of the analytic map

$$g := \phi' \circ f \circ \phi^{-1} \in \mathfrak{m} < X >^{\oplus n'}$$

induced by the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \phi \downarrow \simeq & & \phi' \downarrow \simeq \\ \phi(U) & \overset{g}{\dashrightarrow} & \phi'(U') \end{array}$$

then

$$\begin{array}{rl}
[g_1(X), g_1(Y)]_{F'} := & \text{Step 1} \\
B'(g_1(X), g_1(Y)) - B'(g_1(Y), g_1(X)) = & \text{Step 2} \\
B'(g(X), g(Y)) - B'(g(Y), g(X)) + O(3) = & \text{Step 3} \\
F'(g(X), g(Y)) - F'(g(Y), g(X)) + O(3) = & \text{Step 4} \\
g(F(X,Y) - g(F(Y,X)) + O(3) = & \text{Step 5} \\
g_1(F(X,Y)) - g_1(F(Y,X)) + O(3) = & \text{Step 6} \\
g_1(F(X,Y) - F(Y,X)) + O(3) = & \text{Step 7} \\
g_1(B(X,Y) - B(Y,X)) + O(3) =: & \text{Step 8} \\
g_1([X,Y]_F) + O(3) &
\end{array}$$

Discussion: Tuesday, 11.7.2017, 12.15 p.m.

Problems 11

41. Consider a Lie group G with multiplication

$$m : G \times G \to G, m(x,y) := x \cdot y$$

and inverse

$$\sigma : G \to G, x \mapsto \sigma(x) := x^{-1}.$$

Show for the tangent maps:

i)

$$T_{(e,e)}m : Lie\ G \times Lie\ G \to Lie\ G, (A,B) \mapsto A+B$$

ii)

$$T_e\sigma : Lie\ G \to Lie\ G, A \mapsto -A.$$

Hint: Consider a formal group law of G.

42. Prove: The functor Lie is faithful on the subcategory of connected Lie groups, i.e. for any two morphisms

$$f_1, f_2 : G \to H$$

of connected Lie groups the equation

$$Lie\ f_1 = Lie\ f_2$$

implies $f_1 = f_2$.

43. *Exponential of matrices*. Consider the Lie group

$$G := GL(n, \mathbb{K})$$

and the Lie algebra

$$gl(n, \mathbb{K}) := (M(n \times n, \mathbb{K}), [-,-]).$$

Show:

i) $gl(n,\mathbb{K}) = Lie\ GL(n,\mathbb{K})$.

Hint: Consider a suitable formal group law.

ii) The exponential map

$$exp : gl(n,\mathbb{K}) \to GL(n,\mathbb{K})$$

equals the e-function

$$e : gl(n,\mathbb{K}) \to GL(n,\mathbb{K}), A \mapsto e^A.$$

44. Show: A Lie group G has no small subgroups.

Hint: Choose neighbourhoods V of $0 \in Lie\ G$ and U of $e \in G$ such that $exp|V : V \to U$ is an analytic isomorphism. Consider a subgroup H with $H \subset exp(V/2)$. An arbitrary but fixed $h \in H$ has the form $h = exp\ v_1$, and $h^2 = exp(v_2)$. How do v_1 and v_2 relate? Iterate the argument.

Discussion: Tuesday, 18.7.2017, 12.15 p.m.

DEPARTMENT OF MATHEMATICS
LMU MÜNCHEN
SUMMER TERM 2017

LIE GROUPS

Joachim Wehler

Problems 12

45. Consider a morphism

$$f : G \to H$$

of Lie groups and a Lie-subgroup $H' \subset H$.

Show: The pre-image

$$G' := f^{-1}(H')$$

is a Lie subgroup of G with Lie algebra

$$Lie\ G' = (Lie\ f)^{-1}(Lie\ H').$$

Hint: You may consider a suitable group operation $G \times (H/H') \to (H/H')$ with isotropy group $G_{eH'} = G'$.

