Stein Manifolds

Joachim Wehler

Ludwig-Maximilians-Universität München

Department of Mathematics

Winter Semester 2021/22 - Lecture Notes

DRAFT, Release 4.3

September 4, 2022

I prepared these notes for the participants of my lecture. The course took place at the mathematical department of LMU (Ludwig-Maximilians-Universität) at Munich during the winter semester 2021/2022. Main parts of the texts rely on notes of a lecture of O. Forster [6].

Compared to the oral lecture these written notes contain some additional material. I thank all participants of the course, in particular J. Bartenschlager and L. Schönlinner, who have pointed out errors and typos in the notes.

Please report to

wehler@math.lmu.de

any errors or typos, adding the release of the lecture notes.

Release notes:

- *Release 4.3*: Chapter 4, Section 4.1 Minor additions, correction and clarification of the proof of Theorem 4.28.
- *Release 4.2*: Chapter 4: Minor revisions.
- *Release 4.1*: Chapter 4: Several results added from local Weierstrass theory. Chapter 6, Section 6.3: Several results added. Index updated.
- *Release 4.0*: Chapter 5, 6, 7 minor revisions. Chapter 6, Definition 6.15 added, renumbering.
- *Release 0.1*: Document created, Chapter 1, Section 1.1.

Contents

Introduction

Fig. 0.1 The Stein garden (Aussichtspunkt = viewpoint)

The theory of Stein manifolds originates as generalization of the theory of open, i.e. non-compact Riemann surfaces: Each Riemann surface is either compact or a Stein manifold.

A Stein manifold X is characterized by its wealth of holomorphic functions: There exist enough holomorphic functions f_j on X to define a closed embedding

$$f = (f_1, ..., f_n) : X \hookrightarrow \mathbb{C}^n$$

for a suitable finite $n \in \mathbb{N}$: Hence each Stein manifold X is a closed submanifold of an affine space $\mathbb{C}^n$.

Of course the theory develops the other way round: How to find intrinsic properties of a complex manifold X which allow a closed affine embedding of X? The proper means to answer this question is sheaf cohomology.

In somewhat more detail the content of these lecture notes is as follows:

Chapter 1 carries over concepts and results from 1-dimensional complex analysis to complex analysis of several variables on open subdomains of $\mathbb{C}^n$. We emphasize those results which carry over literally as well as those which are different in several variables.

Chapter 2 generalizes the concepts and results from Chapter 1 to complex manifolds. We introduce the language of sheaves on a complex manifold X. It serves to glue local results to global results on X. As a first example we state the Cousin problems from complex analysis and their solutions as questions to the sheaves of holomorphic resp. meromorphic functions on a complex manifold. Then we solve the Cousin problems for polydiscs.

Cohomology theory in Chapter 3 uses the language of sheaves to measure the obstructions against globalizing local results on manifolds. The chapter introduces Čech cohomology as an example of a cohomology theory for sheaves. We show that each sheaf has a flabby resolution which allows to compute sheaf cohomology by acyclic resolutions, thanks to the abstract de Rahm theorem. As an application we compute the Dolbeault groups for the holomorphic cohomology and show that each Cousin problem is solvable on the polydisc.

Chapter 4 introduces the concept of coherence for module sheaves over the structure sheaf of a complex manifold X. This property is fundamental for the whole theory of complex manifolds. The chapter starts with the necessary results from Local Analytic Theory: The Weierstrass preparation theorem and the Weierstrass division theorem imply a series of algebraic properties of the ring R of convergent power series. From cohomological algebra we recall Hilbert's syszygy theorem for finitely generated modules over R. Locally a coherent sheaf is the cokernel of a morphism of finitely generated $\mathscr{O}$-modules. Hence coherence extends the commutative algebra of finitely generated modules over R to the theory of module sheaves over the structure sheaf of X.

Chapter 5 introduces and proves Cartan's lemma about the splitting of matrices of invertible holomorphic functions defined on the intersection of two adjacent product domains. Using Cartan's lemma on holomorphic matrix functions we carry over the syzygy theorem to coherent sheaves on a polydisc. As an application, coherent sheaves are acyclic on a shrinked polydisc.

The class of Stein manifolds is introduced in Chapter 6. We prove the main results of the theory: Theorem *A* and *B* about the cohomology of coherent sheaves on Stein manifolds. The proof of Theorem *B* starts with the local version for polydiscs. It advances via Runge approximation for coherent sheaves up to the global result. The signpost on this path is the choice of an exhaustion of the Stein manifold by analytic polyhedra. The applications show in which sense the complex analysis of Stein manifolds generalizes the theory of open Riemann surfaces to the higher-dimensional case.

The final Chapter 7 gives an outlook to some more advanced topics from the theory of Stein manifolds, e.g to the embedding theorem, the spectrum of a Stein algebra and duality for coherent sheaves on Stein manifolds. In addition, we make some remarks about the case of singularities, i.e. about generalizing the results to Stein spaces. A last section emphasizes the parallel between Stein spaces from complex analysis and affine schemes from algebraic geometry.

The theory of Stein manifolds is named after Karl Stein (1913-2000). Joint work together with his teacher H. Behnke on Riemann surfaces laid the foundations around the midth of the 20th century. For a historical scetch see [14, Einltg.] and [27].

All smooth and complex manifolds in these lecture notes are assumed to satisfy the second axiom of countability.

Chapter 1
Holomorphic functions of several variables

1.1 Convergent power series

Complex analysis of several variables starts with holomorphic functions defined on open subsets of the n-dimensional complex space $\mathbb{C}^n$. Locally these functions are given by a convergent power series in several variables. Each point $z \in \mathbb{C}^n$ is an n-tuple

$$z = (z_1, ..., z_n)$$

of complex numbers $z_1, ..., z_n$. As a consequence, one has often to consider polyindices

$$i = (i_1, ..., i_n) \in \mathbb{N}^n.$$

As experience shows, it takes some time to become familiar with the polyindex notation. Hence one should practice to read formulas about functions of several variables alike to the familiar formulas from one variable. Many results from the 1-dimensional theory carry over to the n-dimensional theory. But there are some remarkable exceptions where one discovers surprisingly different results. We will emphasize these differences during the course of these notes.

Notation 1.1 (Polyindex and polydisc). *Consider $n \in \mathbb{N}$.*

- Factorial*: For a polyindex $i = (i_1, ..., i_n) \in \mathbb{N}^n$ set*

$$i! := i_1! \cdot ... \cdot i_n! \in \mathbb{N}$$

- Modulus*: For a polyindex $i = (i_1, ..., i_n) \in \mathbb{N}^n$ set*

$$|i| := i_1 + ... + i_n$$

- Exponential*: For a polyindex $i = (i_1, ..., i_n) \in \mathbb{N}^n$ and a point $z = (z_1, ..., z_n) \in \mathbb{C}^n$ define the product*

$$z^i := z_1^{i_1} \cdot ... \cdot z_n^{i_n} \in \mathbb{C}$$

- Disc and polydisc*: In one dimension, for* $a \in \mathbb{C}$ *and* $r \in \mathbb{R}_+^*$ *we denote by*

$$D(a;r) := \{z \in \mathbb{C} : \ |z-a| < r\}$$

the open disc *around a with radius r. In several dimensions, for*

$$a = (a_1, ..., a_n) \in \mathbb{C}^n \text{ and a polyradius } r = (r_1, ..., r_n) \in (\mathbb{R}_+^*)^n$$

we denote by

$$\Delta(a;r) := \prod_{\nu=1,...,n} D(a_\nu; r_\nu) =$$

$$= \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_\nu - a_\nu| < r_\nu, \ \nu = 1, ..., n\} \subset \mathbb{C}^n$$

the open polydisc *around a with polyradius r. The polydisc* $\Delta(a;r)$ *is a product domain; it is the Cartesian product of* 1*-dimensional discs.*

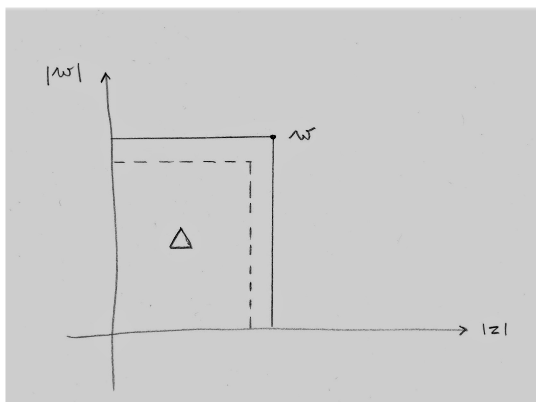


Fig. 1.1 Compact convergence of power series

Proposition 1.2 (Convergence of power series). *Consider a complex power series in several variables* $z = (z_1, ..., z_n) \in \mathbb{C}^n$

$$f(z) := \sum_{i \in \mathbb{N}^n} c_i \cdot z^i, c_i \in \mathbb{C} \text{ for all } i \in \mathbb{N}^n.$$

Assume that the power series is convergent for a given point

$$w = (w_1, ..., w_n) \in \mathbb{C}^n$$

with respect to a suitable ordering of indices. If there exists a polyradius

$$r = (r_1, ..., r_n) \in \mathbb{R}_+^* \text{ with } r_\nu < |w_\nu|, \nu = 1, ...n,$$

then $f(z)$ is absolutely and uniformly convergent in the closed polydisc

$$\overline{\Delta} := \{z \in \mathbb{C}^n : |z_\nu| \le r_\nu,\ \nu = 1, ..., n\},$$

see Figure 1.1. Moreover, $f(z)$ is convergent for all $z \in \overline{\Delta}$ with respect to each ordering of indices, and the value $f(z) \in \mathbb{C}$ is independent from the chosen order.

Proof. For $\nu = 1, ..., n$ set

$$\theta_\nu := \frac{r_\nu}{|w_\nu|} < 1 \text{ and } \theta := (\theta_1, ..., \theta_n).$$

There exists a constant M such that for all $z \in \overline{\Delta}$ and all $i \in \mathbb{N}^n$

$$|c_i \cdot w^i| \le M$$

The estimate

$$|z_\nu| \le r_\nu = \frac{r_\nu}{|w_\nu|} \cdot |w_\nu| = \theta_\nu \cdot |w_\nu|,\ \nu = 1, ..., n,$$

implies

$$|c_i \cdot z^i| \le |c_i \cdot w^i| \cdot \theta^i \le M \cdot \theta^i$$

The formula for the geometric series

$$\sum_{i \in \mathbb{N}^n} M \cdot \theta^i = M \cdot \frac{1}{(1-\theta_1) \cdot ... \cdot (1-\theta_n)}$$

shows that the assumptions of the rearrangement theorem are satisfied, and finishes the claim of the theorem. □

We define *holomorphic* functions as *analytic* functions, i.e. as functions which locally expand into a convergent power series.

Definition 1.3 (Holomorphic function). Consider an open set $U \subset \mathbb{C}^n$.

1. A function

$$f : U \to \mathbb{C}$$

is *holomorphic* if f expands into a convergent power series locally around each point of U, i.e. for each point $a \in U$ exists a neighbourhood $V \subset U$ of a and a power series convergent in V

$$\sum_{i \in \mathbb{N}^n} c_i \cdot (z-a)^i$$

such that for all $z \in V$

$$f(z) = \sum_{i \in \mathbb{N}^n} c_i \cdot (z-a)^i$$

2. A map
$$F=(F_1,...,F_k):U\to\mathbb{C}^k$$
is *holomorphic* iff each component function $F_j,\ j=1,...,k$, is holomorphic.

Alike to functions of one complex variable also for functions f of several complex variables $z=(z_1,...,z_n)\in\mathbb{C}^n$ the following concepts are equivalent, see [6, § 1]:

- Analyticity of f, i.e. locally expandable in a convergent power series.
- Existence of continuous partial derivatives of f with respect to all real variables
$$x_\nu \text{ and } y_\nu \text{ with } z_\nu=x_\nu+i\cdot y_\nu,\ \nu=1,...,n,$$
which satisfy the Cauchy-Riemann differential equations
$$\frac{\partial f}{\partial \bar{z}_\nu}=0,\ \nu=1,...,n.$$
Here
$$\frac{\partial}{\partial \bar{z}_\nu}:=\frac{\partial}{\partial x_\nu}+i\cdot\frac{\partial}{\partial y_\nu}$$
- Existence of a complex-linear approximation for f at each point (Total complex differentiability).

As a consequence sum and product as well as the composition of holomorphic functions are again holomorphic.

Notation 1.4 (Ring of holomorphic functions). *For an open set $U\subset\mathbb{C}^n$ we denote by $\mathscr{O}(U)$ the ring of holomorphic functions on U. It is a commutative ring with unit, even a $\mathbb{C}$-algebra.*

Remark 1.5 (Partial holomorphy). Each holomorphic function is continuous due to the compact convergence from Proposition 1.2. Due to the same proposition each holomorphic function of several variables is holomorphic in each variable separately, i.e. if
$$f:U\to\mathbb{C},\ U\subset\mathbb{C}^n \text{ open},$$
is a holomorphic function of several variables and if $a=(a_1,...,a_n)\in U$, then for each $\nu=1,...,n$ exists a radius $r_\nu>0$ such that
$$U_\nu:=\{a_1\}\times...\times\{a_{\nu-1}\}\times\{z_j\in\mathbb{C}:\ |z-a_\nu|<r_\nu\}\times\{a_{\nu+1}\}\times...\times\{a_n\}\subset U$$
and the restriction

$$f|U_\nu : U_\nu \to \mathbb{C}$$

is holomorphic as a function of the distinguished single complex variable z_ν.

Theorem 1.6 (Osgood's lemma). *Consider an open set $U \subset \mathbb{C}^n$ and a continuous function*

$$f : U \to \mathbb{C}$$

Then f is holomorphic in the sense of Definition 1.3 iff f is holomorphic in each variable separately.

Proof. Due to Remark 1.5 only the direction "$\Leftarrow$" needs a proof. Assume that f is holomorphic as a function of each variable separately.

Holomorphy is a local property. Consider a point $a \in U$ and choose a polydisc

$$\Delta := \Delta(a;r)$$

with $\overline{\Delta} \subset U$. For each index tuple $K = (k_1,...,k_n) \in \mathbb{N}^n$ set

$$e := (1,...,1) \in \mathbb{N}^n$$

and define

$$c_K := \frac{1}{(2\pi i)^n} \cdot \int_{\zeta \in \partial\Delta} \frac{f(\zeta)}{(\zeta - a)^{K+e}}\, d\zeta$$

Note that the integral is a multiple integral

$$d\zeta = d\zeta_1 ... d\zeta_n$$

along the compact set $\partial\Delta \subset \mathbb{C}^n$.

Claim: In Δ the function f expands in a convergent power series around a

$$f(z) = \sum_{K \in \mathbb{N}^n} c_K \cdot (z-a)^K$$

with coefficients

$$c_K = \frac{1}{K!} \cdot \frac{\partial^{|K|} f}{\partial z_1^{k_1} ... \partial z_n^{k_n}}(a)$$

We proof the claim by induction on the number n of variables.

Induction start $n = 1$: The result is one of the main results from complex analysis in one variable: Each holomorphic function f is analytic and has derivaties of each order. Due to the Cauchy integral formula the coefficients of the power series expansion derive from the derivatives of f as claimed above.

Induction step $n - 1 \mapsto n$: Split

$a = (a', a_n) \in U \cap (\mathbb{C}^{n-1} \times \mathbb{C})$, $K = (K', k) \in \mathbb{N}^{n-1} \times \mathbb{N}$, $z = (z', z_n) \in (\mathbb{C}^{n-1} \times \mathbb{C}) \cap \Delta$,

$$e = (e', 1) \text{ with } e' = (1, ..., 1) \in \mathbb{N}^{n-1}$$

and set

$$\Delta' := \Delta \cap \mathbb{C}^{n-1}$$

Define for each $k \in \mathbb{N}$ and $z' \in \Delta'$ the 1-dimensional integral with respect to the last variable ζ_n and depending on the first variables as the parameter z'

$$c_k(z') := \frac{1}{2\pi i} \cdot \int_{|\zeta_n - a_n| = r_n} \frac{f(z', \zeta_n)}{(\zeta_n - a_n)^{k+1}} \, d\zeta_n.$$

The integral is well-defined because the integrand is continuous along the compact path of integration. By induction start

$$f(z', z_n) = \sum_{k=0}^{\infty} c_k(z') \cdot (z_n - a_n)^k$$

and the coefficient $c_k(z')$ depends continuously on z'. Because

$$\frac{\partial f}{\partial \bar{z}_\nu} = 0, \nu = 1, ..., n-1,$$

the function c_k is even partially holomorphic - with respect to all variables of z'. The induction assumption applies to each function $c_k,\ k \in \mathbb{N}$. It provides the convergent power series expansions with respect to z'

$$c_k(z') = \sum_{K' \in \mathbb{N}^{n-1}} c_{(K',k)} \cdot (z' - a')^{K'}$$

and the well-defined $n-1$-dimensional integral

$$c_{(K',k)} := \frac{1}{(2\pi i)^{n-1}} \cdot \int_{\substack{|z_j - a_j| = r_j \\ j=1,...,n-1}} \frac{c_k(\zeta')}{(\zeta' - a')^{K'+e'}} \, d\zeta' =$$

$$= \frac{1}{(2\pi i)^n} \cdot \int_{\substack{|z_j - a_j| = r_j \\ j=1,...,n}} \frac{f(\zeta)}{(\zeta - a)^{K+e}} \, d\zeta$$

satisfies

$$c_{(K',k)} = \frac{1}{K'!} \cdot \frac{\partial^{|K'|} c_k}{\partial z_1^{k_1} ... \partial z_{n-1}^{k_{n-1}}}(a')$$

Due to the induction start

$$c_k(a') = \frac{1}{k!} \cdot \frac{\partial^k f}{\partial z_n^k}(a)$$

we eventually get

$$c_K = c_{(K',k)} = \frac{1}{K!} \cdot \frac{\partial^{|K|} f}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}(a)$$

□

The proof of Theorem 1.6 contains in particular the Cauchy integral formula Corollary 1.7.

Corollary 1.7 (Cauchy integral formula). *Consider an open set $U \subset \mathbb{C}^n$ and a holomorphic function*

$$f : U \to \mathbb{C}.$$

Then all partial derivatives of f exist as holomorphic functions and can be represented by the following integral formula: For each point $a \in U$ and for each polydisc

$$\Delta := \Delta(a;\ r) \text{ with closure } \overline{\Delta} \subset U$$

holds for all $z \in \Delta$ and each polyindex $j = (j_1, ..., j_n) \in \mathbb{N}^n$ the Cauchy integral formula

$$\frac{\partial^{|j|} f}{\partial z_1^{j_1} \ldots \partial z_n^{j_n}}(z) = \frac{j!}{(2\pi i)^n} \cdot \int_{\zeta \in \partial \Delta} \frac{f(\zeta)}{(\zeta - z)^{j+e}} \, d\zeta$$

with the shorthand

$$\zeta = (\zeta_1, ..., \zeta_n),\ d\zeta = d\zeta_1 ... d\zeta_n,\ e = (1, ..., 1) \in \mathbb{N}^n$$

Proof. See the inductive proof of Theorem 1.6. □

Remark 1.8 (Hartogs' theorem). It is a remarkable result due to Hartogs that in Theorem 1.6 one can dismiss the assumption that f is continuous, see [22, Chap. III, Theor. of Hartogs], [19, Theor. 2.2.8].

1.2 First applications of Cauchy's integral formula

Definition 1.9 (Domain). A *domain* $G \subset \mathbb{C}^n$ is a non-empty, open and connected subset of $\mathbb{C}^n$.

Theorem 1.10 (Identity theorem). *Consider a domain $G \subset \mathbb{C}^n$. Two holomorphic functions*

$$f,\ g : G \to \mathbb{C}$$

are equal if they coincide on a non-empty open subset of G.

Proof. By definition the set of local coincidence

$$X := \{x \in G : \text{ For a suitable neighbourhood } U \text{ of } x \text{ holds } f|U = g|U\}$$

is open and non-empty. The set X is also closed with respect to the topology of G: Consider an arbitrary point

$$z_0 \in \partial X \cap G.$$

We have to show $z_0 \in X$, i.e. for a polydisc Δ around z_0 holds

$$f|\Delta = g|\Delta :$$

Choose a polydisc $\Delta \subset G$ centered around z_0. Because

$$\Delta \cap X \neq \emptyset$$

there exists a point

$$a \in \Delta \cap X.$$

For arbitrary $z \in \Delta$ the function Φ of the single complex variable t

$$\phi(t) := f(a + t \cdot (z - a)) - g(a + t \cdot (z - a))$$

is holomorphic in a connected, complex neighbourhood in $\mathbb{C}$ of the real interval $[0, 1]$, see Figure 1.2.

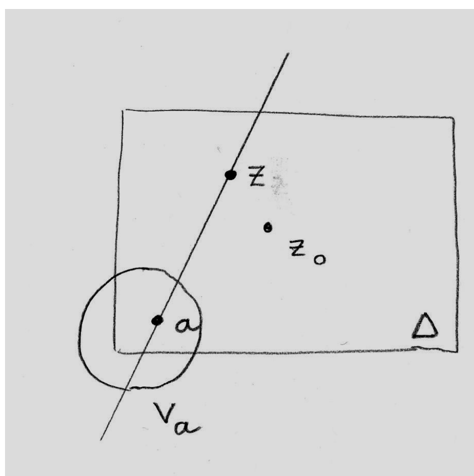


Fig. 1.2 Identity theorem

Because both functions f and g coincide by assumption in a suitable neighbourhood $V_a \subset \mathbb{C}^n$ of a, the identity theorem of complex analysis in one variable implies

$$\phi = 0.$$

Hence in particular

$$\phi(1) = 0, \text{ i.e. } f(z) = g(z).$$

Because $z \in \Delta$ has been choosen arbitrarily, there holds

$$f|\Delta = g|\Delta.$$

It implies $z_0 \in X$ by definition of X. As a consequence $X \subset G$ is also closed in G, and the connectedness of G implies

$$X = G.$$

□

Corollary 1.11 (Integral domain). *For a domain $G \subset \mathbb{C}^n$ the ring $\mathscr{O}(G)$ of holomorphic functions is an integral domain.*

Proof. We have to show for $f, g, \in \mathscr{O}(G)$:

$$f \cdot g = 0 \implies f = 0 \text{ or } g = 0.$$

If $f \neq 0$ then there exists a point $a \in G$ with $f(a) \neq 0$. By continuity $f|V \neq 0$ for a suitable neighbourhood V of a. Hence $g|V = 0$. Theorem 1.10 implies $g = 0$. □

The conclusion of Corollary 1.11 does not hold for an open subset $G \subset \mathbb{C}^n$ with at least two connected components.

Corollary 1.12 (Open map). *Consider an open subset $U \subset \mathbb{C}^n$ and a holomorphic map*

$$f : U \to \mathbb{C}$$

which is not constant on any component of U. Then f is an open map.

Proof. Consider an arbitrary point $a \in U$. Choose an open polydisc $\Delta \subset U$ around a. Theorem 1.10 implies the existence of a point $b \in \Delta$ with

$$f(a) \neq f(b)$$

If $L \subset \mathbb{C}^n$ denotes the complex line passing through a and b then the restriction

$$f|L \cap \Delta$$

is a non-constant holomophic function of one variable. The open mapping theorem for holomorphic functions of one variable implies that

$$f(L \cap \Delta)$$

is an open neighbourhood of $f(a)$ in $\mathbb{C}$, and as a consequence also the superset

$$f(\Delta) \supset f(\Delta \cap L).$$

□

Corollary 1.13 (Maximum modulus theorem). *Consider a domain $G \subset \mathbb{C}^n$ and a holomorphic function $f \in \mathscr{O}(G)$ which attains the maximum of its modulus at a point $a \in G$, i.e. for all $z \in G$ holds*

$$|f(a)| \geq |f(z)|$$

Then f is constant in G.

Proof. The proof is indirect. Assume that f is not constant. Then $|f(a)| \neq 0$ which implies

$$|f(a)| = \sup |f(G)| =: r > 0$$

Corollary 1.12 implies that $f(G) \subset \mathbb{C}$ is open. Moreover

$$f(G) \subset \overline{\Delta}_r(0) \subset \mathbb{C},$$

and therefore

$$\partial\Delta_r(0) \cap f(G) = \emptyset.$$

But

$$f(a) \in \partial\Delta_r(0) \cap f(G),$$

a contradiction. □

The following Theorem 1.14 shows the existence of domains

$$B \subset \mathbb{C}^n,\ n \geq 2,$$

such that each holomorphic function $f \in \mathscr{O}(B)$ extends holomorphically to a strictly larger domain in $\mathbb{C}^n$, see Figure 1.3.
This property is in striking contrast to complex analysis of one variable: For each domain $B \subset \mathbb{C}$ exists a holomorphic function $f \in \mathscr{O}(B)$ which does not extend holomorphically across a certain boundary point $z_0 \in \partial B$.

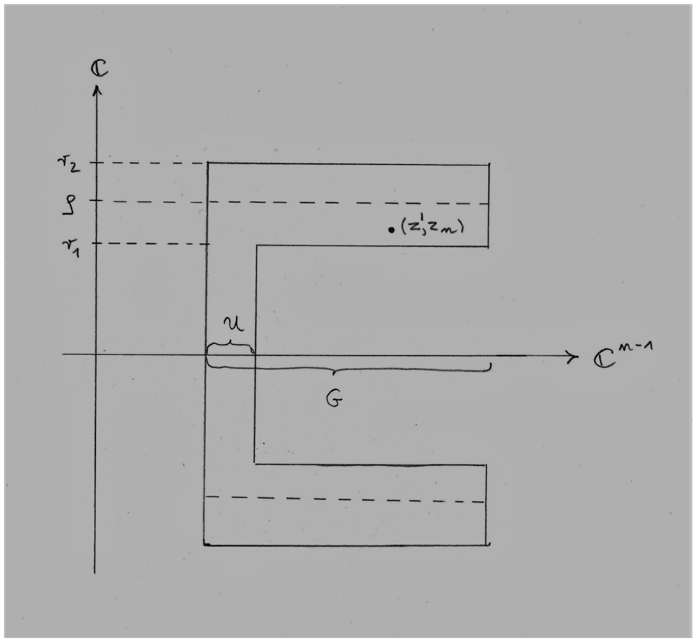


Fig. 1.3 Hartogs' continuity theorem

Theorem 1.14 (Hartogs' continuity theorem). *Let $G \subset \mathbb{C}^{n-1}$, $n \geq 2$, be a domain. Consider the following geometric situation: A non-empty open set $U \subset G$, and two radii $0 < r_1 < r_2$ defining the 1-dimensional complex domains*

$$S := \{z \in \mathbb{C} : |z| < r_2\} \text{ (disc) and } R := \{z \in \mathbb{C} : r_1 < |z| < r_2\} \text{ (annulus)},$$

see Figure 1.3. Set

$$H := U \times S \cup G \times R$$

Then each holomorphic function

$$f \in \mathscr{O}(H)$$

extends uniquely to a holomorphic function

$$\tilde{f} \in \mathscr{O}(G \times S).$$

Proof. Choose an intermediate radius $r_1 < \rho < r_2$, and define for a each $z = (z', z_n) \in G \times R$

$$\tilde{f}(z) := \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{f(z', \zeta)}{\zeta - z_n} \, d\zeta$$

The resulting function $\tilde{f}$ is holomorphic on

$$G \times \{z_n \in \mathbb{C} : \ |z_n| < \rho\}$$

On the non-empty open set

$$\tilde{U} := U \times \{z_n \in \mathbb{C} : \ |z_n| < \rho\} \subset U \times S$$

holds

$$\tilde{f}|\tilde{U} = f|\tilde{U}.$$

Theorem 1.10 implies $\tilde{f}|(U \times S \ \cup \ G \times R) = f$. $\square$

The domain

$$U \times S \cup G \times R$$

from Figure 1.3 is named a *Hartogs figure* in $\mathbb{C}^n$.

Corollary 1.15 (Kugelsatz). *For $n \geq 2$ consider two polyradii*

$$r = (r_1, ..., r_n) \text{ and } \tilde{r} = (\tilde{r}_1, ..., \tilde{r}_n) \text{ with } r_j < \tilde{r}_j \text{ for all } j = 1, ...n.$$

Then the canonical restriction

$$\mathscr{O}(\Delta(\tilde{r})) \to \mathscr{O}(\Delta(\tilde{r}) \setminus \overline{\Delta}(r))$$

is surjective, see Figure 1.4

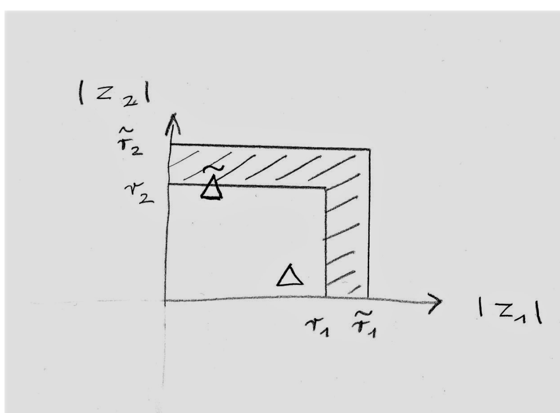


Fig. 1.4 Kugelsatz

Theorem 1.16 (Weierstrass' convergence theorem). *Consider an open set $U \subset \mathbb{C}^n$ and a compact convergent sequence $(f_\nu)_{\nu \in \mathbb{N}}$ of holomorphic functions*

$$f_\nu : U \to \mathbb{C},\ \nu \in \mathbb{N}.$$

- *There exists the limit*

$$f := \lim_{\nu \to \infty} f_\nu$$

as a holomorphic function

$$f : U \to \mathbb{C}$$

- *For each polyindex $j = (j_1, ... j_n) \in \mathbb{N}^n$ also the sequence*

$$(f^{(j)})_{\nu \in \mathbb{N}} := \left(\frac{\partial^{|j|} f_\nu}{\partial z_1^{j_1} ... \partial z_n^{j_n}} \right)_{\nu \in \mathbb{N}}$$

of partial derivatives is compact convergent in U with limit $f^{(j)}$.

Proof. i) It is well-known that the limit of a compact convergent sequence of continuous functions exists as a continuous function. In addition, Weierstrass' convergence theorem from complex analysis of one variable implies that the limit f is holomorphic in each variable separately. Theorem 1.6 implies that f is holomorphic as a function of several variables.

ii) In order to prove the compact convergence of the sequence of partial derivatives it suffices to prove the claim for the partial derivatives

$$\left(\frac{\partial f_\nu}{\partial z_k} \right)_{\nu \in \mathbb{N}},\ k = 1, ..., n,$$

of order $= 1$ and to apply the result in an iterative way.