46. Consider a Lie group G. Show:

i) The following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{Ad} & GL(Lie\ G) \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle e} \\ Lie\ G & \xrightarrow{ad} & gl(Lie\ G) \end{array},$$

i.e. for all $x \in Lie\ G$

$$Ad(exp\ x) = e^{ad\ x} \in GL(Lie\ G).$$

ii) For all $x, y \in Lie\ G$:

$$exp\ x \cdot exp\ y \cdot (exp\ x)^{-1} = exp(e^{ad\ x}(y)) \in G.$$

Hint: Consider the commutative diagram which defines $Ad\ g,\ g \in G$, as the tangent map of ϕ_g. Apply part i).

47. For fixed $\alpha \in \mathbb{R}$ consider the 1-parameter group

$$j : \mathbb{R} \to T^2, t \mapsto [(t, \alpha \cdot t)],$$

of the real torus

$$T^2 := \mathbb{R}^2/\mathbb{Z}^2.$$

The image $H := j(\mathbb{R})$ is a line in T^2 with slope α. Show for irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$:

i) The map j is an injective immersion.

ii) The subgroup $H \subset T^2$ is not a Lie subgroup.

Hint: For an indirect proof you may represent H as a homogeneous space with respect to the group operation

$$\phi : \mathbb{R} \times H \to H,\ (t,h) \mapsto j(t) + h.$$

48. The center of a Lie group G is defined as

$$Z(G) := \{g \in G : g \cdot h = h \cdot g \ for\ all\ h \in G\}.$$

The center of a Lie algebra L is defined as

$$Z(L) := \{x \in L : [x,y] = 0 \ for\ all\ y \in L\}.$$

Assume G connected and show:

i) The center of G is the kernel of the adjoint representation:

$$Z(G) = ker\ [Ad : G \to GL(Lie\ G)].$$

In particular $Z(G) \subset G$ is a Lie subgroup.

ii) The Lie algebra of the center of G is the center of the Lie algebra of G, i.e.

$$Lie\ (Z(G)) = Z(Lie\ G).$$

iii) The Lie group G is Abelian if and only if its Lie algebra *Lie G* is Abelian.

Discussion: Tuesday, 25.7.2017, 12.15 p.m.

Selected Solutions

1 i) Assume X to be Hausdorff. The diagonal $\Delta \subset X \times X$ is closed if its complement is open. A point

$$(x,y) \in (X \times X) \setminus \Delta$$

satisfies $x \neq y$. According to the Hausdorff property of X disjoint neighbourhoods U of x and V of y exist. In particular

$$U \times V$$

is a neighbourhood of (x,y) in $X \times X$ with

$$U \times V \subset (X \times X) \setminus \Delta.$$

Thus the complement of Δ is open and Δ is closed.

Concerning the opposite direction: Assume Δ to be closed. Then its complement is open: Any point

$$(x,y) \in (X \times X) \setminus \Delta,$$

i.e. satisfying $x \neq y$, has a neighbourhood, without restriction a product neighbourhood $U \times V$, disjoint from Δ. Then U is a neighbourhood of x, V is a neighbourhood of y and

$$U \cap V = \emptyset.$$

Therefore X is a Hausdorff space.

ii) Consider a Hausdorff space X and a compact subset $K \subset X$. We claim: The complement $X \setminus K$ is open.

Consider an arbitrary but fixed point $x \in X \setminus K$. For any point $y \in K$ a pair of disjoint neighbourhoods $U(y)$ of y and $V(y)$ of x exist. The family

$$U(y)_{y \in K}$$

is an open covering of K. Compactness of K implies the existence of a finite subcovering

$$K \subset \bigcup_{\nu=1}^{n} U(y_\nu).$$

The finite intersection

$$V := \bigcap_{\nu=1}^{n} V(y_\nu)$$

is a neighbourhood of x with $K \cap V = \emptyset$. Therefore $X \setminus K$ is open and K is closed.

iii) Consider a compact Hausdorff space X and a closed subset $A \subset X$. Consider an open covering $\mathscr{U}$ of A. Define the open set $U := X \setminus A$ and consider the open covering

$$\mathscr{U} \cup \{U\}$$

of X. Compactness of X provides the existence of a finite subcovering. It contains a finite subcovering $\mathscr{U}' \subset \mathscr{U}$ of A. Therefore A is compact.