For given $k = 1, ..., n$ consider an arbitrary compact $K \subset U$. There exist a radius $r > 0$ and a compact K' with

$$K \subset K' \subset U$$

such that for all $z = (z_1, ..., z_n) \in K$ and $\zeta \in \mathbb{C}$ with $|\zeta| < r$ also

$$(z_1, ..., z_{k-1}, z_k + \zeta, z_{k+1}, ..., z_n) \in K'$$

Then each holomorphic function $g \in \mathscr{O}(U)$ satisfies according to the 1-dimensional Cauchy integral formula for $z \in K$

$$\frac{\partial g}{\partial z_k}(z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \frac{g(z_1, ..., z_{k-1}, z_k + \zeta, z_{k+1}, ..., z_n)}{\zeta^2}\, d\zeta$$

The substitution

$$\zeta = r \cdot e^{i\phi}$$

shows: The norm with respect to the compact sets K and K' satisfies the estimate

$$\left\| \frac{\partial g}{\partial z_k} \right\|_K \leq \frac{1}{r} \cdot \|g\|_{K'}$$

Applying this estimate to each function

$$g := f_\nu - f,\ \nu \in \mathbb{N},$$

proves the compact convergence of the sequence of all derivatives of order $= 1$ as a consequence of the compact convergence of the sequence $(f_\nu)_{\nu \in \mathbb{N}}$. □

Definition 1.17 (Fréchet space).

1. A *topological vector space* is a vector space such that addition and scalar multiplication are continuous functions. In the following the base field $\mathbb{C}$ is provided with its Euclidean topology.

2. A *seminorm* on a complex vector space V is a map

$$p : V \to \mathbb{R}_+$$

satisfiying:

i) For all $\lambda \in \mathbb{C}$ and for all $\mathrm{v} \in V$

$$p(\lambda \cdot \mathrm{v}) = |\lambda| \cdot p(\mathrm{v})$$

ii) For all $\mathrm{v}_1, \mathrm{v}_2 \in V$

$$p(\mathrm{v}_1 + \mathrm{v}_2) \leq p(\mathrm{v}_1) + p(\mathrm{v}_2) \text{ (Triangle inequality).}$$

A seminorm p is a *norm* if

$$p(\mathrm{v}) = 0 \implies \mathrm{v} = 0.$$

3. A topological complex vector space V is a *Fréchet space* if V is a complete Hausdorff space with the topology defined by a countable family $(p_\nu)_{\nu \in \mathbb{N}}$ of seminorms, i.e. a neighbourhood basis of $0 \in V$ is given by the finite intersections of sets of the form

$$V(\nu, \varepsilon) := \{\mathrm{v} \in V :\ p_\nu(\mathrm{v}) < \varepsilon\},\ \varepsilon > 0,\ \nu \in \mathbb{N}.$$

Apparently the concept of Fréchet spaces generalizes the concept of Banach spaces by replacing a fixed norm by a countable family of seminorms. A sequence

$$(f_\nu)_\nu \in \mathbb{N}$$

of elements of a Fréchet space V is a *Cauchy sequence* if for each neighbourhood of zero $W \subset V$ exists $N \in \mathbb{N}$ such that for all $\nu,\ \mu \geq N$

$$f_\nu - f_\mu \in W.$$

Each Fréchet space V is metrizable, e.g. by the metric

$$d(f,g) := \sum_{\nu=0}^{\infty} \frac{1}{2^\nu} \cdot \frac{p_\nu(f-g)}{1+p_\nu(f-g)},\ f,g \in V.$$

Remark 1.18 (Fréchet space).

1. *Subspace and quotient*: Consider a Fréchet space V: If $W \subset V$ is a closed subspace then also W and V/W are Fréchet spaces with the induced topologies.

2. *Countable product of Fréchet space*: For a countable family $(V_i)_{i \in I}$ also the product
$$V := \prod_{i \in I} V_i$$
becomes a Fréchet space when provided with the product topology.

3. *Open mapping theorem*: A continuous surjective linear map $\phi : V \to W$ between two Fréchet spaces is an open map.

A good introduction to Fréchet spaces are the first chapters of Rudin's book [25], see also [17, App. B, Theor. 6].

Proposition 1.19 (Topology of compact convergence). *Consider an open set $U \subset \mathbb{C}^n$. Provided with the topology of compact convergence both vector spaces*

$$\mathscr{C}(U) := \{f : U \to \mathbb{C} :\ f \text{ continuous}\}$$

and

$$\mathscr{O}(U) := \{f : U \to \mathbb{C} :\ f \text{ holomorphic}\}$$

are Fréchet spaces. Moreover

$$\mathscr{O}(U) \subset \mathscr{C}(U)$$

is a closed subspace. For open $V \subset U$ the restriction map

$$\mathscr{O}(U) \to \mathscr{O}(V),\ f \mapsto f|V,$$

is continuous.

Proof. One chooses an exhaustion $(U_\nu)_{\nu\in\mathbb{N}}$ of U by relatively compact subsets

$$U_\nu \subset\subset U_{\nu+1} \subset\subset U,\ \nu \in \mathbb{N},$$

and defines the seminorms

$$p_\nu : \mathscr{C}(U) \to \mathbb{R}_+,\ p_\nu(f) := \|f\|_{U_\nu} := sup\ \{|f(z)| :\ z \in U_\nu\}$$

The finite intersections of the sets

$$V(\nu,\varepsilon) := \{f \in \mathscr{C}(U) :\ p_\nu(f) < \varepsilon\},\ \nu \in \mathbb{N},\ \varepsilon > 0,$$

are a neighbourhood basis of $0 \in \mathscr{C}(U)$ of a Fréchet topology on $\mathscr{C}(U)$.

A sequence of continuous functions on U is convergent in the Fréchet topology of $\mathscr{C}(U)$ iff the functions are uniformly convergent on compact subsets of U. Here one uses that the sequence $(U_\nu)_\nu$ is an exhaustion of U by relatively compact open subsets. The subspace

$$\mathscr{O}(U) \subset \mathscr{C}(U),$$

provided with the subspace topology, is closed as a consequence of Weierstrass' convergence theorem, Theorem 1.16. Hence $\mathscr{O}(U)$ is a Fréchet space with respect to the topology of compact convergence. The continuity of the restriction maps is a direct consequence of the definition of the topology, because a compact subset of V is also a compact subset of U. □

1.3 Dolbeault's lemma

Dolbeault's lemma deals with the exterior differential operator d'' on smooth and holomorphic differential forms on polydiscs. It will be the basis for many computations of cohomology groups in later chapters.

We recall some basic results about differential forms defined in an open set

$$U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}.$$

We split the complex coordinates $z = (z_1,...,z_n) \in \mathbb{C}^n$ in real and imaginary part

$$z_\nu = x_\nu + i \cdot x_{n+\nu},\ \nu = 1,...,n.$$

Remark 1.20 (Differential forms and exterior derivation).

- *Smooth functions*: We denote by $\mathscr{E}(U)$ the complex vector space of smooth functions

$$f : U \to \mathbb{C}$$

- *Smooth alternating differential forms*: For $1 \le r \le 2n$ we denote by $\mathscr{E}^r(U)$ the complex vector space of differential forms on U of degree $= r$. Each differential form $\omega \in \mathscr{E}^r(U)$ has with respect to real coordinates $(x_1, ..., x_{2n}) \in \mathbb{R}^{2n}$ the form

$$\omega = \sum_{1 \le j_1 < ... < j_r \le 2n} f_{j_1,...,j_r} \cdot dx_{j_1} \wedge ... \wedge dx_{j_r}$$

with elements $f_{j_1,...,j_r} \in \mathscr{E}^r(U)$. Due to the complex structure on $U \subset \mathbb{C}^n$ the vector space $\mathscr{E}^r(U)$ splits as the direct sum

$$\mathscr{E}^r(U) = \bigoplus_{\substack{1 \le p,q \le n \\ p+q=r}} \mathscr{E}^{p,q}(U)$$

with $\mathscr{E}^{p,q}(U)$ the vector space of all (p,q)-forms. A differential form $\omega \in \mathscr{E}^{p,q}(U)$ has the form

$$\omega = \sum_{\substack{1 \le j_1 < ... < j_p \le n \\ 1 \le j_1 < ... < j_q \le n}} f_{j_1 ... j_p j_1 ... j_q} \; dz_{j_1} \wedge ... \wedge dz_{j_p} \wedge d\bar{z}_{j_1} \wedge ... d\bar{z}_{j_q}$$

with $f_{j_1 ... j_p j_1 ... j_q} \in \mathscr{E}(U)$. Here

$$dz_\nu = dx_\nu + i \cdot dx_{n+\nu} \text{ (holomorphic differential)}$$

and

$$d\bar{z}_\nu = dx_\nu - i \cdot dx_{n+\nu} \text{ (antiholomorphic differential)}$$

We often use the shorthand

$$\omega = \sum_{I,J} f_{IJ} \; dz_I \wedge d\bar{z}_J \in \mathscr{E}^{p,q}(U)$$

- *Linear differential operators*: There are $\mathbb{C}$-linear differential operators

$$d' : \mathscr{E}(U) \to \mathscr{E}^{1,0}(U), \; d'f := \sum_{\nu=1}^{n} \frac{\partial f}{\partial z_\nu} dz_\nu$$

and

$$d'' : \mathscr{E}(U) \to \mathscr{E}^{0,1}(U), \; d''f := \sum_{\nu=1}^{n} \frac{\partial f}{\partial \bar{z}_\nu} d\bar{z}_\nu$$

and

$$d := d' + d'' : \mathscr{E}(U) \to \mathscr{E}^1(U)$$

A function $f \in \mathscr{E}(U)$ is holomorphic if and only if $d''f = 0$.

- *Exterior derivation of higher order*: For $p, q \in \mathbb{N}$ the linear differential operators from the previous step extend to $\mathbb{C}$-linear differential operators

$$d' : \mathscr{E}^{p,q}(U) \to \mathscr{E}^{p+1,q}(U)$$

and

$$d'' : \mathscr{E}^{p,q}(U) \to \mathscr{E}^{p,q+1}(U)$$

as follows: For

$$\omega = f \cdot dz_I \wedge d\bar{z}_J \in \mathscr{E}^{p,q}(U)$$

define

$$d'\omega := d'f \wedge dz_I \wedge d\bar{z}_J \in \mathscr{E}^{p+1,q}(U) \text{ and } d''\omega := d''f \wedge dz_I \wedge d\bar{z}_J \in \mathscr{E}^{p,q+1}(U)$$

and extend the definition by $\mathbb{C}$-linearity.

Proposition 1.21 (Exterior derivation as a chain map). *For an open set $U \subset \mathbb{C}^n$ the composition of exterior derivations satisfies*

$$d' \circ d' = d'' \circ d'' = 0$$

Proof. Consider a single summand

$$\omega = f \cdot dz_J \wedge d\bar{z}_K \in \mathscr{E}^{p,q}(U)$$

Compute

$$d'\omega = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \wedge dz_J \wedge d\bar{z}_K$$

and

$$d'(d'\omega) = \sum_{k=1,j=1}^{n} \frac{\partial^2 f}{\partial z_k \partial z_j} dz_k \wedge dz_j \wedge dz_J \wedge d\bar{z}_K =$$

$$= \sum_{1 \le k < j \le n}^{n} \left(\frac{\partial^2 f}{\partial z_k \partial z_j} - \frac{\partial^2 f}{\partial z_j \partial z_k} \right) dz_k \wedge dz_j \wedge dz_J \wedge d\bar{z}_K = 0$$

and analogously for d''. The vanishing depends on the following fact: For smooth functions the higher partial derivations do not depend on the order of differentiation. □

Definition 1.22 (Holomorphic differential forms). Consider an open set $U \subset \mathbb{C}^n$. The elements

$$\omega = \sum_I f_I \, dz_I := \sum_{1 \le j_1 < \ldots < j_p \le n} f_{j_1 \ldots j_q} \, dz_{j_1} \wedge \ldots \wedge dz_{j_p} \in \mathscr{E}^{p,0}(U) \text{ with } f_{j_1 \ldots j_q} \in \mathscr{O}(U)$$

form the vector space $\Omega^p(U)$ of *holomorphic differential* p-forms.

Note

$$\Omega^p(U) = ker\ [d'' : \mathscr{E}^{p,0}(U) \to \mathscr{E}^{p,1}(U)]$$

Remark 1.23 (Fréchet topology on the vector spaces of smooth differential forms). Consider an open set $U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$.

1. *Smooth functions*: After choosing an exhaustion $(U_j)_{j\in\mathbb{N}}$ of U by relatively compact open sets one defines the family of seminorms on $\mathscr{E}(U)$

   $$p(j,k) : \mathscr{E}(U) \to \mathbb{R}_+$$

   with

   $$p(j,k)(f) := \left\{ sup \left| \frac{\partial^{k_1+...+k_{2n}} f}{\partial x_1^{i_1} ... \partial x_{2n}^{k_{2n}}}(z) \right| : z \in U_j \text{ and } k_1 + ... + k_{2n} \le k \right\},\ j,\ k \in \mathbb{N}$$

   The family of seminorms provides $\mathscr{E}(U)$ with the structure of a Fréchet space.

2. *Smooth differential forms*: The Fréchet structure on $\mathscr{E}(U)$ carries over to the coefficients of the differential forms, and eventually to the vector spaces $\mathscr{E}^{p,q}(U)$ and $\mathscr{E}^r(U)$.

3. *Holomorphic differential forms*: Similarly, the Fréchet structure on $\mathscr{O}(U)$ from Proposition 1.19 carries over to a Fréchet structure on $\Omega^p(U), p \ge 1$.

4. One checks that all Fréchet structures are independent from the choice of the exhaustion.

There are closed inclusions as closed Fréchet spaces

$$\Omega^p(U) \subset \mathscr{E}^{p,0}(U) \text{ and } \mathscr{O}(U) \subset \mathscr{E}(U) \subset \mathscr{C}(U).$$

We now enter into Dolbeault theory. Consider an open set $U \subset \mathbb{C}^n$. Recall that a differential form $\omega \in \mathscr{E}^{p,q}(U)$ is named

- *closed* with respect to d'' if

  $$d''\omega = 0 \in \mathscr{E}^{p,q+1}(U)$$

- and *exact* with respect to d'' if

  $$\omega = d''\sigma$$

  for a suitable $\sigma \in \mathscr{E}^{p,q-1}(U)$.

Due to the chain condition

$$d'' \circ d'' = 0$$

each exact form is closed. Dolbeault's lemma investigates the reverse direction: Which properties of U ensure that each d''-closed ω form is also d''-exact? The Dolbeault groups

$$\frac{ker\ [d'' : \mathscr{E}^{p,q}(U) \to \mathscr{E}^{p,q+1}(U)]}{im\ [d'' : \mathscr{E}^{p,q-1}(U) \to \mathscr{E}^{p,q}(U)]}$$

measure the obstructions against exactness on U.

From the viewpoint of the differential operator d'' one has to integrate for given ω the differential equation on U

$$d''\sigma = \omega.$$

Hence the basic differential equation from Dolbeault theory is the inhomogeneous linear differential equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

for a given smooth function $g = g(z,s)$ with 1-dimensional variable z and parameter s. If g has compact support with respect to the variable z then a solution f can be obtained by an integral formula with the Cauchy kernel, see Theorem 1.24.

Theorem 1.24 (Solving the $\overline{\partial}$-problem). *Consider an open $U \subset \mathbb{C}^n$ and a smooth function*

$$g : \mathbb{C} \times U \to \mathbb{C}.$$

Assume that for each $s \in U$ the restriction to the first variable

$$g(-,s) : \mathbb{C} \to \mathbb{C}$$

has compact support. Then the function

$$f := \mathbb{C} \times U \to \mathbb{C},\ f(z,s) := \frac{1}{2\pi i} \cdot \int_{\mathbb{C}} \frac{g(\zeta,s)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}$$

is smooth and satisfies

$$\overline{\partial} f := \frac{\partial f}{\partial \bar{z}} = g$$

Proof. i) *The integrand has no singularities*: Fix $z \in \mathbb{C}$ and introduce polar coordinates in $\mathbb{C}$

$$\zeta = z + re^{i\phi}$$

such that

$$d\zeta \wedge d\overline{\zeta} = -2ir\ dr \wedge d\phi$$

Then

$$\frac{1}{\zeta - z} d\zeta \wedge d\overline{\zeta} = -2ie^{-i\phi}\ dr \wedge d\phi$$

Because of the compact support of $g(-,s)$, for each $(z,s) \in \mathbb{C} \times U$ the integral

$$\frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g(\zeta,s)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$$

is well-defined and finite.

ii) *Computing the* $\overline{\partial}$*-derivative*: After translating ζ to $\zeta + z$ to remove the parameter z from the denominator we get

$$f(z,s) := \frac{1}{2\pi i} \cdot \int_{\mathbb{C}} \frac{g(\zeta,s)}{\zeta - z} d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \cdot \int_{\mathbb{C}} g(\zeta + z, s)\ \frac{d\zeta \wedge d\overline{\zeta}}{\zeta}$$

and compute

$$\frac{\partial f}{\partial \overline{z}}(z,s) = \frac{1}{2\pi i} \cdot \int_{\mathbb{C}} \frac{\partial g(z+\zeta,s)}{\partial \overline{z}}\ \frac{d\zeta \wedge d\overline{\zeta}}{\zeta}$$

We now retranslate $\zeta + z$ to ζ, evaluate at the point $(a,s) \in \mathbb{C} \times U$, use the compact support of g with respect to the variable ζ, and apply Stokes' theorem

$$\frac{\partial f}{\partial \overline{z}}(a,s) = \frac{1}{2\pi i} \cdot \int_{\mathbb{C}} \frac{\partial g}{\partial \overline{\zeta}}(\zeta,s)\ \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - a} =$$

$$= \frac{1}{2\pi i} \cdot \lim_{\varepsilon \downarrow 0} \int_{|\zeta - a| \geq \varepsilon} \frac{\partial g}{\partial \overline{\zeta}}(\zeta,s)\ \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - a} = -\frac{1}{2\pi i} \cdot \lim_{\varepsilon \downarrow 0} \int_{|\zeta - a| \geq \varepsilon} d\left(\frac{g(\zeta,s)}{\zeta - a} d\zeta\right) =$$

$$= \frac{1}{2\pi i} \cdot \lim_{\varepsilon \downarrow 0} \int_{|\zeta - a| = \varepsilon} \frac{g(\zeta,s)}{\zeta - a} d\zeta = \frac{1}{2\pi} \cdot \lim_{\varepsilon \downarrow 0} \int_0^{2\pi} g(a + \varepsilon \cdot e^{i\theta}, s)\ d\theta = g(a,s)$$

Here the exterior derivative is computed as

$$d\left(\frac{g(\zeta,s)}{\zeta - a} d\zeta\right) = d'\left(\frac{g(\zeta,s)}{\zeta - a} d\zeta\right) + d''\left(\frac{g(\zeta,s)}{\zeta - a} d\zeta\right) =$$

$$= d''\left(\frac{g(\zeta,s)}{\zeta - a} d\zeta\right) = \frac{\overline{\partial} g \cdot (z-a)}{(z-a)^2} d\overline{\zeta} \wedge d\zeta = -\frac{\overline{\partial} g}{z - a} d\zeta \wedge d\overline{\zeta}$$

□

As a consequence of the solution of the $\overline{\partial}$-problem in Theorem 1.24 the following Proposition 1.25 shows: For a polydisc each d''-closed differential form is also d''-exact - but possibly one has to shrink the domain of the solution.

Proposition 1.25 (Solving the d''-equation after shrinking). *Consider an open polydisc $\Delta \subset \mathbb{C}^n$ and a concentric, relatively compact polydisc*

$$\Delta' \subset\subset \Delta.$$

For each pair $p,q \in \mathbb{N},\ q \geq 1$, and for each closed differential form

$$\omega \in \mathscr{E}^{p,q}(\Delta) \text{ with } d''\omega = 0$$

exists a smooth differential form

$$\sigma \in \mathscr{E}^{p,q-1}(\Delta')$$

satisfying

$$d''\sigma = \omega|\Delta'.$$

Proof. The proof relies on Theorem 1.24. It proceeds by induction on the highest index ν of the differentials $d\bar{z}_\nu$ which enter into the definition of ω: For each $\nu = 0,...,n$ define

$$A_\nu(\Delta) := \left\{ \omega \in \mathscr{E}^{p,q}(\Delta) : \ \omega = \sum_I \sum_{J \leq \nu} a_{IJ}\ dz_I \wedge d\bar{z}_J \right\}$$

Here

$$J \leq \nu$$

indicates that each tuple $(j_1,...,j_q) \in J$ satisfies

$$j_s \leq \nu \text{ for } s = 1,...,q.$$

We prove by induction on ν the statement $\mathscr{A}(\nu)$: The claim of the proposition holds for all polydiscs Δ and all $\omega \in A_\nu(\Delta)$.

Induction start $\nu = 0$: Then $A_0(\Delta) = \{0\}$ because $q \geq 1$. Set

$$\sigma := 0 \in \mathscr{E}^{p,q-1}(\Delta').$$

Induction step $\nu - 1 \mapsto \nu$: We choose a concentric polydisc Δ'' with

$$\Delta' \subset\subset \Delta'' \subset\subset \Delta.$$

The polyradii

$$r',\ r'',\ r \text{ of } \Delta' \subset\subset \Delta'' \subset\subset \Delta$$

satisfy

$$r' < r'' < r.$$

We decompose a given $\omega \in A_\nu(\Delta)$

$$\omega = \sum_I \sum_{J \leq \nu} f_{IJ}\, dz_I \wedge d\bar{z}_J \text{ with } d''\omega = 0$$

as

$$\omega = d\bar{z}_\nu \wedge \alpha + \beta$$

with

$$\alpha = \sum_I \sum_{J \leq \nu-1} a_{IJ}\, dz_I \wedge d\bar{z}_J \in \mathscr{E}^{p,q-1}(\Delta)$$

and

$$\beta \in A_{\nu-1}(\Delta)$$

Then

$$d''\omega = \sum_I \sum_{J \leq \nu} \left(\sum_{m=1}^{n} \frac{\partial f_{IJ}}{\partial \bar{z}_m}\, d\bar{z}_m \wedge dz_I \wedge d\bar{z}_m \right)$$

The assumption $d''\omega = 0$ implies for $m > \nu$

$$\frac{\partial f_{IJ}}{\partial \bar{z}_m} = 0,$$

i.e. the coefficients f_{IJ} depend holomorphically on the final variables

$$z_{\nu+1}, ..., z_n.$$

Each coefficent a_{IJ} agrees up to sign with a coefficent $f_{I'J'}$ for a suitable pair of indices $I'J'$. Because

$$\Delta'' \subset\subset \Delta$$

there exists a smooth function $\tilde{a}_{IJ} \in \mathscr{E}(\mathbb{C}^n)$ with

$$supp\ \tilde{a}_{IJ} \subset \Delta \text{ and } \tilde{a}_{IJ}|\Delta'' = a_{IJ}|\Delta''$$

At this point one has to shrink the domain from Δ to Δ'' to extend a_{IJ} to a global smooth function $\tilde{a}_{IJ}$ with compact support. In particular $\tilde{a}_{IJ}$ has compact support with respect to the distinguished variable z_ν and depends holomorphically on the final variables $z_{\nu+1}, ..., z_n$. Hence Theorem 1.24 applies to the function

$$\mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C},\ (\zeta, z') \mapsto \tilde{a}_{IJ}(z_1, ..., z_{\nu-1}, \zeta, z_{\nu+1}, ..., z_n)$$

It provides the smooth function

$$c_{IJ} \in \mathscr{E}(\Delta'')$$

defined as

$$c_{IJ}(z_1, ..., z_n) := \frac{1}{2\pi i} \cdot \int_{\mathbb{C}} \frac{\tilde{a}_{IJ}(z_1, ..., z_{\nu-1}, \zeta, z_{\nu+1}, ..., z_n)}{\zeta - z_\nu}\, d\zeta \wedge d\overline{\zeta}$$

It satisfies

$$\frac{\partial c_{IJ}}{\partial \bar{z}_\nu} = \tilde{a}_{IJ}|\Delta'' = a_{IJ}|\Delta''$$

Like $\tilde{a}_{IJ}$ also the function c_{IJ} depends holomorphically on the variables $z_{\nu+1}, ..., z_n$. We define

$$\gamma := \sum_I \sum_{J \leq \nu-1} c_{IJ}\ dz_I \wedge d\bar{z}_J \in \mathscr{E}^{p,q-1}(\Delta'')$$

Then on Δ''

$$d''\gamma := \sum_I \sum_{J \leq \nu-1} \left(\sum_{m=1}^{\nu} \frac{\partial c_{IJ}}{\partial \bar{z}_m} d\bar{z}_m \wedge dz_i \wedge d\bar{z}_J \right) = d\bar{z}_\nu \wedge \alpha + \delta \in \mathscr{E}^{p,q}(\Delta'')$$

with a form

$$\delta \in A_{\nu-1}(\Delta'')$$

having coefficients not depending on the final variables $z_{\nu+1}, ..., z_n$. Then

$$\omega - d''\gamma = \beta - \delta \in A_{\nu-1}(\Delta'')$$

The induction assumption applies to

$$\beta - \delta \in A_{\nu-1}(\Delta'')$$

because

$$d''(\beta - \delta) = d''\omega - (d'' \circ d'')\gamma = 0$$

The induction assumption provides a form

$$\eta \in \mathscr{E}^{p,q-1}(\Delta')$$

satisfying

$$d''\eta = \beta - \delta = \omega - d''\gamma$$

Set

$$\sigma := \gamma + \eta \in \mathscr{E}^{p,q-1}(\Delta')$$

On Δ'

$$d''\sigma = d''\gamma + d''\eta = \omega$$

which finishes the induction step and terminates the proof. □

The proof of Proposition 1.25 introduces the intermediate polydisc Δ'' to obtain the functions $\tilde{a}_{IJ}$ with compact support. This principle ”obtaining solutions after *shrinking*” has its counterpart in the principle ”extending solutions by *exhaustion*”, see Proposition 1.26. We will see more examples of these two principles in subsequenct passages.

Proposition 1.25 solves the d''-problem on relatively compact polydiscs. The standard method to extend these local solutions to a global solution on arbitrary polydiscs is to exhaust the latter by relatively compact polydiscs. The Mittag-Leffler principle of induction, Proposition 1.26, applies. It extends the local solutions after small changes during each step to a global solution. The global solution is the projective limit of a suitable sequence of compatible local solutions. The existence of the global solution of the d''-problem will be the *Dolbeault lemma* from Theorem 1.27.

Proposition 1.26 (Mittag-Leffler principle of exhaustion). *Consider a sequence*

$$(M_i, d_i)_{i \in \mathbb{N}}$$

of complete metric spaces, $M_0 \neq \emptyset$*, together with continuous maps*

$$\rho_i : M_i \to M_{i-1},\ i \geq 1$$

with dense image. Then the projective limit satisfies

$$\varprojlim (M_i, \rho_i) \neq \emptyset$$

i.e. there exists an infinite ascending sequence of elements

$$(f_i \in M_i)_{i \in \mathbb{N}}$$

which satisfy the compatibility condition

$$\rho(f_i) = f_{i-1},\ i \geq 1.$$

The idea of the proof is to construct the required sequence step by step by induction. Step $k-1$ constructs a chain of lenght $k-1$. At step k each element of the chain is corrected by a difference of order at most $1/2^k$, such that the resulting chain can be continued further by one element, see Figure 1.5.

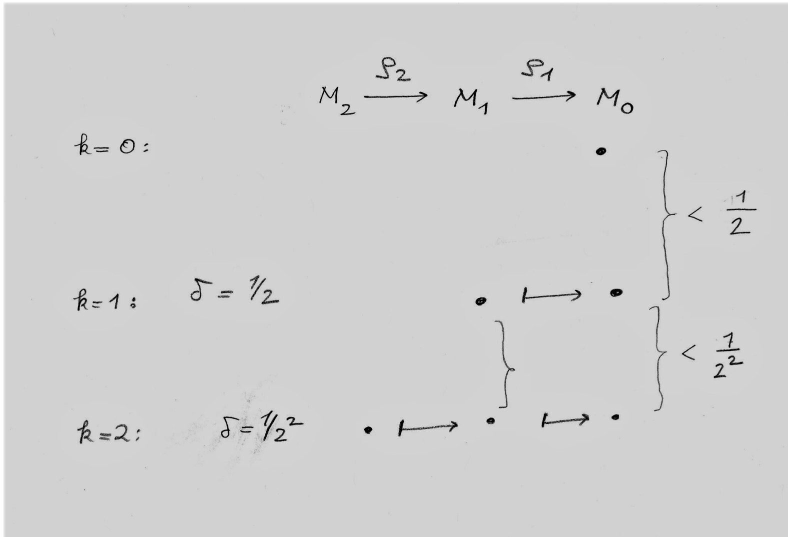


Fig. 1.5 Mittag-Leffer induction, constructions for $k = 0, k = 1$ and $k = 2$

Proof. We prove by induction on $k \in \mathbb{N}$ the statement $\mathscr{A}(k)$: There exists a chain $(g_i^k)_{i=0,...,k}$ of length k of compatible elements $g_i^k \in M_i,\ i = 0,...,k$, satisfying

$$d_*(g_*^k, g_*^{k-1}) < 1/2^k,$$

i.e. for $i = 0,...,k-1$ holds

$$d_i(g_i^k, g_i^{k-1}) < 1/2^k.$$

i) *Induction start* $k = 0$: Choose an arbitrary element $g_0^0 \in M_0$.

ii) *Induction step* $k-1 \mapsto k$: Consider the chain of length $k-1$ formed by the elements

$$(g_i^{k-1})_{i=0,...,k-1}$$

from the induction assumption. Due to the continutity of the restriction maps ρ_i and the resulting continuity of their finite composition there exists

$$0 < \delta < 1/2^k$$

such that each element in the δ-neighbourhood of the final element

$$g_{k-1}^{k-1} \in M_{k-1}$$

maps into the $(1/2^k)$-neighbourhoods of the elements

$$g_i^{k-1},\ i = 0,...,k-2,$$

by the corresponding composition. Because

$$\rho_k : M_k \to M_{k-1}$$

has dense image, one may choose as the seed of the new chain of length k an element

$$g_k^k \in M_k \text{ with } \rho_k(g_k^k) \in M_{k-1}$$

in the δ-neighbourhood of $g_{k-1}^{k-1} \in M_{k-1}$. Defining the elements

$$g_i^k \in M_i,\ i = 0, ..., k-1,$$

as the iterated images of g_k^k ends the induction step. □

Theorem 1.27 (Dolbeault's lemma). *Consider an open polydisc $\Delta \subset \mathbb{C}^n$. For each $p \in \mathbb{N}$ the following sequence of complex vector spaces and morphisms is exact*

$$0 \to \Omega^p(\Delta) \to \mathscr{E}(\Delta)^{p,0} \xrightarrow{d''} \mathscr{E}(\Delta)^{p,1} \xrightarrow{d''} ... \xrightarrow{d''} \mathscr{E}(\Delta)^{p,n} \to 0$$

Exactness of the sequence means that the kernel of each morphism equals the image of the previous morphism. In Theorem 1.27 we relax the condition on the polyradius

$$r = (r_1, ..., r_n)$$

and allow also the component radius $0 < r_j = \infty$.