8 . We choose a symmetric neighbourhood $V = V^{-1}$ of e in G with $V \cap H$ closed in V. We consider an arbitrary but fixed point $x \in \overline{H}$ and claim $x \in H$:
Because $xV \cap H \neq \emptyset$ a point

$$y \in xV \cap H$$

exists. Then

$$x \in yV^{-1} = yV.$$

By translation, $H \subset G$ is locally closed at any point of H, in particular at the point $y \in H$. In particular, yV is a neighbourhood of y with

$$y(V \cap H) = yV \cap H = \overline{yV \cap H}^{yV}$$

the closure taken with respect to yV.

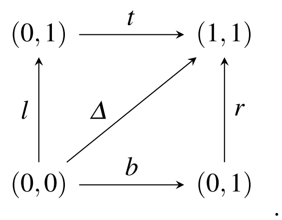
We obtain

$$x \in yV \cap \overline{H} \subset (\overline{yV \cap H}) \cap yV = \overline{yV \cap H}^{yV} = yV \cap H,$$

in particular $x \in H$.

11 . See the book “Edwin Spanier: Algebraic Topology”: Chap. 2, Section 5, Lemma 1 and subsequent reference.

12 . i) (Due to Robert Bruner): Consider the unit square $I \times I$



Choose a homotopy which deformes a parametrization of the diagonal Δ to a parametrization of a path consisting of the bottom edge b followed by the right edge r, fixing the endpoints of Δ:

$$\Phi = (\Phi_1, \Phi_2) : I \times I \to I \times I,$$

i.e.

$$\Phi(t,0) = \Delta(t,t) := (t,t) \in I \times I$$

$$\Phi(t,1) = \begin{cases} (2t,0) & \text{if } t \leq 1/2 \\ (1,2t-1) & \text{if } t \geq 1/2 \end{cases}$$

$$\Phi(0,\tau) = (0,0), \Phi(1,\tau) = (1,1).$$

Define the continuous map

$$\alpha_{12} : I \times I \to G, (t_1,t_2) \mapsto \alpha_1(t_1) \cdot \alpha_2(t_2).$$

Then the continuous map

$$\alpha_{12} \circ \Phi : I \times I \to G$$

satisfies

- $$(\alpha_{12} \circ \Phi)(t,0) = \alpha_{12}(t,t) = \alpha_1(t) \cdot \alpha_2(t) = (\alpha_1 \cdot \alpha_2)(t)$$

- $$(\alpha_{12} \circ \Phi)(t,1) = \alpha_{12}(\Phi_1(t,1), \Phi_2(t,1)) = \alpha_1(\Phi_1(t,1)) \cdot \alpha_2(\Phi_2(t,1)) =$$
$$\left\{ \begin{array}{ll} \alpha_1(2t) \cdot \alpha_2(0) & \text{if } t \leq 1/2 \\ \alpha_1(1) \cdot \alpha_2(2t-1) & \text{if } t \geq 1/2 \end{array} \right\} = (\alpha_1 * \alpha_2)(t)$$

- $$(\alpha_{12} \circ \Phi)(0,\tau) = \alpha_{12}(\Phi(0,\tau)) = \alpha_{12}(0,0) = e$$

- $$(\alpha_{12} \circ \Phi)(1,\tau) = \alpha_{12}(\Phi(1,\tau)) = \alpha_{12}(1,1) = e$$

As a consequence

$$\alpha_1 \cdot \alpha_2 \simeq \alpha_1 * \alpha_2.$$

Choosing in the above argument a homotopy Φ' which deformes a parametrization of the diagonal Δ to a parametrization of the left edge l followed by the top edge t, fixing the endpoints of Δ, proves

$$\alpha_1 \cdot \alpha_2 \simeq \alpha_2 * \alpha_1.$$

Because α_1, α_2 are arbitrary we obtain

$$\alpha_1 \cdot \alpha_2 \simeq \alpha_1 * \alpha_2 \simeq \alpha_2 \cdot \alpha_1$$

which proves that $\pi_1(G,e)$ is Abelian, q.e.d.