We illustrate the idea of the proof of Theorem 1.27 by the special case of the exactness

$$\mathscr{E}^{p,0}(\Delta) \xrightarrow{d''} \mathscr{E}^{p,1}(\Delta) \xrightarrow{d''} \mathscr{E}^{p,2}(\Delta)$$

Consider a given

$$\omega \in \mathscr{E}^{p,1}(\Delta) \text{ with } d''\omega = 0$$

The polydisc $\Delta \subset \mathbb{C}^n$ is not necessarily relatively compact. But there exists an exhaustion of Δ by a sequence $(\Delta_i)_{i\in\mathbb{N}}$ of relatively compact polydiscs $\Delta_{i-1} \subset\subset \Delta_i$. For each $i \in \mathbb{N}$ Proposition 1.25 provides a local solution σ_i on Δ_i satisfying

$$d''\sigma_i = \omega|\Delta_i$$

The family $(\sigma_i)_i$ faces two problems: First, the members of the family are not necessarily compatible with respect to the restriction from Δ_i to Δ_{i-1}. Secondly, in general a given form σ_{i-1} does not even extend approximatively to Δ_i, because the smooth σ_{i-1} has no approximation by Taylor polynomials. The local solutions σ_i transform the second problem into a question about holomorphic p-form on Δ_i. To solve the first problem one shows that the resulting twisted maps

$$\Omega^p(\Delta_i) \to \Omega^p(\Delta_{i-1})$$

have dense image due to the Taylor approximation of the holomorphic coefficient functions. Then the Mittag-Leffler principle of induction solves the problem of compatible extension.

Proof (Theorem 1.27). Recall from Remark 1.23: Each vector space $\mathscr{E}^{p,q}(P)$, P polydisc carries a Fréchet topology, hence it is a complete metric space. Exterior differentiation satisfies

$$d''|\Omega^p = 0 \text{ and } d'' \circ d'' = 0.$$

Hence it remains to show that each closed form is also exact, i.e. if

$$\omega \in \mathscr{E}^{p,q}(\Delta) \text{ with } d''\omega = 0$$

then

- if $q = 0$
$$\omega \in \Omega^p(\Delta)$$
- respectively if $q \geq 1$ exists
$$\sigma \in \mathscr{E}^{p,q-1}(\Delta) \text{ with } \omega = d''\sigma.$$

Choose an arbitrary but fixed $p \in \mathbb{N}$. Exactness at the index $q = 0$ is well known, hence we prove exactness at a given index $q \geq 1$.

i) *Exhaustion by relatively compact concentric polydiscs*: We choose an exhaustion of Δ by a sequence of relatively compact concentric polydiscs

$$\Delta_0 \subset\subset \Delta_1 \subset\subset \ldots \subset\subset \Delta_i \subset\subset \ldots$$

For given $q \geq 1$

$$\omega \in \mathscr{E}^{p,q}(\Delta) \text{ with } d''\omega = 0$$

we consider for each $i \in \mathbb{N}$ the affine vector subspace of $\mathscr{E}^{p,q-1}(\Delta_i)$

$$M_i := M_{i,q} := \left\{\sigma \in \mathscr{E}^{p,q-1}(\Delta_i) : \ d''\sigma = \omega|\Delta_i\right\}, \ i \in \mathbb{N},$$

of solutions over $\Delta_i \subset\subset \Delta$. Due to Proposition 1.25 we can choose for each $i \in \mathbb{N}$ a fixed local solution

$$\sigma_i \in M_i.$$

For $i \geq 1$ let

$$\rho_i : M_i \to M_{i-1}$$

denote the canonical restriction. The proof of the claim is by induction on $q \geq 1$.

ii) *Induction start* $q = 1$: We show for each $i \in \mathbb{N}$ the exactness of

$$\mathscr{E}^{p,0}(\Delta_i) \xrightarrow{d''} \mathscr{E}^{p,1}(\Delta_i) \xrightarrow{d''} \mathscr{E}^{p,2}(\Delta_i)$$

The map

$$\psi_i : \Omega^p(\Delta_i) \to M_i,\ \eta \mapsto \sigma_i + \eta,$$

represents the space M_i of solutions on Δ_i as the affine subspace

$$M_i = \sigma_i + \Omega^p(\Delta_i) \subset \mathscr{E}^{p,0}(\Delta_i):$$

Each element $\sigma \in M_i$ has the form

$$\sigma = \sigma_i + (\sigma - \sigma_i)$$

with

$$\eta := \sigma - \sigma_i \in \Omega^p(\Delta_i) \text{ satisfying } d''\eta = 0.$$

Due to Remark 1.23 each vector space $\Omega^p(\Delta_i)$ carries a Fréchet topology, hence it is a complete metric space. For $i \geq 1$ consider the continuous maps between complete metric spaces

$$\tau_i : \Omega^p(\Delta_i) \to \Omega^p(\Delta_{i-1}),\ \eta \mapsto (\eta + (\sigma_i - \sigma_{i-1}))|\Delta_{i-1}$$

Each map τ_i has dense image: The holomorphic form

$$(\sigma_i - \sigma_{i-1})|\Delta_{i-1} \in \Omega^p(\Delta_{i-1})$$

can be approximated by using the Taylor polynomials of its holomorphic coefficients. Hence each holomorphic form $\sigma \in \Omega^p(\Delta_{i-1})$ can be approximated by using the Taylor polynomials of the coefficents of the holomorphic form

$$(\sigma - (\sigma_i - \sigma_{i-1}))|\Delta_{i-1} \in \Omega^p(\Delta_{i-1}).$$

The following diagram commutes

$$\begin{array}{ccc} \Omega^p(\Delta_i) & \xrightarrow{\tau_i} & \Omega^p(\Delta_{i-1}) \\ \psi_i \downarrow & & \downarrow \psi_{i-1} \\ M_i & \xrightarrow{\rho_i} & M_{i-1} \end{array}$$

The Mittag-Leffler principle, Proposition 1.26, applies to the family

$$(\Omega^p(\Delta_i), \tau_i)_{i \in \mathbb{N}}$$

It provides a compatible family

$$(\eta_i \in \Omega^p(\Delta_i))_{i \in \mathbb{N}} \text{ with } \tau_i(\eta_i) = \eta_{i-1}$$

For each $i \in \mathbb{N}$ define

$$\tilde{\sigma}_i := \psi_i(\eta_i) = \sigma_i + \eta_i \in M_i$$

obtaining the compatibility of the adapted solutions $\tilde{\sigma}_i$

$$\tilde{\sigma}_i|\Delta_{i-1} = (\rho_i \circ \psi_i)(\eta_i) = (\psi_{i-1} \circ \tau_i)(\eta_i) = \psi_{i-1}(\eta_{i-1}) = \tilde{\sigma}_{i-1}$$

The family $(\tilde{\sigma}_i)_{i\in\mathbb{N}}$ defines the form

$$\sigma \in \mathscr{E}^{p,0}(\Delta) \text{ satisfying } \sigma|\Delta_i = \tilde{\sigma}_i,\ i \in \mathbb{N}, \text{ and } d''\sigma = \omega.$$

iii) *Induction step* $q-1 \mapsto q,\ q \geq 2$: We show for each $i \in \mathbb{N}$ the exactness of

$$\mathscr{E}^{p,q-1}(\Delta_i) \xrightarrow{d''} \mathscr{E}^{p,q}(\Delta_i) \xrightarrow{d''} \mathscr{E}^{p,q+1}(\Delta_i)$$

Consider the space of local solutions

$$M_i := M_{i,q} := \left\{\sigma \in \mathscr{E}^{p,q-1}(\Delta_i) :\ d''\sigma = \omega|\Delta_i\right\},\ i \in \mathbb{N},$$

Any pair of local solutions $\sigma_1,\ \sigma_2 \in M_i$ satisfies

$$d''(\sigma_1 - \sigma_2) = 0$$

Hence by induction assumption there exists $\eta \in \mathscr{E}^{p,q-2}(\Delta_i)$ with

$$d''\eta = \sigma_1 - \sigma_2$$

As a consequence, the space M_i is the image of the map

$$\psi_i : \mathscr{E}^{p,q-2}(\Delta_i) \to \mathscr{E}^{p,q-1}(\Delta_i),\ \eta \mapsto \sigma_i + d''\eta,$$

namely the affine subspace

$$M_i = \sigma_i + d''\mathscr{E}^{p,q-2}(\Delta_i) \subset \mathscr{E}^{p,q-1}(\Delta_i)$$

The map ψ_i is surjective: For a given element $\sigma \in M_i$ the form

$$\delta := \sigma - \sigma_i \in \mathscr{E}^{p,q-1}(\Delta_i)$$

satisfies

$$d''\delta = 0.$$

By induction hypothesis exists

$$\eta \in \mathscr{E}^{p,q-2}(\Delta_i) \text{ with } d''\eta = \delta.$$

Therefore

$$\psi_i(\eta) = \sigma_i + d''\eta = \sigma_i + \delta = \sigma.$$

The element

$$\sigma_i - \sigma_{i-1} \in \mathscr{E}^{p,q-1}(\Delta_{i-1}) \text{ satisfies } d''(\sigma_i - \sigma_{i-1}) = 0$$

Hence the induction hypothesis provides a form

$$\gamma_{i-1} \in \mathscr{E}^{p,q-2}(\Delta_{i-1})$$

with

$$d''\gamma_{i-1} = \sigma_i - \sigma_{i-1} \in \mathscr{E}^{p,q-1}(\Delta_{i-1})$$

For each $i \geq 1$ consider the "twisted" continuous map between Fréchet spaces, in particular between complete metric spaces,

$$\tau_i : \mathscr{E}^{p,q-2}(\Delta_i) \to \mathscr{E}^{p,q-2}(\Delta_{i-1}),\ \eta \mapsto \eta|\Delta_{i-1} + \gamma_{i-1},$$

The map has dense image: For each compact set $K \subset \Delta_{i-1}$ choose a smooth function $s \in \mathscr{E}(\Delta_i)$ with

$$supp\ s \subset \Delta_{i-1} \text{ and } s|K = 1.$$

Multiplying by s allows to extend smooth functions on Δ_{i-1} to smooth functions on Δ_i without changing their values on K. The following diagram commutes

$$\begin{array}{ccc} \mathscr{E}^{p,q-2}(\Delta_i) & \xrightarrow{\tau_i} & \mathscr{E}^{p,q-2}(\Delta_{i-1}) \\ \psi_i \downarrow & & \downarrow \psi_{i-1} \\ M_i & \xrightarrow{\rho_i} & M_{i-1} \end{array}$$

The Mittag-Leffler principle, Proposition 1.26, applies to the family

$$(\mathscr{E}^{p,q-2}(\Delta_i),\ \tau_i)_{i\in\mathbb{N}}.$$

It provides a compatible family

$$(\eta_i \in \mathscr{E}^{p,q-2}(\Delta_i))_{i\in\mathbb{N}} \text{ with } \tau_i(\eta_i) = \eta_{i-1}$$

Hence

$$(\sigma_i + d''\eta_i)|\Delta_{i-1} = (\rho_i \circ \psi_i)(\eta_i) = (\psi_{i-1} \circ \tau_i)(\eta_i) = \psi_{i-1}(\eta_{i-1}) = \sigma_{i-1} + d''\eta_{i-1}$$

and the family determines a form

$$\sigma \in \mathscr{E}^{p,q-1}(\Delta) \text{ with } \sigma|\Delta_i = \sigma_i + d''\eta_i,\ i \in \mathbb{N}.$$

As a consequence

$$d''\sigma = \omega.$$

□

Chapter 2
Complex manifolds and sheaves

2.1 Complex manifolds

Definition 2.1 (Topological manifold, chart, complex atlas and complex structure).

1. A *topological manifold* X of real dimension $= k$ is a topological Hausdorff space X such that each point $x \in X$ has an open neighbourhood U with a homeomorphism, named a *chart* around x,

$$\phi : U \xrightarrow{\simeq} V$$

onto an open set $V \subset \mathbb{R}^k$.

2. A *complex atlas* of a topological manifold X of real dimension $2n$, i.e. complex dimension $= n$, is a family $\mathscr{A}$ of charts

$$\mathscr{A} = (\phi_i : U_i \to V_i)_{i \in I}$$

with open subsets

$$V_i \subset \mathbb{C}^n \simeq \mathbb{R}^{2n},$$

such that

•

$$X = \bigcup_{i \in I} U_i$$

• and for all pairs $i, j \in I$ and

$$U_{ij} := U_i \cap U_j$$

the *transition map* of the two charts

$$\psi_{ij} := \phi_i \circ (\phi_j | U_{ij})^{-1} : \phi_j(U_{ij}) \to \phi_i(U_{ij}) \subset \mathbb{C}^n$$

is holomorphic, i.e. each component function of ψ_{ij} is holomorphic on the open set $\phi_j(U_{ij}) \subset \mathbb{C}^n$, see Figure 2.1.

- Two complex atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ of X are *compatible* if their union

$$\mathscr{A}_1 \cup \mathscr{A}_2$$

is again a complex atlas. A maximal set of complex, biholomorphically compatible atlases of X is a *complex structure* Σ on X.

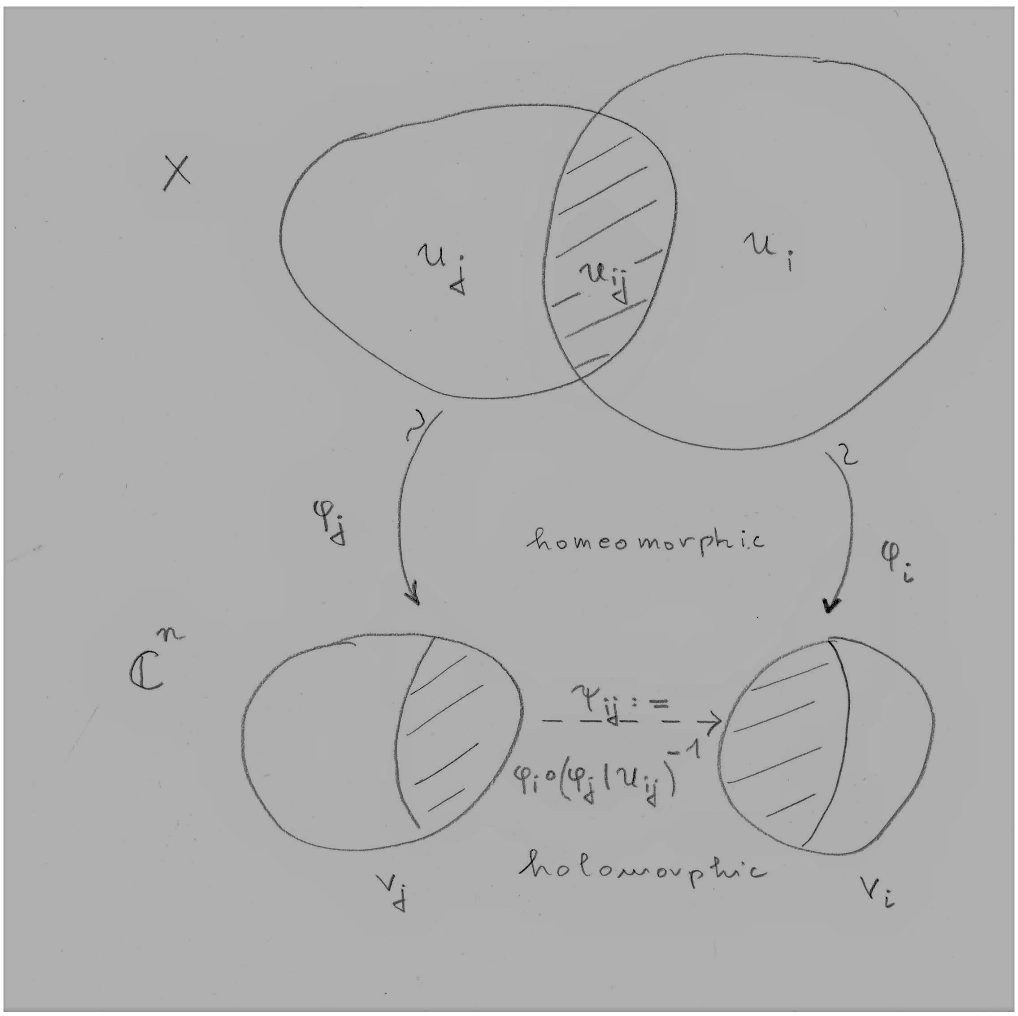


Fig. 2.1 Charts of an atlas defining a complex structure

Definition 2.2 (Complex manifold, holomorphic map).

1. An n-dimensional *complex manifold* is a pair (X,Σ) with a connected, topological manifold X of real dimension $2n$, which has a countable base of the topology, and a complex structure Σ on X.

2. A continuous map
$$f:(X,\Sigma_X)\to(Y,\Sigma_Y)$$
between two complex manifolds is *holomorphic* if for each point $x\in X$ exists a chart around x from an atlas of Σ_X
$$\phi:U\to V,$$
and a chart around $f(x)$ from an atlas of Σ_Y
$$\psi:S\to T,$$
such that $f(U)\subset S$, and the composition
$$\psi\circ f\circ(\phi|U\cap f^{-1}(S))^{-1}:\phi(U\cap f^{-1}(S))\to T\subset\mathbb{C}^m,\ m:=dim_{\mathbb{C}}\,Y,$$
is holomorphic. Note that the definition is independent from the choice of the charts.

3. A holomorphic map $f:X\to Y$ is *biholomorphic* or an isomorphism iff there exists a holomorphic map $g:Y\to X$ such that both maps satisfy
$$g\circ f=id_X \text{ and } f\circ g=id_Y$$

4. A *holomorphic function* on X is a holomorphic map
$$f:(X,\Sigma_X)\to\mathbb{C}.$$
Here $\mathbb{C}$ is considered a complex manifold in the canonical way. For each open set $U\subset X$ the ring of all holomorphic functions on U with respect to addition and multiplication is denoted $\mathscr{O}_X(U)$ or $\mathscr{O}(U)$ for short.

Remark 2.3 (Fundamentals from topology).

1. An open covering $\mathscr{U}$ of a topological space X is *locally finite* if every point $x\in X$ has an open neighbourhood V which meets only finitely many sets of the covering.

2. A Hausdorff space X is *paracompact* if every open covering of X has a locally finite subcovering. Each locally finite covering of a topological space has a subordinate *partition of unity*, see [4, Chap. VIII, Theor. 4.2]

3. Each locally compact, second countable topological space is paracompact.

4. Consider a Hausdorff space X. A subset $U \subset X$ is *relatively compact* in X if the closure $\overline{U} \subset X$ is compact. For two subsets $U_1, U_2 \subset X$ the notation
$$U_1 \subset\subset U_2$$
means: U_1 is relatively compact in X and $\overline{U}_1 \subset U_2$.

5. Each locally-compact topological space X has an *exhaustion* $(U_\nu)_{\nu\in\mathbb{N}}$ by relatively compact open subsets, i.e.
$$U_\nu \subset\subset U_{\nu+1}, \nu \in \mathbb{N}, \text{ and } X = \bigcup_{\nu\in\mathbb{N}} U_\nu.$$

6. Each regular, second countable space is metrizable, see [4, Chap. IX, Cor. 9.2].

7. Each subspace of a metric space is a metric space with respect to the restricted metric.

8. Each metric space is paracompact, see [4, Chap. IX, Theor. 5.3].

9. Each locally-finite covering $\mathscr{U} = (U_i)_{i\in I}$ of a paracompact topological space X has a *shrinking*
$$\mathscr{V} = (V_i)_{i\in I},$$
i.e. the open covering $\mathscr{V}$ satisfies for all $i \in I$
$$V_i \subset\subset U_i,$$
see [29, Satz 9.2.1] or [4, Theor. VII.6.1].

Remark 2.4 (Holomorphic versus smooth).

1. Often one uses for a chart of an n-dimensional complex manifold (X, Σ) the suggestive notation
$$z : U \to V \subset \mathbb{C}^n.$$
Then the decomposition into real part and imaginary part
$$z_\nu = x_\nu + i \cdot y_\nu,\ \nu = 1, ..., n,$$
and the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ defines a chart

$$(x_1, y_1, ..., x_n, y_n) : U \to V \subset \mathbb{R}^{2n}$$

of a *smooth structure* Σ_{smooth} on X: When considering a holomorphic transition function ψ as a function of $2n$ real variables then g has partial derivatives of arbitrary order. Hence the transition function is smooth, i.e. differentiable of class C^∞, and the complex structure Σ induces a smooth structure Σ_{smooth} on X, such that

$$(X, \Sigma_{smooth})$$

is a $2n$-dimensional paracompact smooth manifold.

2. If (X, Σ) is a complex manifold then a map

$$f : X \to \mathbb{C}$$

is *smooth*, if f is smooth on (X, Σ_{smooth}). For an open set $U \subset X$ the ring of all smooth functions on U with respect to addition and multiplication is denoted $\mathscr{E}_X(U)$, or $\mathscr{E}(U)$ for short.

In the following we will denote a complex manifold (X, Σ) simply by X if the details of the complex structure Σ are not relevant.

We now investigate holomorphic maps

$$f : X \to Y$$

between two complex manifolds from a local point of view. The basic objects at a point $x \in X$ are

- the *tangent spaces*

$$T_xX \text{ and } T_{f(x)}Y,$$

  of X at $x \in X$ resp. Y at $f(x) \in Y$, complex vector spaces of dimension equal to $dim_{\mathbb{C}}\, X$ resp. $dim_{\mathbb{C}}\, Y$,

- and the *tangent map* at x

$$T_xf : T_xX \to T_{f(x)}Y,$$

  the complex-linear map which approximates f in a neighbourhood of x.

The local properties of f at a point $a \in X$ derive from the corresponding properties of the tangent map

$$T_xf : T_xX \to T_{f(x)}Y$$

The map f is

- a *local isomorphism*, if it maps sufficiently small open neighbourhoods of a point isomorphically onto neighbourhoods of the image point,

- an *immersion*, if it maps sufficiently small neighbourhoods of a point isomorphically onto a coordinate slice of the image point,

- and a *submersion*, if it splits sufficiently small neighbourhoods of a point in the domain of f as a product, and projects the product onto a neighbourhood of the image point.

Definition 2.5 formalizes these properties, while Proposition 2.6 relates them to the rank of the tangent map.

Definition 2.5 (Local isomorphism, immersion, submersion). Consider a holomorphic map

$$f : X \to Y$$

between two complex manifolds and a point $x \in X$. The map f is named

- *local isomorphism* at x if open neighbourhoods

  $$U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y$$

  exist such that the injections

  $$j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y$$

  extend to the commutative diagram

  $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle j_U}\uparrow & & \uparrow{\scriptstyle j_V} \\ U & \underset{f|U}{\overset{\simeq}{\dashrightarrow}} & V \end{array}$$

  with the biholomorphic restriction

  $$f|U : U \xrightarrow{\simeq} V,$$

- *immersion* at x if open neighbourhoods

  $$U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y, W \text{ of } 0 \text{ in } K^m$$

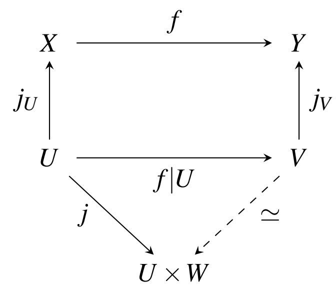
  exist such that the injections

  $$j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y,$$

  and the embedding onto a coordinate slice

  $$j = [U \xrightarrow{\simeq} U \times \{0\} \hookrightarrow U \times W]$$

extend to a commutative diagram



with a biholomorphic map

$$U \times W \xrightarrow{\simeq} V,$$

- *submersion* at x if open neighbourhoods

$$U \text{ of } x \text{ in } X, V \text{ of } f(x) \text{ in } Y, W \text{ of } 0 \text{ in } K^m$$

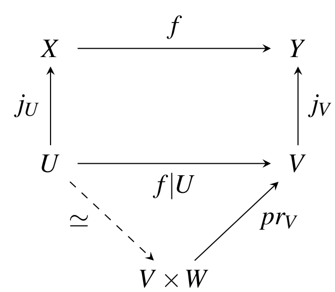
exist such that the injections

$$j_U : U \hookrightarrow X, j_V : V \hookrightarrow Y,$$

and the projection

$$pr_V : V \times W \to V$$

extend to a commutative diagram



with a biholomorphic map

$$U \xrightarrow{\simeq} V \times W,$$

- respectively *local isomorphism*, *immersion*, *submersion* if the corresponding local property holds for each point $x \in X$, see Figure 2.2.

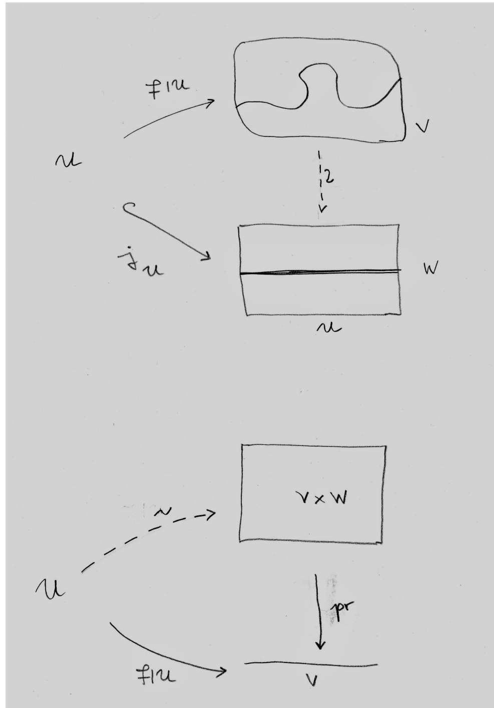


Fig. 2.2 Immersion (top) and submersion (bottom)

Proposition 2.6 (Local isomorphism, immersion, submersion). *A holomorphic map*

$$f: X \to Y$$

between two complex manifolds is at a given point $x \in X$

- *a local isomorphism iff the tangent map $T_x f$ is bijective.*
- *an immersion iff the tangent map $T_x f$ is injective.*
- *respectively a submersion iff the tangent map $T_x f$ is surjective.*

For the proof see [28, Part II, Chap. III.9].

The composition of two immersions is an immersion. The composition of two submersions is a submersion.

Definition 2.7 (Embedding). Consider a holomorphic map

$$f : X \to Y$$

between two complex manifolds and provide the set $f(X) \subset Y$ with the subspace topology.

- The map is an *embedding* if f is an immersion and the induced map $X \to f(X)$ is a homeomorphism.

- If $f(X) \subset Y$ is also closed, then f is named a *closed embedding*.

Theorem 2.8 (Analytic submanifold). *Consider an open set $U \subset \mathbb{C}^n$ and a subset*

$$A \subset U,$$

which is closed with respect to the induced topology of U. For a number $k \in \mathbb{N}$ the following statements are equivalent:

1. Zero set*: Each point $a \in A$ has an open neighbourhood $V \subset U$ and holomorphic functions*
$$f_1, ..., f_{n-k} \in \mathscr{O}(V)$$
such that
$$A \cap V = \{z \in V : \ f_1(z) = ... = f_{n-k}(z) = 0\}$$
and for all $x \in A \cap V$
$$rank \left(\frac{\partial f_\mu}{\partial z_\nu}\right) = n - k$$
2. Parameter representation*: For each point $a \in A$ exists an open neighbourhood $V \subset U$, and an open set $T \subset \mathbb{C}^k$ with holomorphic functions*
$$\phi_1, ..., \phi_k \in \mathscr{O}(T)$$
such that
$$\phi := (\phi_1, ..., \phi_k) : T \to A \cap V \subset \mathbb{C}^n$$
is a homeomorphism satisfying for all $t \in T$
$$rank \left(\frac{\partial \phi_\nu}{\partial t_j}\right) = k$$
3. Locally a k-dimensional affine plane*: For each point $a \in A$ exists an open neighbourhood $V \subset U$, and an open neighbourhood W of the origin in $\mathbb{C}^n$ with a biholomorphic map*
$$\Phi : V \to W$$

satisfying

$$\Phi(A \cap V) = W \cap \{z = (z_1, ..., z_n) \in \mathbb{C}^n : \ z_{k+1} = ... = z_n = 0\}$$

For a proof see [9, § 14]. The proof is analogous to the case of smooth manifolds.

Definition 2.9 (Analytic submanifold). Consider an open set $U \subset \mathbb{C}^n$ and a subset $A \subset U$, closed with respect to the induced topology of U. Then A is an *analytic submanifold* of U, if $A \subset U$ satisfies the equivalent properties from Theorem 2.8.

Theorem 2.8, part 3) shows that each analytic submanifold of an open set $U \subset \mathbb{C}^n$ is a complex manifold. The number $k \in \mathbb{N}$ from Theorem 2.8 is the dimension of A.

2.2 Sheaves and their stalks

We define presheaves (presheaf = Prägarbe) and sheaves (sheaf = Garbe) of Abelian groups first. But the definition and results transfer to other objects of Abelian categories, i.e. to commutative rings R or R-modules and also to the category of sets.

Definition 2.10 (Presheaf of Abelian groups).

1. A *presheaf $\mathscr{F}$ of Abelian groups* on a topological space X is a family of Abelian groups

   $$\mathscr{F}(U),\ U \subset X \text{ open},$$

   and for each pair $V \subset U$ of open subsets of X a homomorphism of Abelian groups

   $$\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$$

   satisfying:

   $$\rho_U^U = id_{\mathscr{F}(U)}$$

   and

   $$\rho_W^V \circ \rho_V^U = \rho_W^U \text{ for } W \subset V \subset U.$$

   The maps ρ_V^U are often named *restrictions* and denoted

   $$f|V := \rho_V^U(f)$$

   for $f \in \mathscr{F}(U)$, $V \subset U$ open.

   The elements of a given Abelian group

$$\mathscr{F}(U),\ U \subset X \text{ open},$$

are named the *sections* of $\mathscr{F}$ on U.

2. A *morphism*

$$f : \mathscr{F} \to \mathscr{G}$$

between two presheaves of Abelian groups with families of restrictions

$$\rho = (\rho_V^U)_{U,V} \text{ resp. } \sigma = (\sigma_V^U)_{U,V}$$

is a family of group homomorphisms

$$f_U : \mathscr{F}(U) \to \mathscr{G}(U),\ U \subset X \text{ open},$$

such that for any pair $V \subset U$ of open subsets of X the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ \rho_V^U \downarrow & & \downarrow \sigma_V^U \\ \mathscr{F}(V) & \xrightarrow{f_V} & \mathscr{G}(V) \end{array}$$

Remark 2.11 (Presheaf as a functor). Consider a fixed topological space X. Denote by $\underline{X}$ the category of open subsets of X:

- Objects of $\underline{X}$ are the open sets $U \subset X$
- and morphisms exist only for pairs of open sets $V \subset U$, and here the only morphism is the injection $V \hookrightarrow U$, i.e. for open sets $U,\ V \subset X$

$$Mor(V,U) := \begin{cases} \{V \hookrightarrow U\} & V \subset U \\ \emptyset & otherwise; \end{cases}$$

Then the presheaves $\mathscr{F}$ of Abelian groups on X are exactly the contravariant functors

$$\mathscr{F} : \underline{X} \to \underline{Ab}$$

to the category $\underline{Ab}$ of Abelian groups. A morphism

$$\mathscr{F} \to \mathscr{G}$$

between two presheaves is a functor morphism (natural transformation) from $\mathscr{F}$ to $\mathscr{G}$.

In general the concept of a presheaf is too weak to support any strong result on complex manifolds. The stronger concept is a *sheaf*. It satisfies two additional sheaf axioms. According to these conditions local sections which coincide on their common domain of definition glue to a unique global section.

Definition 2.12 (Sheaf). Consider a topological space X. A *sheaf* $\mathscr{F}$ of Abelian groups on X is a presheaf of Abelian groups on X, which satisfies the following two sheaf axioms:

For each open $U \subset X$ and for each open covering $\mathscr{U} = (U_i)_{i\in I}$ of U:

1. If two elements $f,\ g \in \mathscr{F}(U)$ satisfy for all $i \in I$
$$f|U_i = g|U_i$$
then
$$f = g,$$
i.e. local equality implies global equality.

2. If a family
$$f_i \in \mathscr{F}(U_i),\ i \in I,$$
satisfies for all $i, j \in I$
$$f_i|U_i \cap U_j = f_j|U_i \cap U_j$$
then an element $f \in \mathscr{F}(U)$ exists satisfying for all $i \in I$
$$f|U_i = f_i,$$
i.e. local sections which agree on the intersections glue to a global element.

A *morphism of sheaves* is a morphism of the underlying presheaves.

If one paraphrases a presheaf as a tool to handle *local* objects, then a sheaf is a tool to glue local objects, which fit together, in order to construct a unique *global* object. If it is not possible to make the parts fit, then cohomology theory is a means to measure the *obstructions*, see Chapter 3.

Definition 2.13 (Subsheaf of Abelian groups). Consider a presheaf $\mathscr{F}$ of Abelian groups on a topological space X.

1. A presheaf of Abelian groups $\mathscr{G}$ on X is a *subpresheaf* of $\mathscr{F}$
   - if for all open sets $U \subset X$
   $$\mathscr{G}(U) \subset \mathscr{F}(U)$$
   is a subgroup, and

- if the restriction maps of $\mathscr{G}$ are induced by the restriction maps of $\mathscr{F}$.

2. If $\mathscr{F}$ is a sheaf, then a sheaf $\mathscr{G}$ is a *subsheaf* of $\mathscr{F}$ if $\mathscr{G}$ is a subpresheaf of $\mathscr{F}$.

Similar to presheaves and sheaves of Abelian groups one defines presheaves and sheaves with other algebraic structures, e.g. rings or modules, see Definition 4.25.

Example 2.14 (Sheaves and presheaves).

1. Let X be a topological space.
   - *Sheaf $\mathscr{C}$ of continuous functions*: For any open set $U \subset X$ define
   $$\mathscr{C}(U) := \{f : U \to \mathbb{C} \mid f \text{ continuous}\}$$
   as the complex vector space of complex-valued, continuous functions on U. The presheaf
   $$\mathscr{C}(U),\ U \subset X \text{ open},$$
   with the restriction of functions
   $$\rho_V^U : \mathscr{C}(U) \to \mathscr{C}(V),\ f \mapsto f|V,\ V \subset U,$$
   is a sheaf. It is named the sheaf $\mathscr{C}$ of continuous functions on X.
   - *Sheaf $\mathbb{Z}$ of locally constant functions*: Consider a topological space X, and provide $\mathbb{Z}$ with its discrete topology. For each open $U \subset X$ define
   $$\mathscr{F}(U) := \{f : U \to \mathbb{Z} \mid f \text{ constant}\}$$
   and take the canonical restriction morphisms. The family
   $$\mathscr{F} := \mathscr{F}(U),\ U \subset X \text{ open},$$
   is a presheaf.

   In general, the presheaf $\mathscr{F}$ is not a sheaf: Assume
   $$X = X_1 \dot{\cup} X_2$$
   with two connected components. Then the family
   $$(f_1, f_2) \in \mathscr{F}(X_1) \times \mathscr{F}(X_2)$$
   with
   $$f_1 := 1 \in \mathbb{Z} \text{ and } f_2 := 2 \in \mathbb{Z}$$
   does not arise from a global section

$$f \in \mathscr{F}(X) = \mathbb{Z}$$

as

$$f_1 = f|X_1 \text{ and } f_2 = f|X_2$$

with a constant section $f \in \mathscr{F}(X)$.

A slight change in the definition of $\mathscr{F}$ provides a sheaf on X: A function on an open set $U \subset X$ is *locally constant* if each point $x \in U$ has a neighbourhood V, such that the restriction $f|V$ is constant. One defines

$$\mathbb{Z}(U) := \{f : U \to \mathbb{Z} \mid f \text{ locally constant}\}.$$

Then

$$\mathbb{Z}(U),\ U \subset X \text{ open},$$

with the canonical restrictions is a sheaf. The sheaf is often denoted $\mathbb{Z}$ like the ring of integers. The context has to clarify whether the symbol denotes the ring of integers or the sheaf of locally constant integer-valued functions.

Similarly one defines the sheaf $\mathbb{C}$ of locally constant complex-valued functions. Note that both sheaves $\mathbb{Z}$ and $\mathbb{C}$ are named *constant* sheaves - although they are defines by locally constant sections.

2. Let X be a complex manifold.

- *Sheaf $\mathscr{O}$ of holomorphic functions*: Consider for each open $U \subset X$ the ring

  $$\mathscr{O}(U) := \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$$

  the ring of holomorphic functions on U. The presheaf

  $$\mathscr{O}(U),\ U \subset X \text{ open},$$

  with the canonical restriction of functions is a sheaf of rings. It is named the sheaf $\mathscr{O}$ of holomorphic functions on X or the *structure sheaf* of X.

- *Sheaf $\mathscr{O}^*$ of holomorphic functions without zeros*: Consider for each open $U \subset X$ the multiplicative Abelian group

  $$\mathscr{O}^*(U) := \{f \in \mathscr{O}(U) : f(x) \neq 0 \text{ for all } x \in U\}.$$

  The presheaf

  $$\mathscr{O}^*(U),\ U \subset X \text{ open},$$

  with the canonical restriction of functions is a sheaf. It is named the sheaf $\mathscr{O}^*$ of holomorphic functions without zeros on X. Apparently $\mathscr{O}^*$ is the sheaf of units of $\mathscr{O}$.

- *Sheaf $\mathscr{E}$ of smooth functions*: Consider for each open $U \subset X$ the ring

$$\mathscr{E}(U) := \{f : U \to \mathbb{C} \mid f \text{ smooth}\}$$

The presheaf

$$\mathscr{E}(U),\ U \subset X \text{ open},$$

with the canonical restriction of functions is a sheaf of rings. It is named the sheaf $\mathscr{E}$ of smooth functions on X or the *smooth structure sheaf*.

Definition 2.15 (Stalk of a presheaf). Consider a presheaf $\mathscr{F}$ of Abelian groups on a topological space X, and a point $x \in X$. The *stalk* $\mathscr{F}_x$ of $\mathscr{F}$ at x is the set of equivalence classes with respect to the following equivalence relation on the union of all $\mathscr{F}(U)$, U open neighbourhood of x:

$$f_1 \in \mathscr{F}(U_1) \sim f_2 \in \mathscr{F}(U_2)$$

if for a suitable open neighbourhood V of x with $V \subset U_1 \cap U_2$

$$f_1|V = f_2|V.$$

Apparently, the stalk $\mathscr{F}_x$ is an Abelian group in a canonical way, and each canonical map

$$\pi_x^U : \mathscr{F}(U) \to \mathscr{F}_x$$

is a group homomorphism. The elements of the stalk $\mathscr{F}_x$ are named the *germs* of $\mathscr{F}$ at x, see Figure 2.3.

The stalk $\mathscr{F}_x$ of a presheaf $\mathscr{F}$ at a point $x \in X$ is the *inductive limit* of the sections from $\mathscr{F}(U)$ for all neighbourhoods U of x:

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U)$$

A sheaf is when you do vertically algebra and horizontally topology, see Figure 2.3.

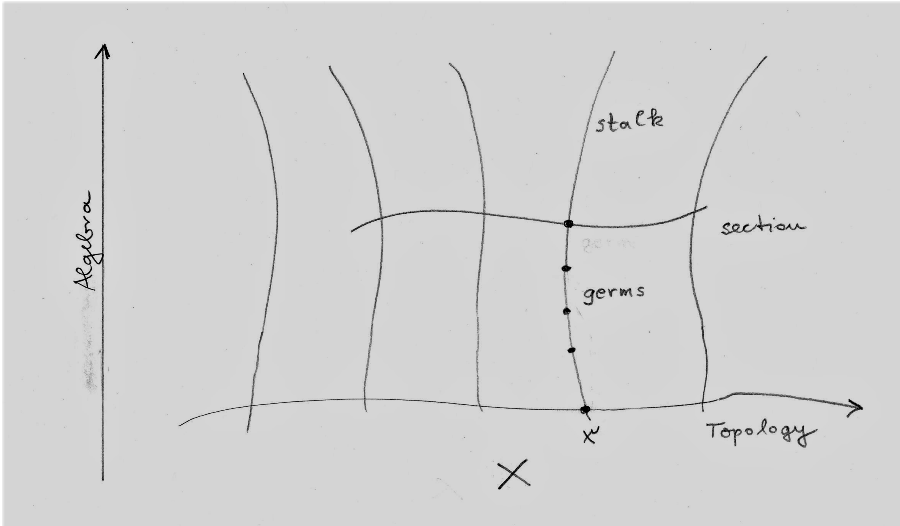


Fig. 2.3 Sheaf on X with section and stalk of germs.

Each subpresheaf of a sheaf is again a sheaf. The quotient of a presheaf $\mathscr{F}$ with respect to a subpresheaf is again a presheaf. But even if $\mathscr{F}$ is a sheaf the quotient subsheaf is not necessarily a sheaf: A family of quotients is not necessarily the quotient of a global sections, see the counter example of the logarithm in Proposition 2.24. Sheafification from Theorem 2.16 attaches to every presheaf a sheaf. The method is used to define the quotient of a sheaf as the sheafification of the quotient presheaf, see Definition 2.17.

Theorem 2.16 (Sheafification of a presheaf). *For each presheaf $\mathscr{F}$ on a topological space exists a sheaf $\hat{\mathscr{F}}$ on X and a presheaf morphism*

$$\mathscr{F} \to \hat{\mathscr{F}}$$

which induces for each $x \in X$ an isomorphism of stalks

$$\mathscr{F}_x \to \hat{\mathscr{F}}_x$$

The sheaf $\hat{\mathscr{F}}$ is named the sheafification *of the presheaf $\mathscr{F}$.*

Proof. i) For an open set $U \subset X$ define the Abelian group

$$w(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x$$

as the Cartesian product of the corresponding stalks. Together with the canonical restrictions

$$w(\mathscr{F})(U) \to w(\mathscr{F})(V),\ V \subset U \text{ open},$$

we obtain a sheaf $w(\mathscr{F})$ on X. The sheaf $w(\mathscr{F})$ is an example of a *flabby* sheaf (deutsch: welke Garbe).

ii) For an open set $U \subset X$ a family of germs

$$\phi = (\phi_x)_{x \in U} \in w(\mathscr{F})(U)$$

is *compatible* if the family locally arises from sections, i.e. if each $x \in U$ has an open neighbourhood $V \subset U$ and a section

$$f \in \mathscr{F}(V) \text{ satisfying } f_y = \phi_y \text{ for all } y \in V.$$

For each open $U \subset X$ define the presehaf

$$\hat{\mathscr{F}}(U) := \{\phi \in w(\mathscr{F})(U) : \ \phi \text{ compatible}\}$$

with the canonical restrictions

$$\hat{\mathscr{F}}(U) \to \hat{\mathscr{F}}(V), \ V \subset U \text{ open.}$$

iii) The presheaf $\hat{\mathscr{F}}$ is a sheaf with the same stalks as $\mathscr{F}$: One checks that the construction via stalks and compatible sections of $w(\mathscr{F})$ satisfies the two sheaf axioms. One also checks that the induced maps on stalks $\mathscr{F}_x \to \hat{\mathscr{F}}_x$ are isomorphisms. □

Definition 2.17 (Quotient sheaf). Consider a topological space X and two sheaves

$$\mathscr{F} \subset \mathscr{G}$$

of Abelian groups. The *quotient sheaf*

$$\mathscr{F}/\mathscr{G}$$

is defined as the sheafification $\mathscr{F}$ of the quotient presheaf

$$U \mapsto \mathscr{F}(U)/\mathscr{G}(U), \ U \subset X \text{ open,}$$

with the canonical restrictions.

A section of the quotient sheaf $\mathscr{F}/\mathscr{G}$ is represented by a family of local sections from $\mathscr{F}$.

A meromorphic function on a complex manifold is a compatible family of quotients of holomorphic functions which are locally defined. Because the denominator functions may have zeros a meromorphic function does not have necessarily a value at each point of definition.

From a general perspective the concept of a meromorphic function uses the algebraic construction of localisation, i.e. taking for rings of holomorphic functions the modules of fractions. Here one has to exclude as denominators those holomorphic functions which vanish identically on a component of their domain of definition. Hence one localises each ring of holomorphic functions with respect to its subring of non-zero-divisors. To obtain a sheaf in the end, one has to sheafify in a final step the algebraic construction. As so often, the whole prodedure is to carry over a construction from commutative algebra to a corresponding construction with sheaves, see [1, Chap. 3].

Definition 2.18 (Sheaf of fractions). Consider a topological space X and a sheaf of rings $\mathscr{R}$ on X.

1. Consider the multiplicatively closed family of sections with non-zero divisor germs

   $$S(U) := \{g \in \mathscr{R}(U) : g_x \in \mathscr{R}_x \text{ non-zero divisor for all } x \in U\},\ U \subset X \text{ open.}$$

   On each $\mathscr{R}(U) \times S(U)$ consider the equivalence relation

   $$(f_1, g_1) \equiv (f_2, g_2) \iff \exists n \in S(U):\ n \cdot (g_2 f_1 - g_1 f_2) = 0 \in \mathscr{R}(U).$$

   Denote by $\mathscr{R}(U)_{S(U)}$ the ring of equivalence classes, and by $\dfrac{f}{g} \in \mathscr{R}(U)_{S(U)}$ the equivalence class of a given pair $(f, g) \in \mathscr{R}(U) \times S(U)$.

2. The sheafification $Q(\mathscr{R})$ of the presheaf

   $$U \mapsto \mathscr{R}(U)_{S(U)},\ U \subset X \text{ open,}$$

   is named the *sheaf of fractions* of $\mathscr{R}$ with respect to the family $S(U)_U$.

Definition 2.19 (Sheaf of meromorphic functions). For a complex n-dimensional manifold X each ring $\mathscr{O}_x,\ x \in X$, is isomorphic to $\mathbb{C}\{z_1, ..., z_n\}$, hence an integral domain. The sheaf of fractions

$$\mathscr{M} := Q(\mathscr{O})$$

is named the *sheaf of meromorphic functions* on X. A section

$$f \in \mathscr{M}(U),\ U \subset X,$$

is named a *meromorphic function* on U.

Hence a meromorphic function on X is a family of fractions $(f_i/g_i)_{i\in I}$ of locally defined holomorphic functions with $g_i \neq 0$ under the identification

$$f_1/g_1 = f_2/g_2 :\Longleftrightarrow g_2 f_1 = g_1 f_2$$

Proposition 2.20 (Field of meromorphic functions). *Consider a complex manifold X. For a domain $G \subset X$ the ring $\mathscr{M}(G)$ of global meromorphic functions is a field.*

Proof. The claim follows because $\mathscr{O}(G)$ is an integral domain, cf. Corollary 1.11. □

The sheaves $\mathscr{O}$ and $\mathscr{M}$ are sheaves of rings. The sheaves $\mathscr{O}^*$ and $\mathscr{M}^*$ are sheaves of multiplicative groups. They are the sheaves of units of repectively $\mathscr{O}$ and $\mathscr{M}$.

Sheaves of locally constant functions like $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are important for homology and cohomology in the context of algebraic topology. While sheaves of holomorphic and meromorphic functions are the basic objects on complex manifolds.

Remark 2.21 (Stalks of the structure sheaf and quotient fields).

1. Let X be a complex manifold. For a point $x \in X$ consider the stalks

$$R := \mathscr{O}_x \text{ and } K := \mathscr{M}_x.$$

   Using a chart around x shows

$$R = \mathbb{C}\{z\}, \text{ the ring of convergent power series with center } = 0,$$

   and

$$K = Q(R), \text{ the quotient field of } R,$$

   a statement about germs.

   The ring R is a local ring, i.e. it has exactly one maximal ideal, namely

$$\mathfrak{m} :=< z_1, ..., z_n > \subset R,$$

   the ideal of non-units of R.

2. In general, the local statement about meromorphic germs does not necessarily generalize to a global statement: On one hand, for $X = \mathbb{C}$ one has

$$\mathscr{M}(X) := (Q(\mathscr{O}))(X) = Q(\mathscr{O}(X))$$

due to the Weierstrass product theorem, a statement about global sections. The same statement holds even for any domain $X \subset \mathbb{C}$. On the other hand, on a compact complex manifold X one has

$$\mathscr{O}(X) = \mathbb{C}$$

but possibly

$$\mathscr{M}(X) \neq Q(\mathscr{O}(X)) = Q(\mathbb{C}) = \mathbb{C}, \text{ e.g., } \mathscr{M}(\mathbb{P}^1) = \mathbb{C}(z).$$

For more advanced results see [24, Kapitel 4* §1.5 Satz, §2.4].

3. Any morphism of presheaves on X

$$f : \mathscr{F} \to \mathscr{G}$$

induces for any $x \in X$ a morphism

$$f_x : \mathscr{F}_x \to \mathscr{G}_x$$

of the corresponding stalks such that the following diagram diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ {\scriptstyle \pi_x^U}\downarrow & & \downarrow{\scriptstyle \tau_x^U} \\ \mathscr{F}_x & \xrightarrow{f_x} & \mathscr{G}_x \end{array}$$

Here the vertical maps are the canonical group homomorphisms from Definition 2.15. In general, these maps are not surjective. But each germ $f_x \in \mathscr{F}_x$ has an open neighbourhood $V \subset X$ and a representative $f \in \mathscr{F}(V)$. The neighbourhood V may depend on the germ f_x.

4. On a complex manifold X sections of a sheaf like the holomorphic structure sheaf $\mathscr{O}$ can be considered at least from the following different topological viewpoints:

- At a point $x \in X$ one considers the *value* $f(x) \in \mathbb{C}$ of a function f which is holomorphic in an open neighbourhood of x.

- At a point $x \in X$ one considers the *germ* $f_x \in \mathbb{C}\{z\}$ of a function f which is holomorphic in an open neighbourhood of x.

- In a given *open neighbourhood* $U \subset X$ of a point $x \in X$ one considers a holomorphic function $f \in \mathscr{O}(U)$.

- One considers a *globally* defined holomorphic function $f \in \mathscr{O}(X)$.

Definition 2.22 (Exact sheaf sequence resp. presheaf sequence). Consider a topological space X.

1. A *sequence of sheaves* on X is a family

$$(f_i : \mathscr{F}_i \to \mathscr{F}_{i+1})_{i\in\mathbb{Z}}$$

of morphisms of sheaves. The family is a *complex* if for all $x \in X$ on the level of stalks the induced family of morphisms of Abelian groups

$$(f_{i,x} : \mathscr{F}_{i,x} \to \mathscr{F}_{i+1,x})_{i\in\mathbb{Z}}$$

satisfies for all $i \in \mathbb{Z}$

$$f_{i,x} \circ f_{i-1,x} = 0.$$

The family is *exact* if for all $x \in X$ on the level of stalks the induced family of morphisms of Abelian groups

$$(f_{i,x} : \mathscr{F}_{i,x} \to \mathscr{F}_{i+1,x})_{i\in\mathbb{Z}}$$

is exact, i.e. if for all $i \in \mathbb{Z}$

$$ker[f_{i,x} : \mathscr{F}_{i,x} \to \mathscr{F}_{i+1,x}] = im[f_{i-1,x} : \mathscr{F}_{i-1,x} \to \mathscr{F}_{i,x}].$$

2. A *short exact sequence of sheaves* is an exact sheaf sequence of the form

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0.$$

3. A morphism of sheaves

$$f : \mathscr{F} \to \mathscr{G}$$

is respectively, *injective* or *surjective* or *bijective* if the corresponding property holds on the level of stalks

$$f_x : \mathscr{F}_x \to \mathscr{G}_x$$

for all $x \in X$.

4. A *sequence of presheaves* on X is a family

$$(f_i : \mathscr{F}_i \xrightarrow{f_i} \mathscr{F}_{i+1})_{i\in\mathbb{Z}}$$

of morphisms between presheaves. The sequence of presheaves is *exact* if for each open $U \subset X$ and each $i \in \mathbb{Z}$ on the level of sections

$$ker\ f_{i,U} = im\ f_{i-1,U}$$

for

$$\mathscr{F}_{i-1}(U) \xrightarrow{f_{i-1,U}} \mathscr{F}_i(U) \xrightarrow{f_{i,U}} \mathscr{F}_{i+1}(U)$$

Remark 2.23 (Exactness of a sheaf sequence).

1. A sequence of *sheaf* morphisms is studied on stalks, while a sequence of *presheaf* morphisms is studied on the level of sections.

2. An exact sequence of sheaves satisfies for all $i \in \mathbb{Z}$

$$f_{i+1} \circ f_i = 0.$$

   A sequence of morphisms between sheaves may be exact when considered as sequence of *sheaves*, but not when considered as sequence of *presheaves*. Exactness of a sheaf sequence is a statement about the induced morphisms of the *stalks*. It is not required that the corresponding sequence of morphisms of the groups of *sections*

$$f_{i,U} : \mathscr{F}_i(U) \to \mathscr{F}_{i+1}(U),\ U \text{ open neighbourhood of } x \in X,\ i \in \mathbb{Z},$$

   is exact.

3. One has to distinguish between a statement on the level of germs and a local statement on the level of sections. It is exactly the task of cohomology theory, see Chapter 3, to measure the difference between exactness on the level of germs and exactness on the level of sections, in particular on the level of global sections.

Proposition 2.24 (Exponential sequence). *The* exponential sequence *on a complex manifold X is the following exact sequence of sheaves of Abelian groups*

$$0 \to \mathbb{Z} \xrightarrow{j} \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$$

Here the morphism j is the canonical inclusion. And the exponential

$$\mathscr{O} \xrightarrow{ex} \mathscr{O}^*$$

is defined for open sets $U \subset X$ as

$$ex_U : \mathscr{O}(U) \to \mathscr{O}^*(U),\ f \mapsto exp(2\pi i \cdot f).$$

Proof. To prove that the sheaf sequence is exact we consider for arbitrary but fixed $x \in X$ the sequence of stalks

$$0 \to \mathbb{Z}_x = \mathbb{Z} \xrightarrow{j_x} \mathscr{O}_x \xrightarrow{ex_x} \mathscr{O}_x^* \to 0$$

i) *Exactness at* $\mathscr{O}_x$: Each holomorphic function f defined on a domain and satisfying

$$e^{2\pi i f} = 1$$

is an integer constant and vice versa.

ii) *Exactness at* $\mathscr{O}_x^*$: The surjectivity of the morphism ex_x follows from the fact, that any holomorphic function without zeros, which is defined on a simply connected domain, has a holomorphic logarithm. □

Note that in general the exponential sequence from Proposition 2.24 is not exact on the level of global sections: For $X = \mathbb{C}^*$ the morphism

$$\mathscr{O}(X) \xrightarrow{ex} \mathscr{O}^*(X),\ f \mapsto exp(2\pi i \cdot f)$$

is not surjective, because the holomorphic function

$$1/z \in \mathscr{O}^*(X)$$

has no logarithm.

Definition 2.25 (Direct image). Consider a continuus map

$$f : X \to Y$$

between topological spaces and a sheaf $\mathscr{F}$ on X. The presheaf on Y

$$(f_*\mathscr{F})(V) := \mathscr{F}(f^{-1}V),\ V \subset Y \text{ open},$$

is a sheaf, named the *direct image* $f_*\mathscr{F}$ of $\mathscr{F}$ with respect to f.

Note. Taking the direct image is a covariant functor:

$$id_*\mathscr{F} = \mathscr{F} \text{ and } (g \circ f)_*\mathscr{F} = g_*(f_*\mathscr{F})$$

Definition 2.26 (Ringed space).

1. A *ringed space* is a pair $(X,\mathscr{R})$ with a topological space X and a subsheaf $\mathscr{R} \subset \mathscr{C}$ of rings of complex-valued continuous functions on X.

2. A *morphism of ringed spaces*

$$(f,\tilde{f}):(X,\mathscr{A})\to(Y,\mathscr{B})$$

is a pair $(f,\tilde{f})$ with a continuous map $f:X\to Y$ and a morphism of sheaves of rings

$$\tilde{f}:\mathscr{B}\to f_*\mathscr{A}$$

3. The morphism $(f,\tilde{f})$ is an *isomorphism of ringed spaces* if there exists a morphism of ringed

$$(g,\tilde{g}):(Y,\mathscr{B})\to(X,\mathscr{A})$$

   - satisfying for the continuous maps

$$g\circ f=id_X,\ f\circ g=id_Y$$

   - and satisfying for the composition of the sheaf morphisms

$$[\mathscr{A}\xrightarrow{\tilde{g}}g_*\mathscr{B}\xrightarrow{g_*(\tilde{f})}g_*(f_*\mathscr{A})=(g\circ f)_*\mathscr{A}=\mathscr{A}]=id_{\mathscr{A}}$$

   and analogously for the composition in the opposite order.

Proposition 2.27 (Local models as ringed spaces).

1. Local model of complex manifolds*: For each open $U\subset\mathbb{C}^n$ the pair $(U,\mathscr{O}_U)$ is a ringed space. If*

$$U\subset\mathbb{C}^n \text{ and } V\subset\mathbb{C}^m$$

are open, then a continuous map

$$\phi:U\to V$$

is holomorphic iff ϕ induces a sheaf morphism

$$\tilde{\phi}:\mathscr{O}_V\to\phi_*\mathscr{O}_U$$

by pullback of holomorphic functions

$$\mathscr{O}_V(W)=\mathscr{O}_{\mathbb{C}^m}(W)\to(\phi_*\mathscr{O}_U)(W)=\mathscr{O}_U(\phi^{-1}W)=\mathscr{O}_{\mathbb{C}^n}(\phi^{-1}W)$$

$$f\mapsto f\circ\phi,\ W\subset V \text{ open}.$$

2. Local model of smooth manifolds*: For each open subset $U\subset\mathbb{R}^n$ the pair $(U,\mathscr{E}_U)$ is a ringed space. If*

$$U\subset\mathbb{R}^n \text{ and } V\subset\mathbb{R}^m$$

are open, then a continuous map

$$\phi:U\to V$$

is smooth iff ϕ induces a sheaf morphism

$$\tilde{\phi} : \mathscr{E}_V \to \phi_* \mathscr{E}_U$$

by pullback of smooth functions.

3. Local model of reduced complex spaces*: Consider an open subset $U \subset \mathbb{C}^n$ and an analytic subset $A \subset U$. Its ideal sheaf $\mathscr{I}_A \subset \mathscr{O}_U$ defines the quotient sheaf*

$$\mathscr{O}_A := \mathscr{O}_U / \mathscr{I}_A,$$

induced from the exact sequence

$$0 \to \mathscr{I}_A \to \mathscr{O}_U \to \mathscr{O}_A \to 0$$

The pair $(A, \mathscr{O}_A)$ is a ringed space.

Proof. 1. If ϕ induces by pullback the sheaf morphisms

$$\mathscr{O}_V \to \phi_* \mathscr{O}_U$$

then in particular for the holomorphic coordinate functions

$$z_j \in \mathscr{O}_V(V),\ j = 1, ..., m :$$

the functions

$$\tilde{\phi}(z_j) = z_j \circ \phi$$

are holomorphic, i.e. all component functions of ϕ are holomorphic. Hence

$$\phi : U \to V$$

is a holomorphic map in the sense of Definition 2.2.

2. The proof is analogous using the real coordinate functions. □

Corollary 2.28 (Complex manifolds as ringed space). *Consider a connected topological manifold X with second countable topology, and on X a subsheaf of rings*

$$\mathscr{R} \subset \mathscr{C}$$

of complex-valued, continuous functions. Then for each open covering $\mathscr{U} = (U_i)_{i \in I}$ of X with homeomorphisms

$$\phi : U_i \xrightarrow{\simeq} V_i,\ V_i \subset \mathbb{C}^n \ open,\ i \in I,$$

are equivalent:

1. *The family* $(\phi_i)_{i\in I}$ *is an atlas of a complex structure on* X *with holomorphic structure sheaf* $\mathcal{O}_X = \mathcal{R}$.

2. *Each homeomorphism*

$$\phi_i : U_i \xrightarrow{\simeq} V_i,\ i \in I,$$

induces by its pullback $\tilde{\phi}_i$ *an isomorphism of ringed spaces*

$$(\phi_i, \tilde{\phi}_i) : (U_i, \mathcal{R}|U_i) \xrightarrow{\simeq} (V_i, \mathcal{O}_{V_i})$$

Corollary 2.28 literally carries over to smooth structures on the topological manifold X and its structure sheaf $\mathcal{E}_X$.

Moreover, a *reduced complex space* is defined by using the local models of analytic sets from Proposition 2.27, see [5, Section 0.14].

2.3 The Cousin problems

Definition 2.29 (Cousin problems). Consider a complex manifold X.

1. An *additive Cousin distribution* on X is an open covering $\mathcal{U} = (U_i)_{i\in I}$ of X and a corresponding family of locally defined meromorphic functions

$$h_i \in \mathcal{M}(U_i), i \in I,$$

such that for all pairs $i, j \in I$ the differences are holomorphic, i.e. they satisfy

$$h_i|U_{ij} - h_j|U_{ij} \in \mathcal{O}(U_{ij}),\ U_{ij} := U_i \cap U_j.$$

A *solution* of the additive Cousin distribution is a global meromorphic function $h \in \mathcal{M}(X)$ with holomorphic differences

$$h|U_i - h_i \in \mathcal{O}(U_i),\ i \in I.$$

2. A *multiplicative Cousin distribution* on X is an open covering $\mathcal{U} = (U_i)_{i\in I}$ of X and a corresponding family of meromorphic functions

$$h_i \in \mathcal{M}^*(U_i), i \in I,$$

such that for all pairs $i, j \in I$ the quotients are holomorphic without zeros, i.e. they satisfy

$$\frac{h_i|U_{ij}}{h_j|U_{ij}} \in \mathcal{O}^*(U_{ij}),\ U_{ij} := U_i \cap U_j,$$

A *solution* of the multiplicative Cousin distribution is a global meromorphic function $h \in \mathscr{M}^*(X)$ with holomorphic quotients without zeros

$$\frac{h|U_i}{h_i} \in \mathscr{O}^*(U_i),\ i \in I.$$

Remark 2.30 shows: The additive resp. multiplicative Cousin problems are the analogue to the problems of Mittag-Leffler resp. Weierstrass from complex analysis in one variable. The Cousin problems ask for a global meromorphic function, which satisfies certain conditions referring to the sheaf $\mathscr{O}$ resp. referring to the sheaf $\mathscr{O}^*$.

Remark 2.30 (Solution of the Cousin problems).

1. Each solution of an additive Cousin distribution defines a global meromophic function
$$h \in \mathscr{M}(X)$$
with the given "principal parts" $(h_i)_{i \in I}$. Each two solutions differ by a holomorphic function from $\mathscr{O}(X)$.

2. Each solution of a multiplicative Cousin distribution defines a global meromophic function
$$h \in \mathscr{M}^*(X)$$
with the same pole and zero orders as $(h_i)_{i \in I}$. Each two solutions differ by a holomorphic function without zeros from $\mathscr{O}^*(X)$

3. The main difference to the 1-dimensional case is the fact that the zeros and poles of holomorphic resp. meromorphic functions of several variables are no longer isolated points.

Our first goal is to solve the Cousin problems for polydics $X \subset \mathbb{C}^n$.

We start the solution with a step *bottom-up*.

Definition 2.31 (Adjacent product domains).