16 . i) Define the continuous map

$$h : S^2 \to S^2, h(x) := x \times f(x) \ (vector\ product).$$

According to Problem 15) a point $x_0 \in S^2$ exists with $h(x_0) = 0$. As a consequence

$$x_0 \| f(x_0),$$

i.e. either $f(x_0) = x_0$ or $f(x_0) = -x_0$.

ii) Assume a topological group structure on S^2 and denote by $e \in S^2$ the neutral element. Choose a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in S^2, x_n \neq e$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} x_n = e.$$

For each $n \in \mathbb{N}$ define the continous map

$$h_n : S^2 \to S^2, y \mapsto x_n \cdot y.$$

If an index $n \in \mathbb{N}$ and a point y_n exist with

$$h_n(y_n) = y_n,\ i.e.\ x_n \cdot y_n = y_n$$

then $x_n = e$, which has been excluded. Therefore part i) implies for each $n \in \mathbb{N}$

$$y_n \cdot x_n = -y_n.$$

Compacteness of S^2 allows to assume the existence of

$$y_0 := \lim_{n \to \infty} y_n.$$

Then on one hand

$$\lim_{n \to \infty} x_n \cdot y_n = e \cdot y_0 = y_0,$$

and on the other hand

$$\lim_{n\to\infty} x_n \cdot y_n = -y_n = -y_0,$$

a contradiction.

17 . Consider the group operation

$$\phi : G \times H \to H, (g,h) \mapsto f(g) \cdot h.$$

The group operation is transitive with

$$ker\ f = G_e$$

the isotropy group of $e \in H$. The induced map

$$\overline{\phi_e} : G/ker\ f \to H$$

is continuous, bijective and open due to the assumptions. Due to the assumption it is a homeomorphism, in particular an open map. If

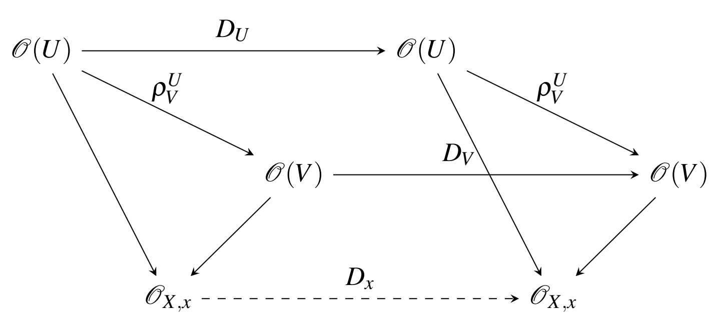
$$\pi : G \to G/ker\ f$$

denotes the canonical projection then

$$f = \overline{\phi_e} \circ \pi : G \to H$$

is the composition of open maps, therefore open itself.

18 . i) Independence of D_x results from the following commutative diagram:



This diagram has to be applied with $V \subset (U_1 \cap U_2)$ in the case of two neighbourhoods U_1 and U_2 of the point $x \in X$.

ii) By definition $D_x([f]) \in \mathscr{O}_{X,x} := [D(f)]$. The product rule

$$D_U(f \cdot g) = D_U(f) \cdot g + f \cdot D_U(g) \in \mathscr{O}(U)$$

implies

$$D_x([f]\cdot[g]) = D_x([f])\cdot[g] + [f]\cdot D_x([g]) \in \mathscr{O}_{X,x}$$

iii) The evaluation map

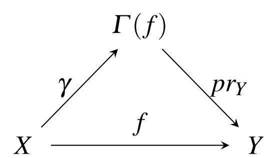
$$D(x)([f]) = \varepsilon(D(x)([f])) = D(x)([f]) \in \mathbb{K}$$

applied to the product rule from part ii) gives

$$D(x)([f]\cdot[g]) = \varepsilon_x((D_x)([f]\cdot[g])) = \varepsilon_x(D_x([f])\cdot[g] + [f]\cdot D_x([g])) =$$

$$= \varepsilon_x(D_x([f]))\cdot\varepsilon_x([g]) + \varepsilon_x([f])\cdot\varepsilon_x(D_x([g])) =$$
$$= D(x)([f])\cdot g(x) + f(x)\cdot D(x)([g]) \in \mathbb{K}.$$

22 . Consider the commutative diagram



with the injective map

$$\gamma : X \to \Gamma(f), x \mapsto (x, f(x)).$$

Equip $\Gamma(f) \subset X \times Y$ with the subspace topology. Then γ is a homeomorphism with inverse map

$$pr_X|\Gamma(f) : \Gamma(f) \to X.$$

We transfer the analytic structure from X to $\Gamma(f)$ via j and obtain an analytic manifold $\Gamma(f)$. It remains to show that the composition

$$j := [X \xrightarrow{\gamma} \Gamma(f) \hookrightarrow X \times Y]$$

is an immersion. The tangent map

$$T_x j = \begin{pmatrix} \mathbb{1} \\ T_f \end{pmatrix}$$

has *rank* $T_x j = dim_x X$. Therefore the tangent map is injective.