1. A *product domain* in an open set $X \subset \mathbb{R}^n$ is a set of the form
$$Q = I_1 \times ... \times I_n \subset X \text{ with intervals } I_j \subset \mathbb{R} \text{ for } j = 1, ..., n.$$
Depending on the type of all intervals one distinguishes *open* product domains resp. *compact* product domains.

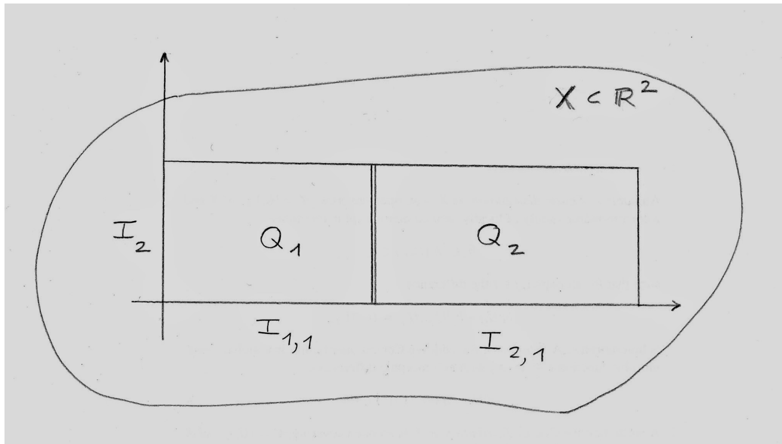


Fig. 2.4 Adjacent product domains

2. Two compact product domains $Q_1, Q_2 \subset X$ are *adjacent* if after renumbering

$$Q_1 = I_{1,1} \times I_2 \times ... \times I_n \text{ and } Q_2 = I_{2,1} \times I_2 \times ... \times I_n$$

with adjacent first intervals

$$I_{1,1} = [a,b],\ I_{2,1} = [b,c] \subset \mathbb{R},$$

see Figure 2.4.

Notation 2.32 (Set of compact product domains). *For an open set $X \subset \mathbb{C}^n$ denote by $\mathcal{Q}(X)$ the set of all compact product domains contained in X.*

Proposition 2.33 (Cousin's principle of induction). *Consider an open set $X \subset \mathbb{R}^N$ and a map*

$$A : \mathcal{Q}(X) \to \{TRUE, FALSE\}.$$

Assume that the truth-function A satisfies the following two properties:

- Small product domains*: Each point $x \in X$ has an open neighbourhood $U \subset X$ with $A(Q) = TRUE$ for all $Q \in \mathcal{Q}(U)$*

- Fusing adjacent product domains*: For each pair of adjacent product domains $Q_1, Q_2 \in Q(X)$ holds*

$$(A(Q_1) = A(Q_2) = TRUE) \implies (A(Q_1 \cup Q_2) = TRUE).$$

Then for all $Q \in \mathcal{Q}(X)$

$$A(Q) = TRUE$$

Proof. For a given product domain

$$Q = I_1 \times ... I_N \in \mathscr{Q}(X)$$

we have to show $A(Q) = TRUE$.

i) *Cutting Q into small adjacent product domains Q' with $A(Q') = TRUE$*: For each $k \in \mathbb{N}$ and for all $j = 1, ..., N$ we split each interval $I_j = [a_j, b_j]$ into k adjacent intervals

$$I_j = I_{j1} \cup ... \cup I_{jk} \text{ with } I_{jl} = [a_j + (l-1) \cdot h_j, a_j + l \cdot h_j]$$

$$h_j := \frac{b_j - a_j}{k},\ l = 1, ..., k,$$

and define for each N-tuple $(i_1, ..., i_N)$ of indices $i_j \leq k$ the small product domains

$$Q_{i_1 ... i_N} := I_{1i_1} \times ... \times I_{Ni_N}$$

from the induced splitting of Q. Due to the first property of the function A there exists a finite open covering $\mathscr{U}$ of the compact set Q such that each element U of the covering $\mathscr{U}$ satisfies

$$A(Q') = TRUE \text{ for all } Q' \in \mathscr{Q}(U).$$

The covering $\mathscr{U}$ of the compact metric space Q has a *Lebesgue number* $\varepsilon > 0$. For large k each of the small product domain $Q_{1_1 ... i_N}$ has diameter $\leq \varepsilon$, and is therefore contained in an element U of the covering $\mathscr{U}$. Choose a fixed $k \in \mathbb{N}$ with this property. Then each of the small product domains

$$Q_{i_1 ... i_N}$$

is contained in an element U of the covering $\mathscr{U}$. As a consequence

$$A(Q_{i_1, ..., i_N}) = TRUE$$

for all N-tuples $(i_1, ..., i_N)$.

ii) *Edge fusing bottom-up*: We now enlarge step by step the size of the product domains $Q' \subset Q$ with $A(Q') = TRUE$. For $m = 0, ..., N$ set

$$Q_{i_1 ... i_m} := I_{1i_1} \times ... \times I_{mi_m} \times I_{m+1} \times ... \times I_N$$

as a product domain, built within Q from m small product domains as first factors and $N - m$ large product domains as last factors. We prove by descending induction on m

$$A(Q_{i_1 ... i_m}) = TRUE \text{ for all tuples } (i_1, ..., i_{m-1})$$

Induction start $m = N$: The result has been proved in part i).

Induction step $m + 1 \mapsto m$: We illustrate the induction step for the induction steps

$$2 \mapsto m = 1 \text{and } 1 \mapsto m = 0$$

in the 2-dimensional case

$$Q = I_1 \times I_2 \subset X \subset \mathbb{R}^2$$

and $k = 3$ partitions, see Figure 2.5: The bold lines indicate the common borders across which the adjacent product domains are fused during the induction step $m \mapsto m-1$. The fusing is possible due to the second property of the function A.

For a formalization of the general step see [6, § 4, Satz 3]. □

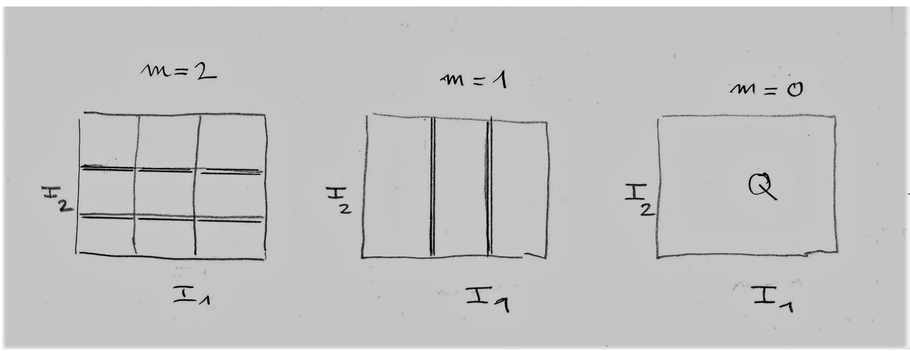


Fig. 2.5 Cousin principle of induction

A first application of Cousin's principle of induction to complex analysis is Proposition 2.34.

Proposition 2.34 (Cartan's lemma for holomorphic functions). *Consider two adjacent product domains*

$$Q',\ Q'' \subset \mathbb{C}^n,$$

and in an open neighbourhood U of $Q' \cap Q''$ a holomorphic function

$$f \in \mathscr{O}(U) \text{ (additive case)}$$

resp. a holomorphic function without zeros

$$g \in \mathscr{O}^*(U) \text{ (multiplicative case).}$$

Then there exist open neighbourhoods

$$\tilde{U}' \supset Q' \text{ and } \tilde{U}'' \supset Q''$$

and an open set $\tilde{U}$ satisfying

$$(\tilde{U}' \cap \tilde{U}'') \supset \tilde{U} \supset (Q' \cap Q'')$$

with

- *holomorphic functions*

$$f' \in \mathscr{O}(\tilde{U}') \text{ and } f'' \in \mathscr{O}(\tilde{U}'')$$

satisfying over $\tilde{U}$

$$f = f' - f''$$

- *resp. holomorphic functions without zeros*

$$g' \in \mathscr{O}^*(\tilde{U}') \text{ and } g'' \in \mathscr{O}^*(\tilde{U}'')$$

satisfying over $\tilde{U}$

$$g = \frac{g'}{g''}$$

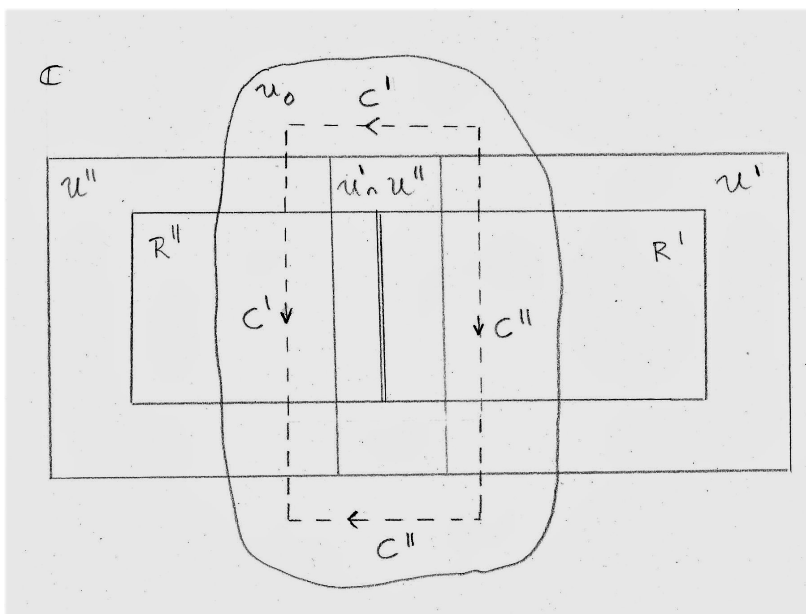


Fig. 2.6 Cartan's lemma for holomorphic functions, view from the first factor $\mathbb{C}$ of $\mathbb{C}^n$

Proof. We may assume the following geometric situation, see Figure 2.6:

$$Q' = R' \times Q \text{ and } Q'' = R'' \times Q$$

with a product domain $Q \subset \mathbb{C}^{n-1}$ and two adjacent product domains $R',\ R'' \subset \mathbb{C}$. Moreover

$$U = U_0 \times V$$

is an open neighbourhood of $Q_1 \cap Q_2$ with $V \subset \mathbb{C}^{n-1}$ open and $U_0 \subset \mathbb{C}$ an open neighbouhood of $Q' \cap Q''$.

i) *Additive case*: Consider the holomorphic function

$$f \in U_0 \times V$$

Compactness of $R' \cap R''$ and openness of U_0 provide a product domain $R \subset \mathbb{C}$ satisfying

$$R' \cap R'' \subset \mathring{R} \subset R \subset U_0 \subset \mathbb{C}$$

Split the boundary as

$$\partial R = C' \cup C''$$

with the orientation from Figure 2.6, and choose open neighbourhoods

$$U' \supset R', U'' \supset R''$$

with

$$C' \cap U' = C'' \cap U'' = \emptyset.$$

For each point $(z,s) \in \mathring{R} \times V$ the Cauchy integral formula with respect to the 1-dimensional integration $d\zeta$ shows

$$f(z,s) = \frac{1}{2\pi i} \cdot \int_{\partial R} \frac{f(\zeta,s)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \cdot \int_{C'} \frac{f(\zeta,s)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \cdot \int_{C''} \frac{f(\zeta,s)}{\zeta - z}\, d\zeta$$

Define the functions

$$f' : U' \times V \to \mathbb{C},\ f'(z,s) := \frac{1}{2\pi i} \cdot \int_{C'} \frac{f(\zeta,s)}{\zeta - z}\, d\zeta,\ \tilde{U}' := U' \times V,$$

and

$$f'' : U'' \times V \to \mathbb{C},\ f''(z,s) := \frac{1}{2\pi i} \cdot \int_{C''} \frac{f(\zeta,s)}{\zeta - z}\, d\zeta,\ \tilde{U}'' := U'' \times V,$$

Both functions are holomorphic due to Theorem 1.6. Due to construction they have on

$$\tilde{U} := (U' \cap U'') \times V$$

the difference

$$f = f' - f''.$$

ii) *Multiplicative case*: The case of the given holomorphic function $g \in \mathscr{O}^*(U_0 \times V)$ reduces to the additive case from part i). Choose

$$U = U_0 \times V$$

simply connected and consider the holomorphic function

$$log\ g \in \mathscr{O}(U_0 \times V)$$

e.g., choosing the principal branch of the holomorphic logarithm in $\mathbb{C}^*$. By part i) there exist holomorphic functions $\tilde{f}'$ in an open neighbouhood of Q' and $\tilde{f}''$ in an open neighbourhood of Q'' satisfying in an open neighbourhood of $Q' \cap Q''$

$$log\ g = \tilde{f}' - \tilde{f}''$$

Then the holomorphic functions without zeros

$$g' := exp\ f' \text{ and } g'' := exp\ \ f''$$

satisfy in an open neighbourhood of $Q' \cap Q''$

$$g = \frac{g'}{g''}.$$

□

As a consequence of Cartan's lemma from Proposition 2.34 one obtains with Proposition 2.35 a solution of the Cousin problems after shrinking.

Proposition 2.35 (Solving the Cousin problems in a an open neighbourhood of a product domain). *Consider an open set $X \subset \mathbb{C}^n$. Then each additive or multiplicative Cousin distribution*

$$\mathscr{C} := (U_i, f_i)_{i \in I}$$

on X has for each each product domain $Q \in \mathscr{Q}(X)$ a solution in a neighbourhood of Q.

Proof. The proof applies Cousin's principle of induction, Proposition 2.33, together with the Cartan Lemma, Proposition 2.34. We consider the map

$$A : \mathscr{Q}(X) \to \{TRUE, FALSE\},$$

defined as

$$A(Q) := \begin{cases} TRUE & \mathscr{C} \text{ is solvable in a neighbourhood of } Q \\ FALSE & otherwise \end{cases}$$

The map A satisfies the assumptions of Proposition 2.33:

- For a given $x \in X$ choose an index $i \in I$ with $x \in U_i$. The function $f_i \in \mathscr{M}(U_i)$ solves the Cousin problem in U_i, in particular for all $Q \in \mathscr{Q}(U_i)$.

- Consider two adjacent product domains $Q',\ Q'' \in \mathscr{Q}(X)$ and assume open neighbourhoods
$$U' \supset Q' \text{ and } U'' \supset Q''$$
with solutions
$$f' \in \mathscr{M}(U'),\ f'' \in \mathscr{M}(U'')$$
of the induced Cousin distributions on U' respectively U''.

i) *Additive case*: On $U' \cap U''$ the function
$$f' - f''$$
is holomorphic because for each $i \in I$
$$(f' - f'')|U' \cap U'' \cap U_i = (f' - f_i) - (f'' - f_i) \in \mathscr{O}(U' \cap U'' \cap U_i)$$
Proposition 2.34 provides two open neighbourhoods
$$V' \supset Q' \text{ and } V'' \supset Q'' \text{ with } V' \cap V'' \subset U' \cap U''$$
with corresponding holomorphic functions
$$g' \in \mathscr{O}(V') \text{ and } g'' \in \mathscr{O}(V'')$$
satisfying on $V' \cap V''$
$$f' - f'' = g' - g'' \text{ i.e. } f' - g' = f'' - g''$$
They define a meromorphic function
$$f = (f' - g', f'' - g'') \in \mathscr{M}(V' \cup V'')$$
which solves $\mathscr{C}$ in the neighbourhood $V' \cup V''$ of $Q' \cup Q''$. Hence
Proposition 2.33 implies
$$A(Q) = TRUE \text{ for all } Q \in \mathscr{Q}(X).$$
ii) *Multiplicative case*: On $U' \cap U''$ the function
$$\frac{f'}{f''}$$
is holomorphic without zeros because for each $i \in I$
$$\frac{f'}{f''} = \frac{f'/f_i}{f''/f_i} \in \mathscr{O}^*(U' \cap U'' \cap U_i)$$
The multiplicative case of Proposition 2.34 provides two open neighbourhoods

$$V' \supset Q' \text{ and } V'' \supset Q'' \text{ with } V' \cap V'' \subset U' \cap U''$$

with corresponding holomorphic functions

$$g' \in \mathscr{O}^*(V') \text{ and } g'' \in \mathscr{O}^*(V'')$$

satisfying on $V' \cap V''$

$$\frac{f'}{f''} = \frac{g'}{g''} \text{ i.e. } \frac{f'}{g'} = \frac{f''}{g''}$$

They define a meromorphic function

$$f = \left(\frac{f'}{g'}, \frac{f''}{g''}\right) \in \mathscr{M}^*(V' \cup V'').$$

Again Proposition 2.33 implies

$$A(Q) = TRUE \text{ for all } Q \in \mathscr{Q}(X).$$

□

Theorem 2.36 (The Cousin problems in polydiscs). *In an n-dimensional polydisc*

$$\Delta(r) := \{(z_1, ..., z_n) \in \mathbb{C}^n : \ |z_\nu| \ < r_\nu, \ \nu = 1, ..., n\}$$

with polyradius

$$r = (r_1, ..., r_n), \ 0 < r_\nu \leq \infty, \ \nu = 1, ..., n,$$

each additive or multiplicative Cousin problem has a solution.

The proof rests on the existence of solutions after shrinking from Proposition 2.35 and the subsequent extension step thanks to the Mittag-Leffler exhaustion from Proposition 1.26.

Proof. Consider a given Cousin distribution on $\Delta(r)$

$$\mathscr{C} = (U_i, \ f_i)_{i \in I}.$$

Choose an exhaustion

$$(X_j)_{j \in \mathbb{N}}$$

of $\Delta(r)$ by relatively compact, open polydiscs satifying $X_j \subset\subset X_{j+1}, \ j \in \mathbb{N}$.

i) *Solution on X_j*: For two polydiscs

$$\Delta \subset\subset \tilde{\Delta} \subset \mathbb{C}^n$$

the Riemann mapping theorem applies to each factor of $\tilde{\Delta}$. It provides a biholomorphic map
$$\phi : \tilde{\Delta} \xrightarrow{\simeq} \tilde{Q}$$
with a product domain
$$\tilde{Q} \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}.$$
The compact subset
$$\phi(\overline{\Delta}) = \overline{\phi(\Delta)} \subset \tilde{Q}$$
has an open neighbourhood
$$Q \subset\subset \tilde{Q},$$
which is a product domain Q. Proposition 2.35 provides a solution of the induced Cousin problem on $\tilde{Q}$ in a neighbourhood of $\overline{Q}$. It pulls back via ϕ to a solution of the original Cousin distribution over Δ.

As a consequence, there exists for each $j \in \mathbb{N}$ a solution of the Cousin distribution $\mathscr{C}$ over the polydisc X_j. For each $j \in \mathbb{N}^*$ denote by M_j the set of solutions of $\mathscr{C}$ on X_j and by
$$\rho_j : M_j \to M_{j-1}$$
the canonical restriction.

ii) *Additive Cousin problem*: For each $j \in \mathbb{N}$ the set M_j is the affine space
$$M_j = f_j + \mathscr{O}(X_j)$$
with an arbitrary, fixed solution $f_j \in M_j$. In particular the map
$$\tau_j : \mathscr{O}(X_j) \to M_j,\ \phi \mapsto f_j + \phi,$$
is bijective. The translation-invariant Fréchet space structure of $\mathscr{O}(X_j)$ carries over to M_j and provides M_j with the structure of a complete metric space. The metric structure is independent from the choice of f_j: Consider a second solution f'_j and the corresponding map
$$\tau'_j : \mathscr{O}(X_j) \to M_j,\ \phi \mapsto f'_j + \phi,$$
Choose a fixed translation-invariant metric δ on $\mathscr{O}(X_j)$. Then the induced metrics d resp. d' satisfy:
$$d(f,g) := \delta(f - f_j, g - f_j) = \delta(f - f_j + (f_j - f'_j), g - f_j + (f_j - f'_j)) =$$
$$= \delta(f - f'_j, g - f'_j) =: d'(f,g)$$
In order to show that each restriction $\rho_j : M_j \to M_{j-1}$ has dense image one considers the commutative diagram

$$\begin{array}{ccc} \mathscr{O}(X_j) & \longrightarrow & \mathscr{O}(X_{j-1}) \\ \tau_j \big\downarrow & & \big\downarrow \tau_{j-1} \\ M_j & \xrightarrow{\rho_j} & M_{j-1} \end{array}$$

with the horizontal map

$$\mathscr{O}(X_j) \to \mathscr{O}(X_{j-1})$$

as the canonical restriction. The latter map is continuous with dense image, because holomorphic functions on X_{j-1} can be approximated on each compact set by their Taylor polynomials, which are defined on all of $\mathbb{C}^n$. By construction of the topologies the restrictions

$$\rho_j : M_j \to M_{j-1}$$

are continuous with dense image. The Mittag-Leffler principle of exhaustion, Proposition 1.26, applies to the sequence $(M_j, \rho_j)_{j \in \mathbb{N}}$ and provides a sequence $(f_j)_{j \in \mathbb{N}}$ of local solutions $f_j \in M_j$ such that

$$\rho_j(f_j) = f_{j-1}$$

for all $j \in \mathbb{N}$. The sequence defines a global meromorphic function

$$f \in \mathscr{M}(\Delta(r))$$

satisfying for all $j \in \mathbb{N}$

$$f|X_j = f_j.$$

iii) *Multiplicative Cousin problem*: Because each polydisc $X_j, j \in \mathbb{N}$, is a simply connected domain, the exponential and its inverse, the logarithm, control the space of local solutions, i.e. the local meromorphic functions which define the Cousin distribution $\mathscr{C}$. For each $j \in \mathbb{N}$ the map

$$\psi_j : \mathscr{O}(X_j) \to M_j,\ \phi \mapsto f_j \cdot e^{\phi},$$

is surjective. The twisted restriction

$$\sigma_j : \mathscr{O}(X_j) \to \mathscr{O}(X_{j-1}),\ \phi \mapsto \phi|X_{j-1} + h_{j-1},$$

with

$$h_{j-1} := log\left(\frac{f_j|X_{j-1}}{f_{j-1}}\right) \in \mathscr{O}(X_{j-1})$$

is continuous. Here h_{j-1} expresses as a quotient the difference between local solutions. The map σ_j has dense image, because holomorphic functions over a given compact $K \subset X_{j-1}$ can be uniformly approximated by their Taylor polynomials. One checks the commutativity of the following diagram

$$\begin{array}{ccc} \mathscr{O}(X_j) & \xrightarrow{\sigma_j} & \mathscr{O}(X_{j-1}) \\ \psi_j \downarrow & & \downarrow \psi_{j-1} \\ M_j & \xrightarrow{\rho_j} & M_{j-1} \end{array}$$

The Mittag-Leffler principle of exhaustion, Proposition 1.26, applies to the sequence of morphism between complete metric spaces

$$(\mathscr{O}(X_j),\ \sigma_j)_{j\in\mathbb{N}}.$$

It provides a sequence $(g_j)_{j\in\mathbb{N}}$ of compatible holomorphic functions

$$g_j \in \mathscr{O}(X_j), \text{ with } \sigma_j(g_j) = \sigma_{j-1},\ j \in \mathbb{N}.$$

The commutativity of the diagram above concludes

$$(f_j \cdot e^{g_j})|X_{j-1} = f_{j-1} \cdot e^{g_{j-1}}$$

Hence the sequence $(f_j \cdot e^{g_j})_{j\in\mathbb{N}}$ defines a global meromorphic function

$$f \in \mathscr{M}^*(\Delta(r)) \text{ with } f|X_j = f_j \cdot e^{g_j},\ j \in \mathbb{N},$$

which solves the multiplicative Cousin problem. □

Remark 2.37 (Solving the Cousin problems).

1. Theorem 2.36 solves the Cousin problems on n-dimensional polydiscs. Cousin's proof constructs the solution bottom-up over neighbourhoods of adjacent product domains (Cousin's principle of induction). For each step Cousin uses a mild version of a result, which was later generalized as Cartan's lemma.

   A second proof for the solvability of Cousin's problems in polydiscs will be given in Corollary 3.24. The second proof relies on Dolbeault's lemma and uses basic results from cohomology theory.

2. Complex analysis in the 20th century was driven by the search for solutions of the Cousin problems for more general complex manifolds X. For an intermediate result see Corollary 3.24. The Corollary gives cohomological conditions for the solvability of the Cousin problems. The final result is Corollary 6.25. The main tool is the cohomological formulation of the Cousin problems as a statement about the cohomology of the structure sheaf $\mathscr{O}_X$ and about the topological cohomology group $H^2(X,\mathbb{Z})$.

Chapter 3
Sheaf cohomology

For a complex manifold X the functor of sections over a fixed open set $U \subset X$

$$\underline{Sh_X} \to \underline{Ab}$$

from the category of sheaves of Abelian groups on X to the category of Abelian groups is *left exact*, i.e. for any exact sequence of sheaf morphisms

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$$

and for any open set $U \subset X$ the sequence of morphisms of Abelian groups of sections

$$0 \to \mathscr{F}(U) \xrightarrow{f_U} \mathscr{G}(U) \xrightarrow{g_U} \mathscr{H}(U)$$

is exact. But in general, for a surjective sheaf morphism g the morphism on sections

$$\mathscr{G}(U) \xrightarrow{g_U} \mathscr{H}(U)$$

is not surjective, see Remark after Proposition 2.24.

Cohomology, or *right-derivation* of the functor of sections, is the means to extend for an exact sequence of sheaf morphisms

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$$

the exact sequence of Abelian groups

$$0 \to \mathscr{F}(U) \xrightarrow{f_U} \mathscr{G}(U) \xrightarrow{g_U} \mathscr{H}(U)$$

to the right-hand. One defines groups

$$H^q(U,\mathscr{F}),\ H^q(U,\mathscr{G}),\ H^q(U,\mathscr{H}),\ q \geq 0,$$

to obtain a long exact sequence in the category of Abelian groups. The length of the extended sequence measures the failing of right-exactness of the functor of sections.

3.1 Čech cohomology groups

Definition 3.1 (Cochains, cocycles, coboundaries and Čech cohomology classes). Consider a topological space X, a presheaf $\mathscr{F}$ of Abelian groups on X and an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X.

- For each $q \in \mathbb{N}$ the q-th *cochain group* of $\mathscr{F}$ with respect to $\mathscr{U}$ is the Abelian group

  $$C^q(\mathscr{U},\mathscr{F}) := \prod_{(i_0...i_q)\in I^{q+1}} \mathscr{F}(U_{i_0} \cap ... \cap U_{i_q}),\ C^{-1}(\mathscr{U},\mathscr{F}) := 0.$$

  Hence a q-cochain is a family

  $$f = (f_{i_0...i_q})_{i_0...i_q \in I^{q+1}}$$

  of sections $f_{i_0...i_q} \in \mathscr{F}(U_{i_0} \cap ... \cap U_{i_q})$ over the $q+1$-fold intersections

  $$U_{i_0...i_q} := U_{i_0} \cap ... \cap U_{i_q}$$

  of the open sets of the covering. The group structure on $C^q(\mathscr{U},\mathscr{F})$ derives from the group structure of the factors.

- For each $q \in \mathbb{N}$ the *coboundary operator*

  $$\delta := \delta^q : C^q(\mathscr{U},\mathscr{F}) \to C^{q+1}(\mathscr{U},\mathscr{F})$$

  is defined for $f \in C^q(\mathscr{U},\mathscr{F})$ as

  $$\delta f := g := (g_{i_0...i_{q+1}})_{(i_0...i_{q+1})\in I^{q+2}}$$

  with the cross sum of restrictions

  $$g_{i_0...i_{q+1}} := \sum_{k=0}^{q+1} (-1)^k \cdot f_{i_0...\hat{i}_k i_{k+1}...i_{q+1}} | U_{i_0...i_k...i_{q+1}}$$

  Here $\hat{i}_k$ means to omit the index i_k.

- For each $q \in \mathbb{N}$ one defines the group of *q-cocycles*

  $$Z^q(\mathscr{U},\mathscr{F}) := ker[C^q(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^q} C^{q+1}(\mathscr{U},\mathscr{F})],$$

the group of *q-coboundaries*

$$B^q(\mathscr{U},\mathscr{F}) := im[C^{q-1}(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^{q-1}} C^q(\mathscr{U},\ \mathscr{F})],$$

and after checking the chain condition, see Remark 3.2,

$$\delta^q \circ \delta^{q-1} = 0$$

the *q-th cohomology group*

$$H^q(\mathscr{U},\mathscr{F}) := \frac{Z^q(\mathscr{U},\mathscr{F})}{B^q(\mathscr{U},\mathscr{F})}$$

Elements from $H^q(\mathscr{U},\mathscr{F})$ are named *q-th Čech cohomology classes* of $\mathscr{F}$ with respect to the covering $\mathscr{U}$. Two cocycles from $Z^q(\mathscr{U},\mathscr{F})$ with determine the same cohomology class in $H^q(\mathscr{U},\mathscr{F})$ are named *cohomologous*.

Remark 3.2 (Cohomology).

1. *Cocycle relation*: The meaning of cocycles of dimension $q = 0,1$ can be stated explicitly as follows:
   - $q = 0$: A family $(f_i)_i \in C^0(\mathscr{U},\mathscr{F})$ is a 0-cocycle iff for all $i,j \in I$

     $$f_j - f_i = 0 \text{ on } U_i \cap U_j,$$

     i.e. if the cochain satisfies on the intersections

     $$U_{ij} := U_i \cap U_j$$

     the cocycle condition

     $$f_i = f_j.$$

     If $\mathscr{F}$ is a sheaf, then 0-cocyles correspond bijectively to global sections $f \in \mathscr{F}(X)$, because

     $$B^0(\mathscr{U},\mathscr{F}) = 0$$

     and because the two sheaf axioms imply that the canonical map

     $$\mathscr{F}(X) \to Z^0(\mathscr{U},\mathscr{F}) = H^0(\mathscr{U},\mathscr{F}),\ f \mapsto (f_i := f|U_i)_{i \in I},$$

     is an isomorphism.
   - $q = 1$: A family $(f_{ij})_{ij} \in C^1(\mathscr{U},\mathscr{F})$ is a 1-cocycle iff for all $i,j,k \in I$

     $$0 = f_{jk} - f_{ik} + f_{ij}$$

     i.e. the cochain satisfies on the 3-fold intersections

$$U_{ijk} := U_i \cap U_j \cap U_k$$

the *cocycle condition*

$$f_{ik} = f_{ij} + f_{jk}.$$

2. *Iteration of the coboundary operator*: One checks that the composition of the coboundary operator from Definition 3.1 satisfies for each $q \in \mathbb{N}$ the chain condition

$$\delta^q \circ \delta^{q-1} = 0,$$

i.e. cochains and coboundary form a *complex* of Abelian groups. For the proof one uses that the sum in the definition of the coboundary operator is an alternating sum. For example for $q = 1$:

$$C^0(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^0} C^1(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^1} C^2(\mathscr{U},\mathscr{F})$$

satisfies

$$\delta^0((f_i)_{i\in I}) = (g_{ij} := f_j - f_i)_{i,j\in I} \in C^1(\mathscr{U},\mathscr{F})$$

and

$$\delta^1((g_{ij})_{i,j\in I}) := ((h_{ijk} = g_{jk} - g_{ik} + g_{ij})_{ijk\in I}) \in C^2(\mathscr{U},\mathscr{F}).$$

As a consequence

$$h_{ijk} = (f_k - f_j) - (f_k - f_i) + (f_j - f_i) = 0.$$

Hence

$$B^1(\mathscr{U},\mathscr{F}) \subset Z^1(\mathscr{U},\mathscr{F})$$

and the first cohomology group is well-defined as

$$H^1(\mathscr{U},\mathscr{F}) := \frac{Z^1(\mathscr{U},\mathscr{F})}{B^1(\mathscr{U},\mathscr{F})}$$

Cocycles and coboundaries are a suitable language to express the Cousin problems from Definition 2.29.

Proposition 3.3 (The Cousin problems as problems for Čech cohomology). *Consider a complex manifold X and an open covering $\mathscr{U} = (U_i)_{i\in I}$ of X.*

1. *An additive Cousin distribution with respect to $\mathscr{U}$ is a 0-cochain with values in the sheaf $\mathscr{M}$*

$$c := (h_i)_{i\in I} \in C^0(\mathscr{U},\mathscr{M})$$

providing a 1-cocycle with values in the sheaf $\mathscr{O}$

$$\Delta c := (h_{ij} = h_j - h_i)_{ij} \in Z^1(\mathscr{U},\mathscr{O})$$

The Cousin distribution c has a solution iff Δc is even a coboundary, i.e. iff

$$\Delta c \in B^1(\mathscr{U}, \mathscr{O}) \text{ or } [\Delta c] = 0 \in H^1(\mathscr{U}, \mathscr{O}).$$

2. *A multiplicative Cousin distribution with respect to* $\mathscr{U}$ *is a* 0*-cochain with values in the sheaf* $\mathscr{M}^*$

$$c := (h_i)_{i \in I} \in C^0(\mathscr{U}, \mathscr{M}^*)$$

providing a 1*-cocycle with values in the multiplicative sheaf* $\mathscr{O}^*$

$$\Delta c := \left(\frac{h_j}{h_i} \right)_{ij} \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

The Cousin distribution c has a solution iff Δc *is even a coboundary, i.e. iff*

$$\Delta c \in B^1(\mathscr{U}, \mathscr{O}^*) \text{ or } [\Delta c] = 0 \in H^1(\mathscr{U}, \mathscr{O}^*).$$

Proof. 1. The cochain

$$\Delta c = (h_{ij})_{ij} \in C^1(\mathscr{U}, \mathscr{O})$$

is a 1-cocycle:

$$h_{ij} + h_{jk} = (h_j - h_i) + (h_k - h_j) = h_k - h_i = h_{ik}$$

Assume that

$$\Delta c \in Z^1(\mathscr{U}, \mathscr{O})$$

is even a coboundary, i.e.

$$\Delta c = \delta^0 g \in B^1(\mathscr{U}, \mathscr{O})$$

with a suitable

$$g = (g_i)_i \in C^0(\mathscr{U}, \mathscr{O}),$$

then on each U_{ij}

$$h_{ij} = h_j - h_i = (\delta g)_{ij} = g_j - g_i$$

i.e.

$$h_j - g_j = h_i - g_i.$$

Hence

$$h := (h_i - g_i)_{i \in I} \in Z^0(\mathscr{U}, \mathscr{M}) = \mathscr{M}(X)$$

is a solution of the additive Cousin distribution. Reverting the argument proves the opposite direction of the claim.

2. Analogous to the first part. □

The question whether a given Cousin distribution has a solution translates into the question: Is the 1-cocycle induced by the Cousin distribution a coboundary? Hence the solvability of a Cousin distribution is equivalent to the vanishing of the induced cohomology class

$$[\Delta c] \in H^1(\mathscr{U},\mathscr{O}) \text{ resp. } [\Delta c] \in H^1(\mathscr{U},\mathscr{O}^*).$$

The cohomological formulation of the Cousin problems provides an example that the first cohomology group

$$H^1(\mathscr{U},\mathscr{F})$$

collects the obstructions against glueing local solutions of a problem to a global solution of the problem. Proposition 3.21 will show that on a smooth manifold all obstructions with values in the smooth structure sheaf or in the sheaf of smooth differential forms vanish. An analoguous result does not hold in the category of holomorphic functions on an arbitrary complex manifold. Hence it is an important result to prove the vanishing theorems on Stein manifolds in Chapter 6.

Our next aim is to remove the dependency of the cohomology on a given open covering of the complex manifold X. We show how to abstract from the covering to obtain a cohomology theory which only depends on X and on the sheaf $\mathscr{F}$.

Definition 3.4 (Refinement). Consider a topological space X and two open coverings of X

$$\mathscr{U} = (U_i)_{i\in I} \text{ and } \mathscr{V} = (V_\alpha)_{\alpha\in A}$$

The covering $\mathscr{V}$ is a *refinement* of the covering $\mathscr{U}$, notation

$$\mathscr{V} < \mathscr{U},$$

if there exists a map, named *refinement map*,

$$\tau : A \to I \text{ such that for all } \alpha \in A : V_\alpha \subset U_{\tau(\alpha)}.$$

The refinement map attaches to each open subset V of the finer covering an open subset U of the coarser covering with $V \subset U$. In general, the refinement map of a refinement $\mathscr{V} < \mathscr{U}$ is not uniquely determined. But any two refinement maps induce homotopic maps between the cochain groups with respect to the coverings $\mathscr{V}$ and $\mathscr{U}$. And due to the homotopy, both maps induce the same map in cohomology.

Proposition 3.5 (Chain homotopy). *Consider a topological space X and a presheaf $\mathscr{F}$ on X. For each pair of refinements*

$$\mathscr{U} = (U_i)_{i\in I} < \mathscr{V} = (V_\alpha)_{\alpha\in A}$$

the refinement map

$$\tau : A \to I$$

induces a family of morphisms

$$t_{\mathscr{V}}^{\mathscr{U}} : H^q(\mathscr{U}, \mathscr{F}) \to H^q(\mathscr{V}, \mathscr{F}),\ q \in \mathbb{N}.$$

These morphisms are independent from the choice of τ.

Proof. i) *Construction of the morphisms*: For each $q \in \mathbb{N}$ the refinement map τ defines a restriction on the level of cochains, namely the group morphism

$$t : C^q(\mathscr{U}, \mathscr{F}) \to C^q(\mathscr{V}, \mathscr{F}),\ \xi = (\xi_{i_0 \ldots i_q}) \mapsto t(\xi) = (\eta_{\alpha_0 \ldots \alpha_q}),$$

with

$$\eta_{\alpha_0 \ldots \alpha_q} := \xi_{\tau(\alpha_0) \ldots \tau(\alpha_q)} | V_{\alpha_0} \cap \ldots \cap V_{\alpha_q}$$

The group morphism is compatible with the coboundary operator, i.e. the following diagram commutes

$$\begin{array}{ccc} C^q(\mathscr{U}, \mathscr{F}) & \xrightarrow{t} & C^q(\mathscr{V}, \mathscr{F}) \\ \delta \downarrow & & \delta \downarrow \\ C^{q+1}(\mathscr{U}, \mathscr{F}) & \xrightarrow{t} & C^{q+1}(\mathscr{V}, \mathscr{F}) \end{array}$$

Hence the morphism induces a group morphism on the level of cocycles

$$t : Z^q(\mathscr{U}, \mathscr{F}) \to Z^q(\mathscr{V}, \mathscr{F}),$$

on the level of coboundaries

$$t : B^q(\mathscr{U}, \mathscr{F}) \to B^q(\mathscr{V}, \mathscr{F}),$$

and on the level of cohomology

$$t : H^q(\mathscr{U}, \mathscr{F}) \to H^q(\mathscr{V}, \mathscr{F}).$$

ii) *Chain homotopy*: Assume two refinement maps

$$\tau_j : A \to I,\ j = 1, 2,$$

and denote by $t_j,\ j = 1, 2$, the induced families of restriction maps. We construct a family of group morphisms

$$h^q : H^{q+1}(\mathscr{U}, \mathscr{F}) \to H^q(\mathscr{V}, \mathscr{F}),\ q \in \mathbb{N},$$

as homotopies between the family t_1 and the family t_2, i.e. such that with respect to the following diagram

$$\begin{array}{ccccc}
C^{q-1}(\mathscr{U},\mathscr{F}) & \xrightarrow{\delta} & C^q(\mathscr{U},\mathscr{F}) & \xrightarrow{\delta} & C^{q+1}(\mathscr{U},\mathscr{F}) \\
 & \swarrow h^q & \downarrow t_1 \mid t_2 & \swarrow h^{q+1} & \\
C^{q-1}(\mathscr{V},\mathscr{F}) & \xrightarrow{\delta} & C^q(\mathscr{V},\mathscr{F}) & \xrightarrow{\delta} & C^{q+1}(\mathscr{V},\mathscr{F})
\end{array}$$

holds

$$t_1 - t_2 = \delta \circ h^q + h^{q+1} \circ \delta.$$

Define

$$h^q : C^q(\mathscr{U},\mathscr{F}) \to C^{q-1}(\mathscr{V},\mathscr{F}),\ f \mapsto h^q(f),$$

with

$$(h^q f)_{\alpha_0 \ldots \alpha_{q-1}} := \sum_{k=0}^{q-1} (-1)^k \cdot f_{\tau_1(\alpha_0)\ldots\tau_1(\alpha_k)\tau_2(\alpha_k)\ldots\tau_2(\alpha_{q-1})}|V$$

for

$$V := V_{\alpha_0} \cap \ldots \cap V_{\alpha_{q-1}} \subset (U_{\tau_1(\alpha_0)} \cap \ldots \cap U_{\tau_1(\alpha_k)} \cap U_{\tau_2(\alpha_k)} \cap \ldots \cap U_{\tau_2(\alpha_{q-1})}),$$

and check

$$t_1 - t_2 = \delta \circ h^q + h^{q+1} \circ \delta : C^q(\mathscr{U},\mathscr{F}) \to C^q(\mathscr{V},\mathscr{F}).$$

As a consequence, for a cocycle $f \in Z^q(\mathscr{U},\mathscr{F})$

$$t_1(f) - t_2(f) = \delta(h^q(f)) + h^{q+1}(\delta(f)) = \delta(h^q(f)) \in B^q(\mathscr{V},\mathscr{F})$$

or

$$[t_1(f)] - [t_2(f)] = 0 \in H^q(\mathscr{V},\mathscr{F}).$$

□

Proposition 3.6 (Injectivity of the refinement). *Consider a topological space X and a sheaf $\mathscr{F}$ on X. Each refinement of two open coverings*

$$\mathscr{V} < \mathscr{U}$$

induces an injection on the level of the first cohomology

$$t_{\mathscr{V}}^{\mathscr{U}} : H^1(\mathscr{U},\mathscr{F}) \to H^1(\mathscr{V},\mathscr{F}).$$

The refinement in higher dimensions is not necessarily injective. Moreover, in general the claim does not hold for a presheaf.

Proof. i) *Refinement*: For $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_\alpha)_{\alpha\in A}$ the refinement map

$$\tau : A \to I \text{ with } V_\alpha \subset U_{\tau(\alpha)}$$

defines on the level of cochains the restriction

$$t : C^1(\mathscr{U},\mathscr{F}) \to C^1(\mathscr{V},\mathscr{F}),\ (\xi_{ij}) \mapsto (\eta_{\alpha\beta}),$$

with

$$\eta_{\alpha\beta} := \xi_{\tau(\alpha)\tau(\beta)}|V_{\alpha\beta},\ V_{\alpha\beta} := V_\alpha \cap V_\beta.$$

The map is also a restriction on the level of cocycles

$$t : Z^1(\mathscr{U},\mathscr{F}) \to Z^1(\mathscr{V},\mathscr{F}).$$

ii) *Injectivity*: Consider a cocycle

$$\xi = (\xi_{ij}) \in Z^1(\mathscr{U},\mathscr{F})$$

and assume that

$$t(\xi) = (\eta_{\alpha\beta}) \in Z^1(\mathscr{V},\mathscr{F})$$

splits as a coboundary

$$(\eta_{\alpha\beta}) = \delta((g_\alpha)) \text{ with } (g_\alpha) \in C^0(\mathscr{V},\mathscr{F}).$$

We have to construct a cochain

$$f = (f_k) \in C^0(\mathscr{U},\mathscr{F}) \text{ with } \xi = \delta f :$$

Consider an arbitrary but fixed index $k \in I$. Using the sheaf axioms for $\mathscr{F}$ we construct a section $f_k \in \mathscr{F}(U_k)$ by gluing local sections on open subsets of U_k: Consider the open covering of U_k induced by $\mathscr{V}$

$$U_k \cap \mathscr{V}$$

and the corresponding cochain

$$(g_\alpha|U_k \cap V_\alpha) \in C^0(U_k \cap \mathscr{V},\mathscr{F}),$$

see Figure 3.1.

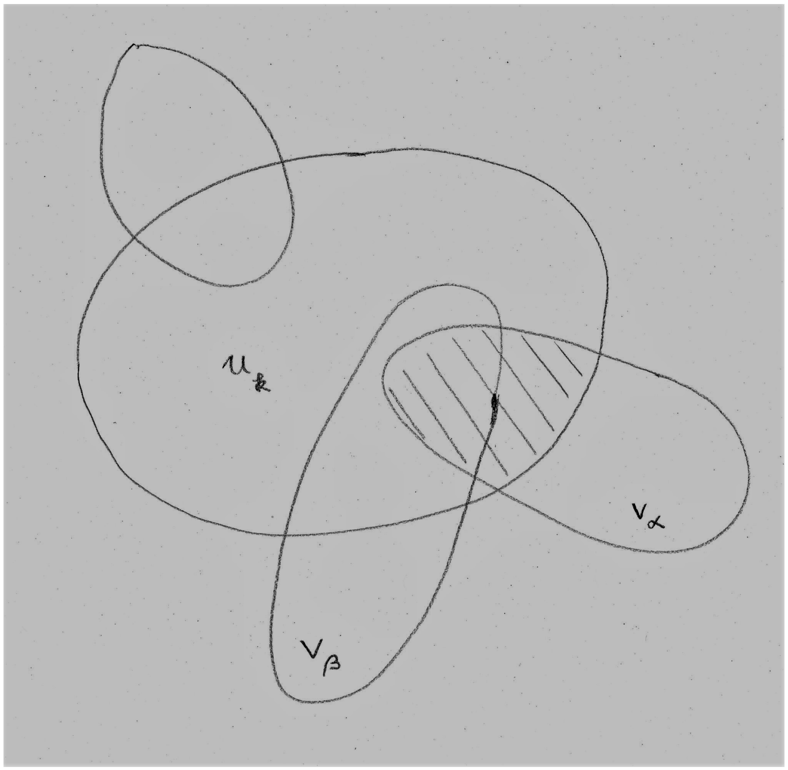


Fig. 3.1 Covering a given U_k by $\mathscr{V}$, dashed $U_k \cap V_\alpha$

On $U_k \cap (V_\alpha \cap V_\beta)$ holds

$$g_\alpha - g_\beta = \eta_{\beta\alpha} = \xi_{\tau(\beta)\tau(\alpha)} = \xi_{\tau(\beta)k} + \xi_{k\tau(\alpha)} = \xi_{\tau(\beta)k} - \xi_{\tau(\alpha)k}$$

Hence for all $\alpha, \beta \in A$

$$g_\alpha + \xi_{\tau(\alpha)k} = g_\beta + \xi_{\tau(\beta)k}$$

The second sheaf axiom for $\mathscr{F}$ provides the section

$$f_k := (g_\alpha + \xi_{\tau(\alpha)k}) \in \mathscr{F}(U_k)$$

Then for each $\alpha \in A$ holds on $(U_i \cap U_j) \cap V_\alpha$

$$\xi_{ij} = \xi_{i\tau(\alpha)} + \xi_{\tau(\alpha)j} = (g_\alpha - f_i) + (f_j - g_\alpha) = f_j - f_i$$

Hence the first sheaf axiom implies on $U_i \cap U_j$

$$\xi_{ij} = f_j - f_i.$$

As a consequence

$$\xi = \delta f.$$

□

For a topological space X and a presheaf $\mathscr{F}$ on X the family of all open coverings of X and restriction maps

$$H^q(\mathscr{U},\mathscr{F}) \xrightarrow{t^{\mathscr{U}}_{\mathscr{V}}} H^q(\mathscr{V},\mathscr{F}) \text{ for } \mathscr{V} < \mathscr{U}$$

is an *inductive family*: The refinement map

$$id : \mathscr{U} \to \mathscr{U}$$

induces the identity $t^{\mathscr{U}}_{\mathscr{U}} = id$, and for refinements

$$\mathscr{W} < \mathscr{V} < \mathscr{U}$$

holds

$$t^{\mathscr{V}}_{\mathscr{W}} \circ t^{\mathscr{U}}_{\mathscr{V}} = t^{\mathscr{U}}_{\mathscr{W}}.$$

Moreover, for each pair of open coverings $\mathscr{U} = (U_i)_i$ and $\mathscr{V} = (V_\alpha)_\alpha$ the covering

$$\mathscr{U} \cap \mathscr{V} := (U_i \cap V_\alpha)_{i,\alpha}$$

satisfies

$$(\mathscr{U} \cap \mathscr{V}) < \mathscr{U} \text{ and } (\mathscr{U} \cap \mathscr{V}) < \mathscr{V}.$$

Definition 3.7 (Čech cohomology as inductive limit). Consider a topological space X and a presheaf $\mathscr{F}$ on X. With respect to the inductive family of all open coverings of X and their refinement maps one defines for each $q \in \mathbb{N}$ the q-th *cohomology group* of X with values in $\mathscr{F}$ as the inductive limit

$$H^q(X,\mathscr{F}) := \varinjlim_{\mathscr{U}} H^q(\mathscr{U},\mathscr{F})$$

The cohomology from Definition 3.7, defined as inductive limit with respect to all open coverings $\mathscr{U}$ of X, is named *Čech cohomology* of X. The corresponding cohomology groups are often written with the Čech accent like in $\check{\mathrm{H}}^q(X,\mathscr{F})$, but we will not use this notation. For cohomology based on resolutions by flabby sheaves see [11, Chap. II, §4.3].

Corollary 3.8 (Injection of the first cohomology). *Consider a topological space X and a sheaf $\mathscr{F}$ on X. For each open covering $\mathscr{U}$ of X the canonical map*

$$H^1(\mathscr{U},\mathscr{F}) \to H^1(X,\mathscr{F})$$

is injective.

Proof. The proof follows from Proposition 3.6. □

3.2 The long exact cohomology sequence

Proposition 3.9 (Long exact cohomology sequence for presheaf morphisms and a fixed covering). *Consider a topological space X and an open covering $\mathscr{U}$ of X. Each short exact sequence of presheaf morphisms on X*

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

induces a sequence of morphisms, named connecting morphisms*,*

$$\delta^* := (\delta_q^* : H^q(\mathscr{U}, \mathscr{H}) \to H^{q+1}(\mathscr{U}, \mathscr{F}))_{q \in \mathbb{N}},$$

such that the following sequence, named long cohomology sequence*, is exact*

$$0 \to H^0(\mathscr{U}, \mathscr{F}) \xrightarrow{\alpha_0} H^0(\mathscr{U}, \mathscr{G}) \xrightarrow{\beta_0} H^0(\mathscr{U}, \mathscr{H}) \xrightarrow{\delta_0^*}$$

$$\xrightarrow{\delta_0^*} H^1(\mathscr{U}, \mathscr{F}) \xrightarrow{\alpha_1} H^1(\mathscr{U}, \mathscr{G}) \xrightarrow{\beta_1} H^1(\mathscr{U}, \mathscr{H}) \xrightarrow{\delta_1^*} \ldots$$

$$\ldots \xrightarrow{\delta_q^*} H^{q+1}(\mathscr{U}, \mathscr{F}) \xrightarrow{\alpha_q} H^{q+1}(\mathscr{U}, \mathscr{G}) \xrightarrow{\beta_q} H^{q+1}(\mathscr{U}, \mathscr{H}) \xrightarrow{\delta_{q+1}^*} \ldots$$

Proof. For each open set $U \subset X$ the assumed exactness of the presheaf morphisms implies the exact sequence

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha_U} \mathscr{G}(U) \xrightarrow{\beta_U} \mathscr{H}(U) \to 0$$

Generalizing the result to cochain groups provides the following commutative diagram with exact rows for each $q \in \mathbb{N}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^q(\mathscr{U}, \mathscr{F}) & \xrightarrow{\alpha_q} & C^q(\mathscr{U}, \mathscr{G}) & \xrightarrow{\beta_q} & C^q(\mathscr{U}, \mathscr{H}) & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & C^{q+1}(\mathscr{U}, \mathscr{F}) & \xrightarrow{\alpha_{q+1}} & C^{q+1}(\mathscr{U}, \mathscr{G}) & \xrightarrow{\beta_{q+1}} & C^{q+1}(\mathscr{U}, \mathscr{H}) & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & C^{q+2}(\mathscr{U}, \mathscr{F}) & \xrightarrow{\alpha_{q+2}} & C^{q+2}(\mathscr{U}, \mathscr{G}) & \xrightarrow{\beta_{q+21}} & C^{q+2}(\mathscr{U}, \mathscr{H}) & \longrightarrow & 0
\end{array}$$

i) *Definition of* δ^*: Define the map

$$\delta_q^* : H^q(\mathscr{U},\mathscr{H}) \to H^{q+1}(\mathscr{U},\mathscr{F})$$

by *descending stairs backwards* along the arrows highlighted in the above diagram to increase level q to level $q+1$: Consider a given class $\xi \in H^q(\mathscr{U},\mathscr{H})$ represented by the cocycle $c \in Z^q(\mathscr{U},\mathscr{H})$. The surjectivity of

$$\beta_q : C^q(\mathscr{U},\mathscr{G}) \to C^q(\mathscr{U},\mathscr{H})$$

provides a cochain $b \in C^q(\mathscr{U},\mathscr{G})$ satisfying

$$\beta_q(b) = c$$

Because

$$0 = \delta(c) = (\delta \circ \beta_q)(b) = \beta_{q+1}(\delta(b))$$

the exactness of the middle row provides a cochain $a \in C^{q+1}(\mathscr{U},\mathscr{F})$ satisfying

$$\alpha_{q+1}(a) = \delta(b)$$

Because

$$\alpha_{q+2}(\delta(a)) = (\alpha_{q+2} \circ \delta)(a) = (\delta \circ \alpha_{q+1})(a) = (\delta \circ \delta)(b) = 0$$

the injectivity of α_{q+2} implies $\delta(a) = 0$, i.e.

$$a \in Z^{q+1}(\mathscr{U},\mathscr{F})$$

is a cocycle. Define now

$$\delta_q^*(\xi) := [a] \in H^{q+1}(\mathscr{U},\mathscr{F})$$

One checks that the class $[a]$ does not depend on the choices made during its definition.

ii) *Exactness at* $H^q(\mathscr{U},\mathscr{H})$: We have to show the exactness of

$$H^q(\mathscr{U},\mathscr{G}) \xrightarrow{\beta_q} H^q(\mathscr{U},\mathscr{H}) \xrightarrow{\delta_q^*} H^{q+1}(\mathscr{U},\mathscr{F})$$

- *Claim ker* $\delta_q^* \subset$ *im* β_q: Consider a given class

  $$\xi = \in H^q(\mathscr{U},\mathscr{H}) \text{ with } \delta_q^*(\xi) = 0 \in H^{q+1}(\mathscr{U},\mathscr{F}).$$

  We have to find a cocycle

  $$\eta \in Z^q(\mathscr{U},\mathscr{G}) \text{ with } \beta_q([\eta]) = \xi.$$

  Choose a representing cocycle

$$c \in Z^q(\mathscr{U}, \mathscr{H}) \text{ with } [c] = \xi.$$

By construction from part i) there exists a cochain

$$b \in C^q(\mathscr{U}, \mathscr{G}) \text{ with } \beta_q(b) = c$$

and a cocycle

$$a \in Z^{q+1}(\mathscr{U}, \mathscr{F}) \text{ with } \alpha_{q+1}(a) = \delta(b) \in B^{q+1}(\mathscr{U}, \mathscr{G})$$

Then

$$\delta_q^*(\xi) = [a] = 0 \implies \exists a' \in C^q(\mathscr{U}, \mathscr{F}) \text{ with } \delta(a') = a \in B^{q+1}(\mathscr{U}, \mathscr{F})$$

The vanishing

$$\delta(b - \alpha_q(a')) = \delta(b) - (\delta \circ \alpha_q)(a') = \delta(b) - \alpha_{q+1}(a) = 0$$

implies

$$b - \alpha_q(a') \in Z^q(\mathscr{U}, \mathscr{G})).$$

Set

$$\eta := [b - \alpha_q(a')] \in H^q(\mathscr{U}, \mathscr{G})$$

Then

$$\beta_q(b - \alpha_q(a')) = \beta_q(b) - (\beta_q \circ \alpha_q)(a') = \beta_q(b) = c.$$

Hence

$$\beta_q([\eta]) = \xi.$$

- *Claim im* $\beta_q \subset$ *ker* δ_q^*: Consider a class $\xi \in H^q(\mathscr{U}, \mathscr{H})$ represented as

$$\xi = [c] \text{ with } c \in Z^q(\mathscr{U}, \mathscr{H})$$

and assume the existence of a cocycle

$$b \in Z^q(\mathscr{U}, \mathscr{G}) \text{ with } \beta_q(b) = c.$$

By construction exists

$$a \in Z^{q+1}(\mathscr{U}, \mathscr{F}) \text{ with } \alpha_q(a) = \delta(b)$$

From

$$\delta(b) = 0 \in C^{q+1}(\mathscr{U}, \mathscr{G})$$

follows due to the injectivity of α_{q+1} the vanishing

$$a = 0 \in Z^{q+1}(\mathscr{U}, \mathscr{F}).$$

Hence

$$\delta_q^*(\xi) = [a] = 0 \in H^{q+1}(\mathscr{U}, \mathscr{F}).$$

iii) *Exactness at* $H^q(\mathscr{U}, \mathscr{G})$: We have to show the exactness of

$$H^q(\mathscr{U}, \mathscr{F}) \xrightarrow{\alpha_q} H^q(\mathscr{U}, \mathscr{G}) \xrightarrow{\beta_q} H^q(\mathscr{U}, \mathscr{H})$$

- *Claim ker* $\beta_q \subset$ *im* α_q: Consider a class

  $$[b] \in H^q(\mathscr{U}, \mathscr{G}) \text{ with } \beta_q([b]) = 0 \in H^q(\mathscr{U}, \mathscr{H})$$

  Then $\beta_q(b) \in B^q(\mathscr{U}, \mathscr{H})$, i.e. there exists

  $$c' \in C^{q-1}(\mathscr{U}, \mathscr{H}) \text{ with } \delta(c') = \beta_q(b)$$

  The surjectivity of β_q provides a cochain

  $$b' \in C^{q-1}(\mathscr{U}, \mathscr{G}) \text{ with } \beta_{q-1}(b') = c'$$

  The commutativity

  $$\delta \circ \beta_{q-1} = \beta_q \circ \delta$$

  implies

  $$\beta_q(b - \delta(b')) = 0.$$

  The exactness at $C^q(\mathscr{U}, \mathscr{G})$ provides an element

  $$a \in C^q(\mathscr{U}, \mathscr{F}) \text{ with } \alpha_q(a) = b - \delta(b')$$

  Using the commutativity

  $$\alpha_{q+1} \circ \delta = \delta \circ \alpha_q$$

  and the injectivity of α_{q+1} shows $\delta(a) = 0$, hence $a \in Z^q(\mathscr{U}, \mathscr{F})$ and

  $$\alpha_q([a]) = [b - \delta(b')] = [b]$$

- *Claim im* $\alpha_q \subset$ *ker* β_q: The claim follows from the cochain property $\beta_q \circ \alpha_q = 0$.

iv) *Exactness at* $H^q(\mathscr{U}, \mathscr{F})$: We have to show the exactness of

$$H^{q-1}(\mathscr{U}, \mathscr{H}) \xrightarrow{\delta_{q-1}^*} H^q(\mathscr{U}, \mathscr{F}) \xrightarrow{\alpha_q} H^q(\mathscr{U}, \mathscr{G})$$

- *Claim ker* $\alpha_q \subset$ *im* δ_{q-1}^*: Consider a class $[a] \in H^q(\mathscr{U}, \mathscr{F})$ with

  $$\alpha_q([a]) = [\alpha_q(a)] = 0 \in H^q(\mathscr{U}, \mathscr{G})$$

  Then the cocycle $\alpha_q(a) \in Z^q(\mathscr{U}, \mathscr{G})$ is a coboundary, i.e.

  $$\alpha_q(a) = \delta(b) \in B^q(\mathscr{U}, \mathscr{G})$$

for a suitable $b \in C^{q-1}(\mathscr{U},\mathscr{G})$. As a consequence

$$\delta(\beta_{q-1}(b)) = (\delta \circ \beta_{q-1})(b) = (\beta_q \circ \delta)(b) = \beta_q(\alpha_q) = (\beta_q \circ \alpha_q)(a) = 0,$$

which implies $\beta_{q-1}(b) \in Z^{q-1}(\mathscr{U},\mathscr{H})$. By construction

$$[a] = \delta^*_{q-1}([\beta_{q-1}(b)])$$

- *Claim im* $\delta^*_{q-1} \subset$ *ker* α_q: Consider a class

$$\delta^*_{q-1}(\xi) = [a] \in H^q(\mathscr{U},\mathscr{H}) \text{ with } a \in Z^q(\mathscr{U},\mathscr{F})$$

satisfying

$$\alpha_q(a) = \delta(b) \text{ for a suitable } b \in C^{q-1}(\mathscr{U},\mathscr{G})$$

Then

$$\alpha_q([a]) = [\alpha_q(a)] = [\delta(b)] = 0 \in H^q(\mathscr{U},\mathscr{G})$$

□

Corollary 3.10 (Long exact cohomology sequence for presheaf morphisms). *Each short exact sequence*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

of presheaf morphisms on a topological space X induces a long exact cohomology sequence

$$0 \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{G}) \to H^0(X,\mathscr{H}) \xrightarrow{\delta_0^*}$$

$$\xrightarrow{\delta^*} H^1(X,\mathscr{F}) \to H^1(X,\mathscr{G}) \to H^1(X,\mathscr{H}) \xrightarrow{\delta_1^*} \ldots$$

$$\ldots \xrightarrow{\delta_q^*} H^{q+1}(X,\mathscr{F}) \to H^{q+1}(X,\mathscr{G}) \to H^{q+1}(X,\mathscr{H}) \xrightarrow{\delta_{q+1}^*} \ldots$$

Proof. One applies the functor *inductive limit* to the long exact sequence from Proposition 3.9. The functor is exact, see [1, Chap. 2, Ex. 19]. □

Lemma 3.11 prepares the proof that on a paracompact space X the long exact cohomology sequence from Corollary 3.10 also originates from an exact sequence of *sheaf* morphisms - not only presheaf morphisms.

Lemma 3.11 (Sheafification and cohomology). *On a paracompact topological space X a presheaf* $\mathscr{F}$ *and its sheafification* $\hat{\mathscr{F}}$ *have the same cohomology, i.e. the canonical map*

$$\mathscr{F} \to \hat{\mathscr{F}}$$

induces for all $q \in \mathbb{N}$ *isomorphisms*

$$H^q(X,\mathscr{F}) \xrightarrow{\simeq} H^q(X,\hat{\mathscr{F}}).$$

Proof. See [23, Satz 33.7], the book uses the term "Garbendatum" which means "presheaf". The proof relies on the following properties:

- For a presheaf $\mathscr{G}$ on a paracompact topological space X with sheafification $\hat{\mathscr{G}} = 0$ all cohomology groups vanish, i.e.

  $$H^q(X, \mathscr{G}) = 0 \text{ for all } q \in \mathbb{N},$$

  see [23, Satz 33.6].