24 . i) For a holomorphic function g of one variable the Cauchy formula in polar coordinates reads

$$g(z) = \frac{1}{2\pi i}\cdot\int_{|\zeta - w| = r} \frac{g(\zeta)\, d\zeta}{\zeta - z} = \frac{1}{2\pi}\cdot\int_0^{2\pi} g(z + r\cdot e^{2\pi i\phi})d\phi$$

One obtains
$$|g(z)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} |g(z + r \cdot e^{2\pi i \phi})| d\phi.$$
If $|g|$ attends a local maximum at z_0 then for small $r > 0$
$$|g(z_0)| \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} |g(z_0 + r \cdot e^{2\pi i \phi})| d\phi \leq \frac{1}{2\pi} \cdot \int_0^{2\pi} |g(z_0)| d\phi = |g(z_0)|$$
or
$$|g(z_0)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} |g(z_0 + r \cdot e^{2\pi i \phi})| d\phi$$
If for at least one $\zeta := z_0 + r \cdot e^{2\pi i \phi}$ holds
$$|g(\zeta)| < |g(z_0)|$$
then the same inequality also holds in a neighbourhood of ζ. As a consequence
$$|g(z_0)| = \frac{1}{2\pi} \cdot \int_0^{2\pi} |g(z_0 + r \cdot e^{2\pi i \phi})| d\phi < \frac{1}{2\pi} \cdot \int_0^{2\pi} |g(z_0)| d\phi = |g(z_0)|,$$
a contradiction. Therefore
$$|g(z)| = |g(z_0)|$$
in a neighbourhood of z_0.

The preceding argument from one variable extends to the Cauchy formula for several variables. It shows: If $|f|$ attends a local maximum at z_0, then $|f|$ is constant in a neighbourhood of z_0. In particular
$$|f|^2 = f \cdot \overline{f}$$
is constant in a neighbourhood of z_0. If $f(z_0) = 0$ then by assumption also $f(z) = 0$ in a neighbourhood of z_0.

Therefore we assume $f(z_0) \neq 0$. Taking the partial derivatives for $j = 1, ..., n$ shows
$$\frac{\partial}{\partial z_j}(f \cdot \overline{f}) = \frac{\partial f}{\partial z_j} \cdot \overline{f} + f \cdot \frac{\partial \overline{f}}{\partial z_j} = 0$$
or
$$\frac{\partial f}{\partial z_j} \cdot \overline{f} = 0,$$
which implies
$$\frac{\partial f}{\partial z_j} = 0$$
in a neighbourhood of z_0. As a consequence f is constant in a neighbourhood of z_0. The identity theorem implies that f is constant in G.

ii) A holomorphic function f on a compact manifold assumes the maximum of its value because $|f|$ is a continuous function. The result now follows from part i).

28 . i) For any $t \neq 0$ the pair

$$r_t := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) \in R,$$

but

$$\lim_{t \to 0} r_t = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \notin R.$$

Therefore $R \subset X \times X$ is not closed.

ii) Assume on the contrary that an analytic structure on X/R with the required properties exists. Then any singleton in X/R would be closed. Therefore also

$$p^{-1}\left(\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \subset X$$

would be closed. But according to part i) the orbit of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$) is not closed, a contradiction.

iii) We have

$$f(A_1) = f(A_2) \iff (A_1, A_2) \in R.$$

Therefore $\overline{f}$ is well-defined and uniquely determined.

29 . Similar to the proof of Corollary 2.33, part 1).

32 . i) The long exact homotopy sequence of the Hopf bundle contains the section

$$\pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3)$$

Because

$$\pi_2(S^3) = \pi_1(S^3) = 0 \textit{ and } \pi_1(S^1) = \mathbb{Z}$$

we obtain $\pi_2(S^2) = \mathbb{Z}$.

ii) Short proof (*S. Hirscher*): If the Hopf bundle were trivial it would induce a homeomorphism $S^3 \simeq S^2 \times S^1$. The latter induces an isomorphism

$$\pi_1(S^3, e) \simeq \pi_1(S^2, e) \times \pi_1(S^1, e),$$

a contradition because $\pi_1(S^3, e) = \{e\}$ but $\pi_1(S^1, e) \simeq \mathbb{Z}$.

Proof using a section: Assume that the Hopf bundle p has a section $s : S^2 \to S^3$. Consider the map

$$p \circ s = id_{S^2} : S^2 \to S^2.$$

On one hand, because π_2 is a functor we have

$$\pi_2(id_{S^2}) = id_{\pi_2(S^2)} : \mathbb{Z} \to \mathbb{Z}$$

using $\pi_2(S^2) = \mathbb{Z}$ from part i). On the other hand, because $\pi_2(S^3) = 0$ the map s is homotopic to a constant map. Therefore also $id_{\pi_2(S^2)}$ is homotopic to a constant map, which implies $\pi_2(id_{S^2}) = 0$, a contradiction.

36 . i) For $n \geq 1$ consider the group of n-th roots of unity.

$$\mu_n := \{\exp(\frac{2\pi i}{n} \cdot k) \in \mathbb{C}^* : k = 0, ..., n-1\}$$

and the short exact sequence of Lie group morphisms

$$\{e\} \to \mu_n \xrightarrow{j} SU(n) \to G := SU(n)/\mu_n \to \{e\}.$$

Here

$$j : \mu_n \to SU(n), e^{i\phi} \mapsto e^{i\phi} \cdot \mathbb{1}.$$

The corresponding long exact homotopy sequence contains the section

$$\{e\} = \pi_1(SU(n), e) \to \pi_1(G, e) \to \pi_0(\mu_n) = \mu_n \to \pi_0(SU(n), e) = \{e\}$$

and proves

$$\pi_1(G, e) \simeq \mu_n \simeq \mathbb{Z}_n.$$

ii) Any finitely generated Abelian group is a finite product

$$H = \prod_{i \in I} H_i$$

of cyclic groups $H_i,\ i \in I$. For each $i \in I$ choose a Lie group G_i with fundamental group $\pi_1(G_i, e) = H_i$ - according to part i) - or $G_i = S^1$ if $H_i = \mathbb{Z}$. Then

$$G := \prod_{i \in I} G_i$$

is a Lie group with

$$\pi_1(G, e) = \pi_1(\prod_{i \in I} G_i, e) = \prod_{i \in I} \pi_1(G_i, e) = H.$$

37 . i) Consider a vector field $A \in \Theta(X)$ and define the derivation

$$B_X := \sum_{j=1}^{n} A_X(z_j) \cdot \frac{\partial}{\partial z_j}$$

and its restrictions

$$B_U := B_X|U, U \subset X \ open.$$

We claim $A = B \in \Theta(X)$: It suffices to prove for all $x \in X$ the equality of tangent vectors

$$A(x) = B(x) \in T_x X.$$

On one hand, the tangent vector $A(x)$ has the form

$$A(x) = \sum_{j=1}^{n} \alpha_j \cdot \left. \frac{\partial}{\partial z_j} \right|_{z=x}$$

with coefficients

$$\alpha_j = A(x)(z_{j_x}) \in \mathbb{K};$$

here $z_{j_x} \in \mathscr{O}_{X,x}$ denotes the germ of the analytic function z_j. On the other hand,

$$B(x) = \sum_{j=1}^{n} (A_X(z_j))(x) \cdot \left. \frac{\partial}{\partial z_j} \right|_{z=x}.$$

The equality

$$A(x)(z_{j_x}) = (A_X(z_j))(x) \in \mathbb{K}$$

results from the commutative diagram, which defines $A(x)$ by choosing a representative for the germ of an analytic function. The equality proves

$$A(x) = B(x).$$

As a consequence $A = B$.

ii) The map

$$\Theta(X) \to \mathscr{O}_X(X)^{\oplus n}, A \mapsto (A_X(z_1), ..., A_X(z_n))$$

is well-defined and $\mathbb{K}$-linear.