- Sheafification is an exact functor on short exact sequences of presheaf morphisms. □

Theorem 3.12 (Long exact cohomology sequence for sheaf morphisms). *Consider a paracompact topological space X. Each short exact sequence of sheaf morphisms on X*

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

induces a long exact cohomology sequence

$$0 \to H^0(X, \mathscr{F}) \xrightarrow{\alpha} H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \xrightarrow{\delta_0^*}$$

$$\xrightarrow{\delta^*} H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to H^1(X, \mathscr{H}) \xrightarrow{\delta_1^*} \dots$$

$$\dots \xrightarrow{\delta_q^*} H^{q+1}(X, \mathscr{F}) \to H^{q+1}(X, \mathscr{G}) \to H^{q+1}(X, \mathscr{H}) \xrightarrow{\delta_{q+1}^*} \dots$$

Note that the given short exact sequence of sheaf morphisms is not necessarily exact when considered sequence of presheaf morphisms. Hence Theorem 3.12 is not a particular case of Corollary 3.10.

Proof. We consider the quotient presheaf $\mathscr{Q}$ on X

$$\mathscr{Q}(U) := coker\ [\mathscr{F}(U) \xrightarrow{\alpha_U} \mathscr{G}(U)] = \mathscr{G}(U)/\alpha_U(\mathscr{F}(U)),\ U \subset X \text{ open},$$

with the induced restrictions, and obtain the short exact sequence of presheaves on X

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{Q} \to 0$$

Corollary 3.10 provides the corresponding long exact cohomology sequence which contains the cohomology groups $H^q(X, \mathscr{Q})$. Because

$$\mathscr{H} = \hat{\mathscr{Q}}$$

Lemma 3.11 concludes

$$H^q(X, \mathscr{H}) = H^q(X, \hat{\mathscr{Q}}) \simeq H^q(X, \mathscr{Q}),$$

which finishes the proof. □

3.3 Acyclic sheaves and applications

Definition 3.13 (Acyclic sheaf, flabby sheaf). Consider a topological space X.

1. A sheaf $\mathscr{F}$ on X is *acyclic* if for all $q \geq 1$

$$H^q(X, \mathscr{F}) = 0.$$

2. A sheaf $\mathscr{F}$ on X is *flabby* or *flasque* (deutsch: *welk*) if for each open set $U \subset X$ the canonical restriction

$$\mathscr{F}(X) \to \mathscr{F}(U),\ f \mapsto f|U,$$

is surjective, i.e. if each section over an open subset extends to all of X.

Proposition 3.14 (Flabby sheaves are acyclic). *Each flabby sheaf $\mathscr{F}$ on a topologial space X satisfies for each open covering $\mathscr{U} = (U_i)_{i \in I}$ of X and for all $q \geq 1$*

$$H^q(\mathscr{U}, \mathscr{F}) = 0.$$

In particular $\mathscr{F}$ is acyclic.

Proof. Assume $q \geq 1$ and consider a cocycle

$$\xi = (\xi_{i_0 \dots i_q})_{(i_0, \dots, i_q) \in I^{q+1}} \in Z^q(\mathscr{U}, \mathscr{F}).$$

i) *Splitting over small open sets*: If $i^* \in I$ and $Y \subset U_{i^*}$ open, then ξ splits over Y: Define

$$\eta := (\eta_{i_0 \dots i_{q-1}})_{(i_0, \dots, i_{q-1})} \in C^{q-1}(\mathscr{U} \cap Y, \mathscr{F})$$

as the family of sections

$$\eta_{i_0, \dots, i_{q-1}} := \xi_{i^* i_0 \dots i_{q-1}}|Y,$$

see Figure 3.2.

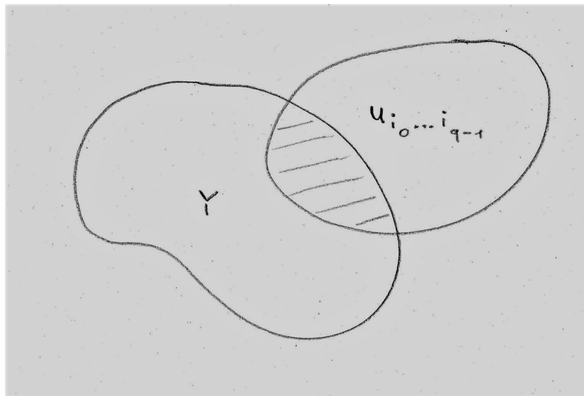


Fig. 3.2 $Y \subset U_{i^*}$, dashed $Y \cap U_{i_0 \dots i_{q-1}}$

Then

$$(\delta\eta)_{i_0,\dots,i_q} = \sum_{\alpha=0}^{q} (-1)^\alpha \cdot \xi_{i^* i_0 \dots \hat{i}_\alpha \dots i_q}$$

The cocycle condition with repect to $\mathscr{U}$

$$0 = (\delta\xi)_{i^* i_0 \dots i_q} = \xi_{i_0,\dots,i_q} - \sum_{\alpha=0}^{q} (-1)^\alpha \cdot \xi_{i^* i_0 \dots \hat{i}_\alpha \dots i_q}$$

implies

$$\sum_{\alpha=0}^{q} (-1)^\alpha \cdot \xi_{i^* i_0 \dots \hat{i}_\alpha \dots i_q} = \xi_{i_0,\dots,i_q},$$

hence

$$\delta\eta = \xi|Y$$

ii) *Maximal splittings due to Zorn's lemma*: Consider the set of local splittings

$$Split(\xi) := \left\{(Y,\eta) : \ Y \subset X \text{ open}, \eta \in C^{q-1}(\mathscr{U} \cap Y, \mathscr{F}) \text{ with } \delta\eta = \xi|Y\right\}$$

Due to part i) the set $Split(\xi)$ is not empty. The set is partially ordered by

$$(Y,\eta) \leq (Y',\eta') :\Longleftrightarrow \ Y \subset Y' \text{ and } \eta'|Y = \eta.$$

For each totally ordered subset $\tilde{S} \subset Split(\xi)$ consider the open set

$$\tilde{Y} := \bigcup \{Y \subset X : \exists \ \eta_Y \text{ with } (Y,\eta_Y) \in \tilde{S}\}$$

The definition of the partial order and the second sheaf axiom provide a cochain on $\tilde{Y}$

$$\tilde{\eta} \in C^{q-1}(\mathscr{U} \cap \tilde{Y}, \mathscr{F})$$

satisfying for all Y with $(Y, \eta_Y) \in \tilde{S}$

$$\tilde{\eta}|Y = \eta_Y \text{ and } \delta\tilde{\eta}|Y = \delta(\tilde{\eta}|Y) = \delta\eta_Y = \xi|Y$$

The first sheaf axiom ensures

$$\delta\tilde{\eta} = \xi|\tilde{Y},$$

which shows that $\tilde{S}$ has the upper bound $(\tilde{Y}, \tilde{\eta})$. Zorn's lemma provides a maximal element

$$(Z, \eta) \in Split(\xi).$$

iii) *Global splitting*: We now apply the assumption that the sheaf $\mathscr{F}$ is flabby. We give an indirect proof that a maximal element $(Z, \eta) \in Split(\xi)$ from part ii) provides a splitting of ξ on X.

If $Z \neq X$ then exists an index $i_0 \in I$ with $U_{i_0} \not\subset Z$. Set

$$Y' := Z \cup U_{i_0}$$

Part i) provides a cochain $a \in C^{q-1}(\mathscr{U} \cap U_{i_0}, \mathscr{F})$ satisfying

$$\delta a = \xi|U_{i_0}$$

As a consequence on $(Z \cap U_{i_0}) \subset U_{i_0}$ holds $\delta(\eta - a) = 0$. We now consider the cocyle

$$\eta - a \in Z^{q-1}(\mathscr{U} \cap (Z \cap U_{i_0}), \mathscr{F}).$$

- *Case* $q \geq 2$: The cocycle

  $$\eta - a \in Z^{q-1}(\mathscr{U} \cap (Z \cap U_{i_0}), \mathscr{F})$$

  does not extend to the cover of Y'

  $$\mathscr{U} \cap Y'$$

  necessarily as a *cocycle*. Because extending sections does not preserve the cocycle condition. Therefore we extend a suitable *cochain* from $C^{q-2}(\mathscr{U} \cap (Z \cap U_{i_0}), \mathscr{F})$.

  Part i) provides an element $\tilde{b} \in C^{q-2}(\mathscr{U} \cap (Z \cap U_{i_0}), \mathscr{F})$ satisfying

  $$\delta\tilde{b} = \eta - a$$

  Because $(U_{i_0} \cap Z) \subset U_{i_0}$ the flabbyness of $\mathscr{F}$ allows to extend $\tilde{b}$ to an element

  $$b \in C^{q-2}(\mathscr{U} \cap U_{i_0}, \mathscr{F}) \text{ with } b|(U_{i_0} \cap Z) = \tilde{b}$$

  We define an element $\eta' \in C^{q-1}(\mathscr{U} \cap Y', \mathscr{F})$ as

$$\eta' := \begin{cases} \eta & \text{on } Z \\ a + \delta b & \text{on } U_{i_0} \end{cases}$$

It satisfies

$$\eta'|Z = \eta \text{ and } \delta\eta' = \xi|Y' \text{ because } \delta(\eta'|U_{i_0}) = \delta a|U_0 = \xi|U_{i_0}$$

- *Case* $q = 1$: By assumption

$$\eta - a \in Z^0(\mathscr{U} \cap (Z \cap U_{i_0}), \mathscr{F}) = \mathscr{F}(Z \cap U_{i_0})$$

The flabbyness of $\mathscr{F}$ extends the section $\eta - a$ to a section

$$b \in \mathscr{F}(U_{i_0}) \text{ with } b|(Z \cap U_{i_0}) = \eta - a$$

Using

$$b \in Z^0(\mathscr{U} \cap U_{i_0}, \mathscr{F}) \subset C^0(\mathscr{U} \cap U_{i_0}, \mathscr{F})$$

we define the cochain $\eta' \in C^1(\mathscr{U} \cap Y', \mathscr{F})$ as

$$\eta' := \begin{cases} \eta & \text{on } Z \\ a + b & \text{on } U_{i_0} \end{cases}$$

obtaining

$$\eta'|Z = \eta \text{ and } \delta\eta' = \xi|Y' \text{ because } \delta(a+b)|Z \cap U_{i_0} = \delta\eta|Z \cap U_{i_0} = \xi|Z \cap U_{i_0}$$

In both cases we obtain a proper extension (Y', η') satisfying

$$(Y', \eta') \geq (Z, \eta) \text{ but } Y' \supset Z,\ Y' \neq Z,$$

a contradiction to the maximality of (Z, η). □

Theorem 3.15 (Canonical flabby resolution). *Each sheaf $\mathscr{F}$ on a topological space X has a* canonical resolution *by flabby sheaves*

$$0 \to \mathscr{F} \to \mathscr{W}^0 \to \mathscr{W}^1 \to \ldots$$

Proof. We construct the resolution from left step by step by induction.

i) Induction start $\mathscr{W}^0$: Set

$$\mathscr{W}^0 := w(\mathscr{F}),$$

the sheaf from the proof of Theorem 2.16. The sheaf $w(\mathscr{F})$ is flabby because each section $s_U \in w(\mathscr{F})(U)$ extends by zero to a global section $s \in w(\mathscr{F})(X)$. The maps

$$\mathscr{F} \to w(\mathscr{F}),\ \phi \mapsto (\phi_x)_{x \in U},\ U \subset X \text{ open},$$

define a morphism of sheaves $\mathscr{F} \to w(\mathscr{F})$ which is injective. Hence the sequence of sheaf morphisms

$$0 \to \mathscr{F} \to \mathscr{W}^0$$

is exact.

ii) Induction step $k \mapsto k+1$: Assume an exact sequence of lenght k of sheaf morphisms with flabby sheaves $\mathscr{W}^i,\ i = 0, ..., k$

$$0 \to \mathscr{F} \to \mathscr{W}^0 \to ... \to \mathscr{W}^{k-1} \xrightarrow{\alpha_{k-1}} \mathscr{W}^k$$

The sheaf

$$coker\ \alpha_{k-1},$$

the sheafification of the corresponding presheaf according to Theorem 2.16, fits into the canonical exact sequence of sheaf morphisms

$$\mathscr{W}^{k-1} \xrightarrow{\alpha_{k-1}} \mathscr{W}^k \xrightarrow{\pi_k} coker\ \alpha_{k-1} \to 0$$

Set

$$\mathscr{W}^{k+1} := w(coker\ \alpha_{k-1})$$

and consider the canonical injection, a sheaf morphism,

$$j_k : coker\ \alpha_{k-1} \hookrightarrow \mathscr{W}^{k+1}$$

Then the sequence

$$\mathscr{W}^{k-1} \xrightarrow{\alpha_{k-1}} \mathscr{W}^k \xrightarrow{\alpha_k := j_k \circ \pi_k} \mathscr{W}^{k+1}$$

is exact and extends the assumed exact sequence of flabby sheaves. □

Proposition 3.16 (Abstract de Rahm theorem). *Consider a paracompact space X and assume that the sheaf $\mathscr{F}$ on X has a resolution by acyclic sheaves $\mathscr{G}^j,\ j \in \mathbb{N}$,*

$$0 \to \mathscr{F} \to \mathscr{G}^0 \to \mathscr{G}^1 \to ...$$

Then the cohomology of $\mathscr{F}$ can be computed as

$$H^q(X, \mathscr{F}) \simeq \frac{ker\ [\mathscr{G}^q(X) \to \mathscr{G}^{q+1}(X)]}{im\ [\mathscr{G}^{q-1}(X) \to \mathscr{G}^q(X)]},\ q \in \mathbb{N}.$$

Here $\mathscr{G}^{-1} := 0$.

Proof. i) *Case $q = 0$*: The resolution implies

$$\mathscr{F} \simeq ker\ [\mathscr{G}^0 \to \mathscr{G}^1].$$

The left exactness of the functor $\Gamma(X, -)$ with

$$\Gamma(X,\mathscr{G}) := \mathscr{G}(X)$$

for a sheaf $\mathscr{G}$ on X ensures the exactness of

$$0 \to \mathscr{F}(X) \to \mathscr{G}^0(X) \to \mathscr{G}^1(X)$$

which implies

$$H^0(X,\mathscr{F}) = \mathscr{F}(X) \simeq ker\ [\mathscr{G}^0(X) \to \mathscr{G}^1(X)]$$

and proves the case $q = 0$.

ii) *Case* $q \geq 1$: The idea of the proof is to split the long exact sequence of sheaf morphisms from the resolution into a family of short exact sequences. For each short exact sequence one considers the long exact cohomology sequence, and takes into account that the sheaves $\mathscr{G}^n$ are acyclic. For $n \in \mathbb{N}$ set

$$\mathscr{K}^n := ker\ [\mathscr{G}^n \to \mathscr{G}^{n+1}]$$

and consider the canonical short exact sequence of sheaf morphisms

$$0 \to \mathscr{K}^n \to \mathscr{G}^n \to \mathscr{K}^{n+1} \to 0.$$

Theorem 3.12 provides for each $q \geq 1$ the following section of its long exact sequence

$$0 = H^q(X,\mathscr{G}^n) \to H^q(X,\mathscr{K}^{n+1}) \xrightarrow{\delta_q^*} H^{q+1}(X,\mathscr{K}^n) \to H^{q+1}(X,\mathscr{G}^n) = 0.$$

Hence the connection morphisms are isomorphisms

$$\delta_q^* : H^q(X,\mathscr{K}^{n+1}) \simeq H^{q+1}(X,\mathscr{K}^n),$$

which implies successively for $q \geq 1$

$$H^q(X,\mathscr{F}) = H^q(X,\mathscr{K}^0) \simeq H^{q-1}(X,\mathscr{K}^1) \simeq H^1(X,\mathscr{K}^{q-1}),$$

and for

$$0 \to \mathscr{K}^{q-1} \to \mathscr{G}^{q-1} \to \mathscr{K}^q \to 0$$

the exact sequence

$$H^0(X,\mathscr{G}^{q-1}) \to H^0(X,\mathscr{K}^q) \xrightarrow{\delta^*} H^1(X,\mathscr{K}^{q-1}) \to 0 = H^1(X,\mathscr{G}^{q-1})$$

The last exact sequence implies

$$H^1(X,\mathscr{K}^{q-1}) \simeq \frac{H^0(X,\mathscr{K}^q)}{im\ [H^0(X,\mathscr{G}^{q-1}) \to H^0(X,\mathscr{K}^q)]}$$

Inserting the result into the second last isomorphy implies

$$H^q(X,\mathscr{F}) \simeq \frac{H^0(X,\mathscr{K}^q)}{im\ [H^0(X,\mathscr{G}^{q-1}) \to H^0(X,\mathscr{K}^q)]} = \frac{ker\ [\mathscr{G}^q(X) \to \mathscr{G}^{q+1}(X)]}{im\ [\mathscr{G}^{q-1}(X) \to \mathscr{G}^q(X)]}$$

Here we use the apparent equalities

$$H^0(X,\mathscr{K}^q) = ker\ [H^0(X,\mathscr{G}^q) \to H^0(X,\mathscr{G}^{q+1})]$$

and

$$im\ [\mathscr{G}^{q-1}(X) \to \mathscr{K}^q(X)] = im\ [\mathscr{G}^{q-1}(X) \to \mathscr{G}^q(X)]$$

□

Corollary 3.17 (Cohomology via flabby resolutions). *Consider a paracompact topological space X and a sheaf $\mathscr{F}$ on X. Each resolution of $\mathscr{F}$ by flabby sheaves*

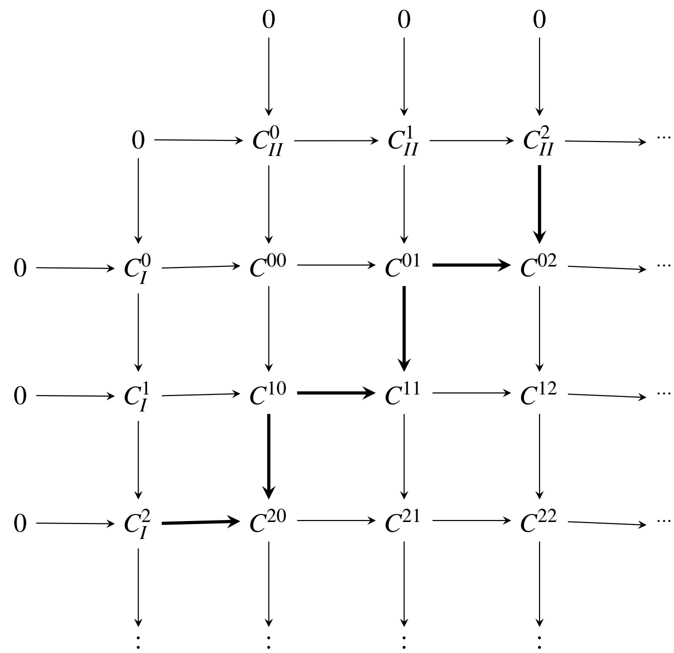
$$0 \to \mathscr{F} \to \mathscr{W}^0 \to \mathscr{W}^1 \to \ldots$$

induces for each $q \in \mathbb{N}$ an isomophism

$$\alpha_q : \frac{ker\ [\mathscr{W}^q(X) \to \mathscr{W}^{q+1}(X)]}{im\ [\mathscr{W}^{q-1}(X) \to \mathscr{W}^q(X)]} \xrightarrow{\simeq} H^q(X,\mathscr{F}),\ \mathscr{W}^{-1} = 0.$$

Proof. The proof follows from Proposition 3.16, because flabby sheaves are acyclic due to Proposition 3.14. □

Lemma 3.18 (Navigation in a double complex). *Consider a double complex of morphisms of Abelian groups*



Assume: With the possible exception of the first column and the first row all other columns and rows are exact . Then the cohomology of the first column is isomorphic to the cohomology of the first row, i.e. for all $q \in \mathbb{N}$

$$H_I^q := \frac{ker\ [C_I^q \to C_I^{q+1}]}{im\ [C_I^{q-1} \to C_I^q]} \simeq H_{II}^q := \frac{ker\ [C_{II}^q \to C_{II}^{q+1}]}{im\ [C_{II}^{q-1} \to C_{II}^q]}$$

Proof. All squares of a double complex are commutative. The proof of the lemma is by stair climbing from the first column to the first row. The above diagram highlights the path on the level of cochains to define the morphism

$$H_I^q \to H_{II}^q$$

for $q = 2$, and also to define its inverse. □

Next we deal with the question: Under which assumptions can the cohomology of a topological space be already computed by considering only a single distinguished covering?

Theorem 3.19 (Leray's theorem). *Consider a metrizable topological space X with an open covering* $\mathscr{U} = (U_i)_{i \in I}$ *and a sheaf* $\mathscr{F}$ *on X. Assume: On each intersection*

$$U_{i_0 \dots i_q} := U_{i_0} \cap \dots \cap U_{i_q},\ q \in \mathbb{N},\ i_0, \dots, i_q \in I^{q+1},$$

the restricted sheaf

$$\mathscr{F}|U_{i_0\ldots i_k}$$

is acyclic. Then for all $q \in \mathbb{N}$ the canonical maps

$$\rho_q : H^q(\mathscr{U},\mathscr{F}) \to H^q(X,\mathscr{F})$$

are isomorphisms.

Theorem 3.19 makes the assumption that X is metrizable. The assumption ensures that the open sets of the covering $\mathscr{U}$ and their finite intersections are also metrizable and therefore paracompact. Note: In general paracompactness is not inherited by open subspaces. The assumption of metrizability is satisfied in the context of complex analysis, because each complex manifold is a regular space and has to satisfy the axiom of second countability, see also Remark 2.3.

Proof. i) *Construction of a double complex*: Theorem 3.15 provides a flabby resolution of $\mathscr{F}$

$$0 \to \mathscr{F} \to \mathscr{W}^0 \to \mathscr{W}^1 \to \ldots$$

The corresponding morphisms extend to a double complex of global sections and cochain groups: The horizontal maps derive from the morphisms of the flabby resolution, the vertical maps are the coboundary maps.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathscr{W}^0(X) & \longrightarrow & \mathscr{W}^1(X) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(\mathscr{U},\mathscr{F}) & \longrightarrow & C^0(\mathscr{U},\mathscr{W}^0) & \longrightarrow & C^0(\mathscr{U},\mathscr{W}^1) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1(\mathscr{U},\mathscr{F}) & \longrightarrow & C^1(\mathscr{U},\mathscr{W}^0) & \longrightarrow & C^1(\mathscr{U},\mathscr{W}^1) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Because all sheaves $\mathscr{W}^j,\ j \in \mathbb{N}$, are flabby, Proposition 3.14 ensures that with the possible exception of the first column all other columns are exact, while the first column with the cochain groups of $\mathscr{F}$ is a complex.

ii) *Exactness of the rows with index* ≥ 2: For given $q \in \mathbb{N}$ and tuple $(i_0, ..., i_q) \in I^{q+1}$ consider the intersection of the covering sets

$$U_\iota := U_{i_0} \cap ... \cap U_{q+1}$$

The open set U_ι is metrizable because X is assumed to be metrizable. In particular U_ι is paracompact. The restriction of the original resolution to U_ι

$$0 \to \mathscr{F}|U_\iota \to \mathscr{W}^0|U_\iota \to \mathscr{W}^1|U_\iota \to ...$$

is a flabby resolution of the restricted sheaf $\mathscr{F}|U_\iota$. Proposition 3.16 implies for $j \in \mathbb{N}$

$$H^j(U_\iota, \mathscr{F}) = \frac{ker\ [\mathscr{W}^j(U_\iota) \to \mathscr{W}^{j+1}(U_\iota)]}{im\ [\mathscr{W}^{j-1}(U_\iota) \to \mathscr{W}^j(U_\iota)]},\ \mathscr{W}^{-1} := 0$$

Due to the acyclicity of $\mathscr{F}|U_\iota$ holds for each $j \geq 1$

$$H^j(U_\iota, \mathscr{F}) = 0$$

Hence the sequence

$$0 \to \mathscr{F}(U_\iota) \to \mathscr{W}^0(U_\iota) \to \mathscr{W}^1(U_\iota) \to ...$$

is exact. Varying $q \in \mathbb{N}$ and the tuple $(i_0, ..., i_q) \in I^{q+1}$ implies the exactness of the complex of cochain groups of the corresponding row

$$0 \to C^q(\mathscr{U}, \mathscr{F}) \to C^q(\mathscr{U}, \mathscr{W}^0) \to C^q(\mathscr{U}, \mathscr{W}^1) \to ...$$

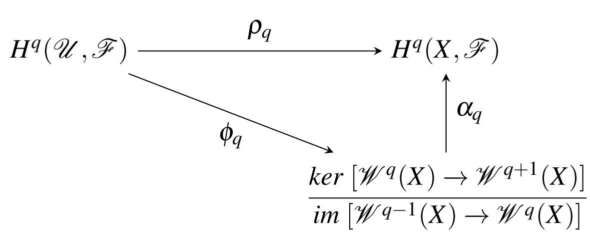
iii) *Navigation in the double complex*: Corollary 3.17 provides for each $q \in \mathbb{N}$ an isomorphism

$$\alpha_q : \frac{ker\ [\mathscr{W}^q(X) \to \mathscr{W}^{q+1}(X)]}{im\ [\mathscr{W}^{q-1}(X) \to \mathscr{W}^q(X)]} \xrightarrow{\simeq} H^q(X, \mathscr{F})$$

Lemma 3.18 applies to the double complex from part i) and provides isomorphisms

$$\phi_q : H^q(\mathscr{U}, \mathscr{F}) \xrightarrow{\simeq} \frac{ker\ [\mathscr{W}^q(X) \to \mathscr{W}^{q+1}(X)]}{im\ [\mathscr{W}^{q-1}(X) \to \mathscr{W}^q(X)]},\ q \in \mathbb{N},$$

such that the following diagram commutes



Because α_q and ϕ_q are isomorphisms, also ρ_q is an isomorphism. $\square$

Theorem 3.20 (Acyclicity of the smooth structure sheaf). *The structure sheaf $\mathscr{E}$ of a smooth manifold $(X,\mathscr{E})$ is acyclic with respect to each open covering*

$$\mathscr{U} = (U_i)_{i\in I}$$

of X, i.e.

$$H^q(\mathscr{U},\mathscr{E}) = 0,\ q \geq 1.$$

In particular

$$H^q(X,\mathscr{E}) = 0,\ q \geq 1.$$

Proof. i) *Partition of unity*: Because X is paracompact there exists a partition of unity $(\eta_i)_{i\in I}$ subordinate to $\mathscr{U}$, i.e. the family of smooth functions

$$\eta_i \in \mathscr{E}(X),\ i \in I,$$

satisfies

- *Range*: For each $i \in I$
$$\eta_i(X) \subset [0,1] \subset \mathbb{R}$$
- *Adapted to $\mathscr{U}$*: $supp\ \eta_i \subset\subset U_i,\ i \in I$.
- *Locally finite*: The family $(supp\ \eta_i)_{i\in I}$ is locally finite.
- *Decomposing the identity*:
$$\sum_{i\in I} \eta_i = 1$$
The sum is well-defined because the family $(supp\ \eta_i)_{i\in I}$ of supports is locally finite.

ii) *Cocycles are coboundaries*: Consider a given cocycle

$$\xi = (\xi_{i_0,\dots,i_q}) \in Z^q(\mathscr{U},\mathscr{E})$$

Set

$$U_{i_0\dots i_{q-1}} := U_{i_0} \cap \dots \cap U_{i_{q-1}}$$

with last index i_{q-1}. For each $i \in I$ the function

$$\eta_i \cdot \xi_{ii_0\dots i_{q-1}} \in \mathscr{E}(U_i \cap U_{i_0\dots i_{q-1}})$$

extends by zero to a smooth function on $U_{i_0\dots i_{q-1}}$, see Figure 3.3. Here one uses

$$supp\ \eta_i \subset\subset U_i.$$

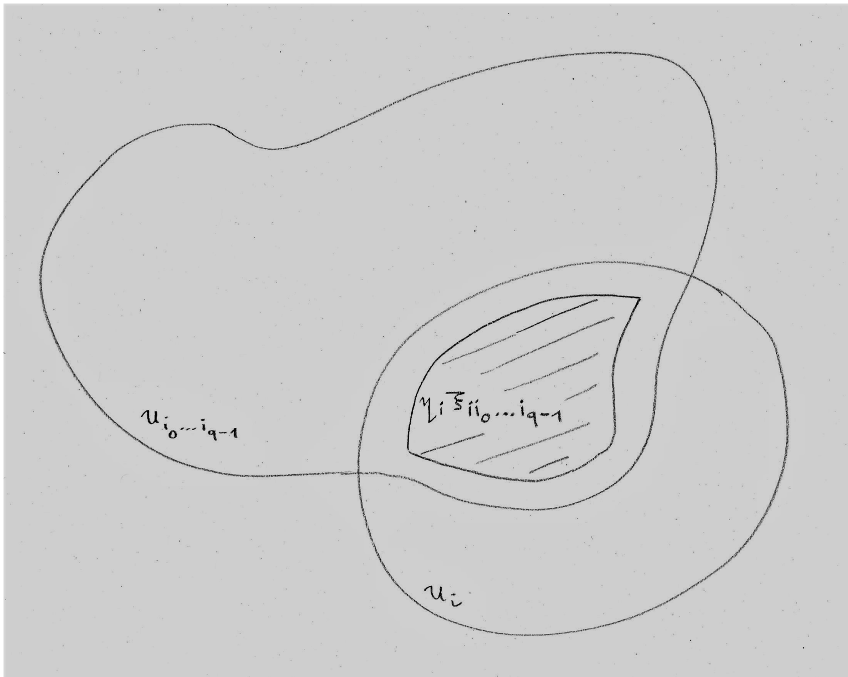


Fig. 3.3 Extending a section from $U_i \cap U_{i_0 \dots i_{q-1}}$ to $U_{i_0 \dots i_{q-1}}$ after shrinking

Varying $i \in I$ provides a covering of $U_{i_0 \dots i_{q-1}}$. The extended functions define the $q-1$-cochain

$$\zeta = (\zeta_{i_0,\dots,i_{q-1}})_{(i_0,\dots,i_{q-1}) \in I^q} \in C^{q-1}(\mathscr{U}, \mathscr{E})$$

with

$$\zeta_{i_0,\dots,i_{q-1}} := \sum_{i \in I} \eta_i \cdot \xi_{ii_0 \dots i_{q-1}}$$

Note that the last sum is well-defined.