The map is injective:
If $A = (A_U)_{U \subset X\ open}$ and $A(z_j) = 0$ for all $j = 1, ..., n$, then $A_X = 0$ according to part i). For all $j = 1, ..., n$: For any open subset $U \subset X$

$$A_U(z_j|U) = A_X(z_j)|U = 0.$$

Therefore $A(x)(z_j) = 0$ for all $x \in X$. Using the power series expansion of a germ $f \in \mathscr{O}_{X,x}$ around $x \in X$ we obtain $A_x = 0$. Therefore $A = 0$.

The map is surjective: An arbitrary n-tuple $(A_1, ..., A_n) \in \mathscr{O}_X(X)^{\oplus n}$ has the pre-image $A := (A_U)_{U \subset X\ open}$ with

$$A_U := \sum_{j=1}^{n} (A_j|U) \cdot \frac{\partial}{\partial z_j}$$

48 . i) Claim $ker\ Ad \subset Z(G)$: Consider $g \in ker\ Ad$. Then

$$Lie\ \phi_g = Ad\ g = id.$$

The morphism of Lie groups ϕ_g is uniquely determined by its tangent map. Therefore we have $\phi_g = id_G$, i.e. $g \in Z(G)$.

Claim $Z(G) \subset ker\ Ad$: If $g \in Z(G)$ then $\phi_g = id_G$. As a consequence

$$Lie\ \phi_g = Ad\ g = id,$$

i.e. $g \in\ ker\ Ad$.

ii) Claim $Z(Lie\ G) \subset Lie\ (Z(G)$: Consider $X \in Z(Lie\ G)$, i.e. for all $t \in \mathbb{K}$

$$[ad\ tX : Lie\ G \rightarrow Lie\ G] = 0.$$

From Problem 46 results

$$Ad\ exp(tX) = exp(ad\ tX) = exp(0) = id_G.$$

As a consequence $g := exp(tX)$ satisfies

$$T_e\phi_g = id$$

or

$$\phi_g = id_G,$$

i.e. for all $t \in \mathbb{K}$

$$exp(tX) \in Z(G).$$

Due to the characterisation of $Lie(Z(G))$ we obtain

$$X \in Lie(Z(G)).$$

Claim $Lie(Z(G)) \subset Z(Lie\ G)$: Consider $X \in Lie(Z(G))$. Then $exp(tX) \in Z(G)$ for all $t \in \mathbb{K}$, i.e. for all $t \in \mathbb{K}, g \in G$,

$$exp(tX) \cdot g \cdot \exp(tX)^{-1} = g.$$

In particular, for arbitrary $Y \in Lie\ G$ we have according to Problem 46, part ii)

$$exp(tX) \cdot exp(Y) \cdot \exp(tX)^{-1} = exp(Y) = exp(e^{ad\ tX}(Y)).$$

Because the exponential map is a local isomorphism we obtain for small $t \in \mathbb{K}$ and also $Y \in Lie\ G$ sufficiently small:

$$Y = e^{ad\ tX}(Y)$$

which implies $(ad\ tX)(Y) = 0$ by comparing coefficients with respect to t. Therefore

$$ad\ X = 0$$

i.e.

$$X \in Z(Lie\ G).$$

iii) Assume G Abelian. Then $\phi_g = id$ for all $g \in G$, hence $Ad\ g = id$ for all $g \in G$. Part i) implies $Z(G) = G$. And part ii) implies

$$Lie\ G = Lie\ Z(G) = ker\ ad,$$

i.e.

$$0 = ad : Lie\ G \to gl(Lie\ G).$$

i.e. *Lie G* Abelian.

For the opposite direction assume *Lie G* is Abelian. Then

$$0 = ad : Lie\ G \to gl(Lie\ G),$$

which implies

$$Ad\ g = id \in GL(Lie\ G)$$

for all $g \in G$ according to the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{Ad} & GL(Lie\ G) \\ {\scriptstyle exp}\uparrow & & \uparrow{\scriptstyle e} \\ Lie\ G & \xrightarrow{ad} & gl(Lie\ G) \end{array}$$

because the morphism of Lie groups Ad is uniquely determined by its tangent map

$$ad = Lie\ Ad.$$

The equation $Ad\ g = id_{GL(Lie\ G)}$ implies $\phi_g = id_G$ because ϕ_g is uniquely determined by its tangent map $Ad\ g$. As a consequence, each $g \in G$ belongs to $Z(G)$, i.e. G is Abelian.