Claim: $\delta\zeta = \xi$. Compute

$$(\delta\zeta)_{i_0 \dots i_q} := \sum_{k=0}^{q} (-1)^k \cdot \zeta_{i_0 \dots \hat{i}_k \dots i_q} = \sum_{k=0}^{q} (-1)^k \cdot \left(\sum_{i \in I} \eta_i \cdot \xi_{ii_0 \dots \hat{i}_k i_q} \right) =$$

$$= \sum_{i \in I} \eta_i \cdot \left(\sum_{k=0}^{q} (-1)^k \cdot \xi_{ii_0 \dots \hat{i}_k \dots i_q} \right)$$

The cocycle condition $\delta\xi = 0$ implies

$$0 = \xi_{i_0 \dots i_q} - \sum_{k=0}^{q} (-1)^k \xi_{ii_0 \dots \hat{i}_k \dots i_q}$$

Hence

$$(\delta \zeta)_{i_0 \dots i_q} = \sum_{i \in I} (\eta_i \cdot \xi_{i_0 \dots i_q}) = \xi_{i_0 \dots i_q} \cdot \sum_{i \in I} \eta_i = \xi_{i_0 \dots i_q}$$

□

Proposition 3.21 (Sheaf of smooth differential forms). *On a smooth manifold X the sheaves $\mathscr{E}^{p,q}$ of smooth (p,q)-forms are acyclic with respect to each open covering $\mathscr{U}$ of X.*

The proof is analogous to the proof of Theorem 3.20.

The most prominent link between a complex manifold and its underlying smooth structure is Dolbeault's theorem, Theorem 3.22.

Theorem 3.22 (Dolbeault's theorem). *On a complex manifold X for all indices $p, q \in \mathbb{N}$*

$$H^q(X, \Omega^p) \cong \frac{ker\ [\mathscr{E}^{p,q}(X) \xrightarrow{d''} \mathscr{E}^{p,q+1}(X)]}{im\ [\mathscr{E}^{p,q-1}(X) \xrightarrow{d''} \mathscr{E}^{p,q}(X)]}$$

Proof. The sheaves $\mathscr{E}^{p,q}$ are acyclic due to Proposition 3.21. Dolbeault's lemma, Theorem 1.27, implies: For each $p \in \mathbb{N}$ the sequence of sheaf morphisms

$$0 \to \Omega^{p,0} \to \mathscr{E}^{p,0} \xrightarrow{d''} \mathscr{E}^{p,1} \xrightarrow{d''} \dots \xrightarrow{d''} \mathscr{E}^{p,n} \to 0,\ n := dim\, X,$$

is exact. Hence the claim follows from the abstract de Rahm theorem, Proposition 3.16.
□

Corollary 3.23 (Acyclicity of the sheaf of holomorphic differential forms on a polydisc). *The structure sheaf $\mathscr{O}$ and all sheaves Ω^p on an open polydisc $\Delta \subset \mathbb{C}^n$ are acyclic.*

Proof. The Corollary follows from Theorem 3.22 and Dolbeault's lemma, Theorem 1.27.
□

Corollary 3.24 (Solution of the Cousin problems). *Consider a complex manifold X.*

1. If

$$H^1(X, \mathscr{O}) = 0$$

then each additive Cousin problem on X is solvable.

2. If

$$H^1(X,\mathscr{O}) = 0 \text{ and } H^2(X,\mathbb{Z}) = 0$$

then each multiplicative Cousin problem on X is solvable.

3. On the polydisc $\Delta \subset \mathbb{C}^n$ each additive and each multiplicative Cousin problem is solvable.

Proof. 1. Due to Definition 2.29: An additive Cousin distribution

$$c := (h_i)_{i\in I}$$

with respect to an open covering $\mathscr{U} = (U_i)_{i\in I}$ is a cochain $h \in C^0(\mathscr{U},\mathscr{M})$. By assumption its coboundary is holomorphic, i.e.

$$\delta h \in Z^1(\mathscr{U},\mathscr{O}).$$

The canonical map

$$H^1(\mathscr{U},\mathscr{O}) \to H^1(X,\mathscr{O})$$

is injective due to Proposition 3.6. Hence

$$H^1(X,\mathscr{O}) = 0 \implies H^1(\mathscr{U},\mathscr{O}) = 0$$

Proposition 3.3 implies that the Cousin distribution c is solvable.

2. The exponential sequence on X, see Proposition 2.24,

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$$

provides the following segment of the long exact cohomology sequence

$$H^1(X,\mathscr{O}) \xrightarrow{ex} H^1(X,\mathscr{O}^*) \xrightarrow{\delta_1^*} H^2(X,\mathbb{Z})$$

If the two cohomology groups at the left and right end vanish then

$$H^1(X,\mathscr{O}^*) = 0.$$

Analogous to part 1) Proposition 3.6 implies for each open covering $\mathscr{U}$ of X

$$H^1(\mathscr{U},\mathscr{O}^*) = 0.$$

Proposition 3.3 implies that each multiplicative Cousin distribution on X is solvable.

3. Corollary 3.23 states

$$H^1(\Delta,\mathscr{O}) = 0$$

Because Δ is contractible also $H^2(\Delta,\mathbb{Z}) = 0$. Hence the claim follows from part 1) and 2). □

Corollary 3.24 gives a second proof for the solvability of the Cousin problems in polydiscs. Theorem 2.36 gave a first proof.

Applying cohomology theory to the exponential sequence provides a logarithm on simply connected domains $X \subset \mathbb{C}^n$. Hence Corollary 3.25 provides a second proof for this result, independent from covering theory.

Corollary 3.25 (Existence of a logarithm). *Consider a complex manifold X with*

$$H^1(X,\mathbb{Z}) = 0.$$

Then each holomorphic function $f \in \mathscr{O}^*(X)$ has a holomorphic logarithm, i.e. there exists $g \in \mathscr{O}(X)$ satisfying

$$e^{2\pi i \cdot g} = f.$$

Proof. The exponential sequence on X

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$$

provides the following segment of the long exact cohomology sequence

$$H^0(X,\mathscr{O}) \xrightarrow{ex} H^0(X,\mathscr{O}^*) \xrightarrow{\delta_0^*} H^1(X,\mathbb{Z})$$

The vanishing $H^1(X,\mathbb{Z}) = 0$ implies the surjectivity of

$$H^0(X,\mathscr{O}) \xrightarrow{ex} H^0(X,\mathscr{O}^*)$$

□

Remark 3.26 (Cohomology theory). Consider a topologcial space X.

1. A *cohomology theory* for sheaves of Abelian groups on X is a family of covariant functors
$$(H^q(X,-))_{q\in\mathbb{N}} : \underline{Sh}_X \to \underline{Ab}$$
which satisfies the following properties:
   - *Connecting morphisms*: For each short exact sequence of sheaf morphisms on X
   $$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$
   exists a family $\delta = (\delta_q^*)_{q\in\mathbb{N}}$ of morphisms
   $$\delta_q^* : H^q(X,\mathscr{H}) \to H^{q+1}(X,\mathscr{F})$$
   such that the induced long cohomology sequence - analogous to Theorem 3.12 - is exact and functorial with respect to morphisms between short exact sequences of sheaf morphisms.

- *Normalization*: There is a canonical isomorphism of global sections

$$H^0(X,\mathscr{F}) \xrightarrow{\simeq} \Gamma(X,\mathscr{F}) := \mathscr{F}(X)$$

- *Acyclicity*: For a flabby sheaf $\mathscr{F}$ on X

$$H^q(X,\mathscr{F}) = 0,\ q \geq 1.$$

2. A cohomology theory for sheaves of Abelian groups on X is uniquely determined up to canonical isomorphisms, see [23, Satz 19.3].

3. For each sheaf $\mathscr{F}$ of Abelian groups on X the canonical flabby resolution from Theorem 3.15

$$0 \to \mathscr{F} \to \mathscr{W}^\bullet$$

provides a cohomology theory for sheaves of Abelian groups on X by defining for $q \in \mathbb{N}$

$$H^q(X,\mathscr{F}) := \frac{ker\ [\mathscr{W}^q(X) \to \mathscr{W}^{q+1}(X)]}{im\ [\mathscr{W}^{q-1}(X) \to \mathscr{W}^q(X)]},\ \mathscr{W}^{-1} := 0,$$

see [11, Chap. II, §4].

4. For paracompact X also Čech cohomology from Definition 3.7 is a cohomology theory.

Chapter 4
Local theory and coherence of sheaves

The chapter introduces the concept of coherent $\mathcal{O}$-modules over the structure sheaf of a complex manifold. The concept allows to extend results from the local analytic geometry of a given stalk of a coherent sheaf to all stalks in a neighbourhood. Insofar coherence allows to sheafify commutative algebra.

4.1 The ring of convergent power series and finitely generated modules

The present section deals with some results from Local Analytic Geometry which are needed for the global theory of coherent sheaves.

We start with the investigation of the local ring

$$R_n := \mathbb{C}\{z_1, ..., z_n\}$$

of convergent power series in n complex variables and its maximal ideal

$$\mathfrak{m} := < z_1, ..., z_n > \subset R.$$

As usual, a *unit* $u \in R_n$ is an element with an inverse $u^{-1} \in R_n$. The maximal ideal $\mathfrak{m}$ is the ideal of all non-units. The ring R_n is an integral domain as a consequence of the identiy theorem, analogous to Corollary 1.11.
In the 1-dimensional case the ring of convergent power series

$$R_1 = \mathbb{C}\{z\}$$

is a principal ideal domain: Each non-zero ideal $\mathfrak{a} \subset R_1$ is generated by a well defined monomial $z^k,\ k \in \mathbb{N}$. Because each $f \in R_1,\ f \neq 0$, has a unique representation

$$f(z) = \sum_{j=k}^{\infty} a_j \cdot z^j = z^k \cdot \sum_{j\in\mathbb{N}} a_{j+k} \cdot z^j$$

with $a_k \neq 0$. The number $k \in \mathbb{N}$ is the order of the zero of f in $0 \in \mathbb{C}$.

The situation is different for $n \geq 2$: A non-zero ideal $\mathfrak{a} \subset R_n$ is not necessary a principal ideal. And a non-zero principal ideal $\mathfrak{a} \subset R_n$ is not neessarily generated by monomials.

First we investigate the question about the generators of non-zero principal ideals in R_n. It's good practice in complex analysis to visualize an ideal by its zero set. Hence we take a geometric view point and ask for the zero set of a holomorphic function, or more precisely: For the zero set of the germ of a holomorphic function. It turns out that the zero set displays an *algebraic* character. This property is brought out by the concept of a Weierstrass polynomial and its role in the Weierstrass theorems, Theorem 4.7 and 4.9.

Definition 4.1 (Power series distinguished in z_n and Weierstrass polynomial).

1. A power series $f \in R_n$ is *distinguished in z_n* of order $k \in \mathbb{N}^*$ if its restriction satisfies
$$f(0,...,0,z_n) = \sum_{\nu=k}^{\infty} c_n \cdot z_n^{\nu},\ c_n(0,...,0) \neq 0,$$
i.e if the restriction $f(0,...,0,-) \in R_1$ has a zero of order k in $0 \in \mathbb{C}$.

2. A polynomial $f \in R_{n-1}[z_n]$ is a *Weierstrass polynomial in z_n* of degree $k \in \mathbb{N}$ if
$$f(z_1,...,z_n) = \sum_{j=1}^{k} a_j(z_1,...,z_{n-1}) \cdot z_n^j \in R_{n-1}[z_n]$$
with coefficients $a_j \in R_{n-1}$ satisfying
$$a_j(0,...,0) = 0,\ j = 0,...,k-1,\ \text{while}\ a_k = 1 \in \mathbb{C}.$$

Hence a power series $f \in R_n$ is distinguished in z_n of order k if the restriction

$$f(0,...,0,z_n)$$

has in $0 \in \mathbb{C}$ a zero of exact order k. The identity theorem from complex analysis of one variable implies that $f \in R_n$ is distinguished in z_n for a suitable order $k \in \mathbb{N}$ if and only if the restriction $f(0,...,0,-) \in R_1$ is not zero.

Each Weierstrass polynomial $f \in R_n$ of degree k is a monic polynomial distinguished in z_n of degree k. All coefficients

$$a_j \in R_{n-1},\ j = 0,...,k-1,$$

vanish at the origin of $\mathbb{C}^{n-1}$. The polynomial character in z_n with coefficients from R_{n-1} makes a Weierstrass polynomial a suitable tool to prove results about R_n by induction on n: As a first result Theorem 4.7 shows that a power series from R_n distinguished in z_n of degree k agrees up to a unit from R_n with a Weierstrass polynomial from $R_{n-1}[z_n]$ of degree k.

Figure 4.1 shows the zeros of a Weierstrass polynomial of degree k as branches over the variables $(z_1,...,z_{n-1})$, which vary in a neighbourhood of $0 \in \mathbb{C}^{n-1}$. The zeros form a branched covering over an open set in $\mathbb{C}^{n-1}$.

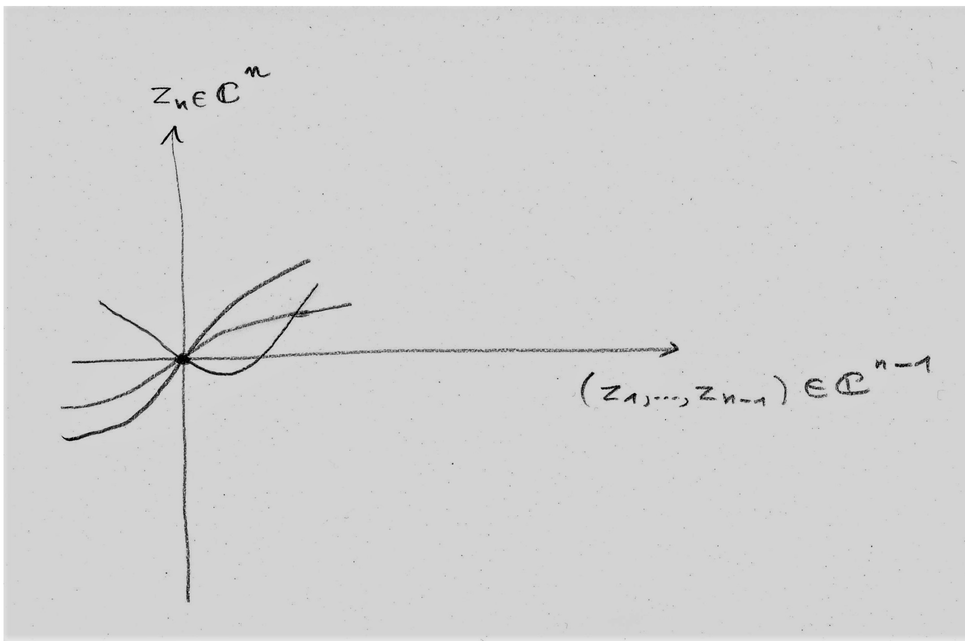


Fig. 4.1 Zero set of a Weierstrass polynomial

A counter example against being distinguished in one of the variables is the element

$$f(z_1,z_2) := z_1 \cdot z_2 \in R_2$$

It satisfies $f \neq 0$, but it is neither distinguished in z_1 not in z_2. Lemma 4.2 shows how this shortage can easily be cured by a linear coordinate transformation.

Lemma 4.2 (Linear coordinate transformation). *For each finite set of non-zero power series*

$$f_1,...,f_r \in R_n$$

exists a linear isomorphism

$$\phi : \mathbb{C}^n \to \mathbb{C}^n$$

such that for each $j = 1, ..., r$ *the transformed power series*

$$f_j \circ \phi \in R_n$$

is distinguished in z_n *with a degree* $k_j \geq 1$.

Proof. First, choose an open neighbourhood $U \subset \mathbb{C}^n$ of $0 \in \mathbb{C}^n$ and a point $a \in U \setminus \{0\}$ satisfying the following property: For each $j = 1, ..., r$ the power series f_j defines a holomorphic function in U and $f_j(a) \neq 0$. Secondly, choose a linear isomophism

$$\phi : \mathbb{C}^n \to \mathbb{C}^n$$

which maps the z_n-axis to the line passing through a and the origin. □

The main input from complex analysis for the local theory is the Cauchy formula in the form of the Laurent splitting in annuli, depending on holomorphic parameters, see Theorem 4.3.

Theorem 4.3 (Laurent splitting in annuli). *Consider a domain* $U \subset \mathbb{C}^{n-1}$ *with* $0 \in U$, *two radii* $0 < r < R < \infty$, *and a holomorphic function*

$$f : U \times (\Delta(R) \setminus \overline{\Delta}(r)) \to \mathbb{C},\ (t,z) \mapsto f(t,z).$$

1. *There exist two uniquely determined holomorphic functions*

$$f_1 : U \times \Delta(R) \to \mathbb{C} \text{ and } f_2 : U \times (\mathbb{C} \setminus \overline{\Delta}(r)) \to \mathbb{C}$$

with the following properties:

- *On* $U \times (\Delta(R) \setminus \overline{\Delta}(r))$ *the function* f *splits additively as*

$$f = f_1 + f_2$$

- *For each* $t \in U$ *holds*

$$\lim_{z \to \infty} f_2(t,z) = 0$$

2. *For each* $t \in U$ *and for each* ρ *with* $r < \rho < R$ *both functions are defined by the same Cauchy kernel: For* $|z| < \rho$

$$f_1(t,z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{f(t,\zeta)}{\zeta - z}\, d\zeta,$$

and for $|z| > \rho$

$$f_2(t,z) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{f(t,\zeta)}{\zeta - z}\, d\zeta$$

3. The functions f_1 and f_2 have the convergent power series

$$f_1(t,z) = \sum_{\nu=0}^{\infty} c_\nu(t) \cdot z^\nu,\ |z| < R,\ \textit{and}\ f_2(t,z) = \sum_{\nu=1}^{\infty} c_{-\nu}(t) \cdot z^{-\nu},\ |z| > r,$$

with coefficents $c_\nu(t), \nu \in \mathbb{Z},\ t \in U$, obtained for arbitrary $\rho > 0$ with $r < \rho < R$ as

$$c_\nu(t) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{f(t,\zeta)}{\zeta^{\nu+1}}\, d\zeta$$

Proof. For fixed $t \in U$ the proof uses from complex analysis of a single variable the unique Laurent representation by the Cauchy formula for holomorphic functions defined in an annulus, see [30, Chap. 3]. To prove the uniqueness: Assume an analogous second pair f_1' and f_2' with

$$f = f_1' + f_2'$$

Then on $U \times (\Delta(R) \setminus \overline{\Delta}(r))$ holds

$$f_1 - f_1' = -(f_2 - f_2')$$

Hence the family $(f_1 - f_1',\ -(f_2 - f_2'))$ defines a function h on $U \times \mathbb{C}$ satisfying for each $t \in U$

$$\lim_{z \to \infty} h(t,z) = 0$$

The maximum principle for one complex variable implies that for each $t \in U$ the holomorphic function $h(t,-)$ vanishes. Hence $h = 0$ and $f_j = f_j', j = 1,2$.

Because the integrand of the Cauchy integral from Theorem 4.3 depends holomorphically on the parameter $t \in U$, both functions f_1 and f_2 are holomophic. □

Remark 4.4 (Counting zeros). Consider two radii $0 < r < R$ and a holomorphic function of one variable

$$f : \Delta(R) \to \mathbb{C}$$

which has no zeros in the annulus $\Delta(R) \setminus \overline{\Delta(r)}$. Then the number k of zeros of f can be computed for any $\rho > 0$ with $r < \rho < R$ by the integral formula

$$k = \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta.$$

For the proof see [30, Chap. 6].

We use this equality from Remark 4.4 to define the global order for the holomorphic functions considered in the following.

Definition 4.5 (Order with respect to z_n). Consider a domain $U \subset \mathbb{C}^{n-1}$, two radii $0 < r < R$ and a holomorphic function

$$f : U \times \Delta(G) \to \mathbb{C}$$

with no zeros in $U \times (\Delta(R) \setminus \overline{\Delta}(r))$. The number

$$k := \frac{1}{2\pi i} \cdot \int_{|\zeta|=\rho} \frac{\frac{\partial f}{\partial z_n}(t,\zeta)}{f(t,\zeta)(t,\zeta)} \, d\zeta, \; r < \rho < R,$$

is named the *order of f in $U \times \Delta(R)$ with respect to z_n.*

For each $t \in U$ the value of the integral is an integer which depends continuously on t. Because U is connected, the value is constant.

Theorem 4.6 (Weierstrass preparation theorem: Analytic version). *Consider a domain $U \subset \mathbb{C}^{n-1}$, two radii $0 < r < R$, and a holomorphic function*

$$f : U \times \Delta(R) \to \mathbb{C}$$

with no zeros in $U \times (\Delta(R) \setminus \overline{\Delta}(r))$. Denote by k the order of f with respect to z_n. Then exist

- *a monic polynomial in z_n of degree k*

$$p : U \times \mathbb{C}, \; p(t, z_n) = z_n^k + \sum_{j=0}^{k-1} a_j(t) \cdot z_n^j$$

  with holomorphic coefficients

$$a_j : U \to \mathbb{C}, \; j = 1, ..., k-1,$$

- *and a holomorphic function*

$$u : U \times \Delta(R) \to \mathbb{C}^*$$

such that in $U \times \Delta(R)$ holds

$$f = u \cdot p.$$

Both functions p and g are uniquely determined by this equation.

The following proof is due to Stickelberger (1887). It uses the Laurent splitting.

Proof. i) *Existence*: Because the function f has order $= k$ with respect to z_n the induced function

$$U \times (\Delta(R) \setminus \overline{\Delta}(r)) \to \mathbb{C},\ (t, z_n) \mapsto \frac{1}{z_n^k} \cdot f(t, z_n),$$

has a well-defined logarithm: During one loop of

$$z_n \in \Delta(R) \setminus \overline{\Delta}(r)$$

the logarithm of the first factor changes by the additive summand $-2k\pi i$ and the second factor by the additive summand $2k\pi i$. Theorem 4.3 provides the Laurent splitting of the function

$$F : U \times (\Delta(R) \setminus \overline{\Delta}(r)) \to \mathbb{C},\ F(t, z_n) := log\ \left(\frac{1}{z_n^k} \cdot f(t, z_n)\right),$$

in the form

$$F = F_1 + F_2$$

with two holomorphic fuctions

$F_1 : U \times \Delta(R) \to \mathbb{C}$ and $F_2 : U \times (\mathbb{C} \setminus \overline{\Delta}(r)) \to \mathbb{C}$ satisfying $\lim_{z_n \to \infty} F_2(t, z_n) = 0,\ t \in U.$

Taking the exponential of F provides the multiplicative splitting

$$f(t, z_n) = exp\ F_1(t, z_n) \cdot z_n^k \cdot exp\ F_2(t, z_n)$$

On one hand, define

$$u := exp\ F_1(t, z_n)$$

which is holomorphic on $U \times \Delta(R)$. The function has no zeros and is therefore invertible. On the other hand, the result

$$\lim_{z_n \to \infty} exp\ F_2(t, z_n) = 1 \text{ due to } \lim_{z_n \to \infty} F_2(t, z_n) = 0,\ t \in U,$$

implies

$$\lim_{z_n \to \infty} exp\ F_2(t, z_n) = 1$$

and for $(t, z_n) \in U \times (\Delta(R) \setminus \overline{\Delta}(r)) :$

$$exp\ F_2(t, z_n) = 1 + \sum_{\nu=1}^{\infty} a_\nu(t) \cdot z_n^{-\nu}$$

resp.

$$z_n^k \cdot exp\ F_2(t, z_n) = z_n^k \cdot \left(1 + \sum_{\nu=1}^{\infty} a_\nu(t) \cdot z_n^{-\nu}\right)$$

Define the polynomial in z_n of degree $= k$

$$p(t,z_n) := z_n^k + \sum_{\nu=1}^{k} a_\nu(t) \cdot z_n^{k-\nu}$$

and the rest

$$h(t,z_n) := \sum_{\nu=1}^{\infty} a_{k+\nu}(t) \cdot z_n^{-\nu}.$$

Then

$$f = u \cdot (p+h)$$

The representation

$$\left(\frac{f}{u} - p\right) - h = 0$$

is a Laurent splitting of zero. The uniqueness of the Laurent splitting, see Theorem 4.3, implies the vanishing of both summands, hence $h = 0$ and

$$f = u \cdot p$$

ii) *Uniqueness of the factors*: For each $t \in U$ the two holomorphic functions of one variable

$$f(t,-) : \Delta(R) \to \mathbb{C} \text{ and } p(t,-) : \Delta(R) \to \mathbb{C}$$

have the same k zeros. The monic polynomial $p(t,-) \in \mathbb{C}[z_n]$ of degree k is uniquely determined by its zeros. In the open complement of the zeros holds

$$\frac{f(t,-)}{p(t,-)} = u(t,-)$$

Hence the identity theorem determines $u(t,-)$ and a posteriori u. □

Theorem 4.7 states: Each power series from R_n distinguished in z_n is associated in R_n to a Weierstrass polynomial from $R_{n-1}[z_n]$.

Theorem 4.7 (Weierstrass preparation theorem: Algebraic version). *For each power series $f \in R_n$ distinguished in z_n of order k exist*

- *a Weierstrass polynomial $p \in R_{n-1}[z_n]$ of degree k*
- *and a unit $u \in R_n$*

such that

$$f = u \cdot p.$$

Proof. The given element $f \in R_n$ can be represented by a holomorphic function

$$f : U \times \Delta(R) \to \mathbb{C}$$

with $U \subset \mathbb{C}^{n-1}$ a domain containing $0 \in \mathbb{C}^{n-1}$ and a radius $R > 0$. Because the restriction $f(0,...,0,z_n)$ has a zero at $0 \in \mathbb{C}$ of finite order k, we may assume - possibly after shrinking U and R - a second radius $r > 0$ with $0 < r < R$ such that the restriction

$$f|U \times (\Delta(R) \setminus \overline{\Delta}(r))$$

satisfies the assumptions of Theorem 4.6. The theorem provides a holomorphic function

$$u : U \times \Delta(R) \to \mathbb{C}^*$$

and a polynomial

$$p : U \times \mathbb{C} \to \mathbb{C},\ p(t,z_n) = z_n^k + \sum_{j=1}^{k-1} a_j \cdot z_n^j,$$

with

$$f = u \cdot p$$

And both functions are uniquely determined by these properties. □

Lemma 4.8 (Multiples of a Weierstrass polynomial). *Consider a Weierstrass polynomial $p \in R_{n-1}[z_n]$ of degree k and a polynomial $f \in R_{n-1}[z_n]$ satisfying*

$$f = q \cdot p$$

with a power series $q \in R_n$. Then also q is a polynomial, i.e.

$$q \in R_{n-1}[z_n]$$

Proof. Set

$$R := R_{n-1}.$$

We apply the Euclidean algorithm of division in the polynomial ring $R[z_n]$ for dividing out the monic polynomial $p \in R[z_n]$. We obtain

$$f = q \cdot p + r$$

with suitable polynomials $q,\ r \in R[z_n]$ and $deg\ r < k$. There exist $\varepsilon > 0$ and a polyradius $\rho > 0$ such that

- all coefficients of the polynomials f, p, r are holomorphic in $\Delta(\rho) \subset \mathbb{C}^{n-1}$,
- and for each $t \in \Delta(\rho)$ all zeros z_n of the Weierstrass polynomial $p(t,-)$ satisfy $|z_n| < \varepsilon$.

As a consequence, for each $t \in \Delta(\rho)$ the function $g(t,-)$ has at least k zeros, and a posteriori also $r(t,-)$. But

$$deg\ r(t,-) \leq k-1,$$

which implies $r(t,-) = 0$. As a consequence

$$f = q \cdot p$$

which proves the claim. □

Theorem 4.9 proves the analogue of Euclidean division for power series distinguished in z_n. The theorem is due to H. Späth (1929) though it is generally named after Weierstrass.

Theorem 4.9 (Weierstrass division theorem). *Consider a power series $f \in R_n$ distinguished in z_n of order k.*

Then for each power series $\phi \in R_n$ exist

- *a power series $q \in R_n$*
- *and a polynomial $r \in R_{n-1}[z_n]$ of degree $< k$*

such that

$$\phi = q \cdot f + r.$$

Both elements q and r are uniquely determined.

Proof. i) *Uniqueness*: Two different representations of ϕ provide a representation of zero

$$0 = q \cdot f + r$$

with $q \in R_n$ and $r \in R_{n-1}[z_n]$ a polynomial of degree $< k$. We consider the elements $f,\ q,\ r$ as holomorphic functions

$$U \times \Delta(R) \to \mathbb{C},\ U \subset \mathbb{C}^{n-1} \text{ a domain with } 0 \in U,\ R > 0.$$

If $r \neq 0$ then exists $t \in U$ such that the polynomial $r(t, z_n) \in \mathbb{C}[z_n]$ does not vanish identically and therefore has at most $k-1$ zeros. But the function

$$-r(t, -) = q(t, -) \cdot f(t, -) \in \mathbb{C}\{z_n\}$$

has at least k zeros, a contradiction. As a consequence $r = 0$, which implies $q = 0$ because R_n is a domain of integrity.

ii) *Reduction to the case of f being a Weierstrass polynomial*: Theorem 4.7 applies to f and provides a representation

$$f = u \cdot p$$

with a unit $u \in R_n$ and a Weierstrass polynomial $p \in R_{n-1}[z_n]$ of degree k. It suffices to prove the claim for the particular case of a Weierstrass polynomial

$$f = p \in R_{n-1}[z_n]:$$

Because a representation

$$\phi = q \cdot p + r$$

implies

$$\phi = u^{-1} \cdot (u \cdot p) + r = u^{-1} \cdot f + r$$

with the unit $u^{-1} \in R_n$.

iii) *Proof for f a Weierstrass polynomial*: We consider f and ϕ as holomorphic functions

$$U \times \Delta(R) \to \mathbb{C}$$

with $U \subset \mathbb{C}^{n-1}$ a domain with $0 \in U$ and with suitable radii $0 < r < R$ such that f has no zeros in

$$V := U \times (\Delta(R) \setminus \overline{\Delta}(r)).$$

The choice of such radii is possible, because $f(0, z_n) \in \mathbb{C}[z_n]$ has a zero in $0 \in \mathbb{C}$ of positive order, which is therefore an isolated zero. Theorem 4.3 applies to the holomorphic function

$$\frac{\phi}{f} : V \to \mathbb{C}$$

and provides the Laurent splitting

$$\frac{\phi}{f} = q + f_2$$

with the two holomorphic summands

$$q : U \times \Delta(R) \to \mathbb{C}$$

and

$$f_2 : U \times (\mathbb{C} \setminus \overline{\Delta}(r)) \to \mathbb{C} \text{ satisfying } \lim_{z_n \to \infty} f_2(t, z_n) = 0,\ t \in U,$$

Expanding

$$f(t, z_n) = z_n^k + \sum_{j=1}^{k} a_j(t) \cdot z_n^{k-j}$$

and

$$f_2(t, z_n) = \sum_{\nu=1}^{\infty} c_\nu(t) \cdot z_n^{-\nu}$$

implies

$$f(t, z_n) \cdot f_2(t, z_n) = \left(z_n^k + \sum_{j=1}^{k} a_j(t) \cdot z_n^{k-j} \right) \cdot \left(\sum_{\nu=1}^{\infty} c_\nu(t) \cdot z_n^{-\nu} \right) =$$

$$= \Big(b_1(t) \cdot z_n^{k-1} + \ldots + b_k(t) \Big) + \sum_{\nu=-1}^{-\infty} b_\nu(t) \cdot z_n^{\nu}$$

with holomorphic coefficients

$$b_\nu : U \to \mathbb{C},\ \nu \in \mathbb{Z}.$$

Splitting the last equation into the two summands

$$r(t,z_n) := \sum_{\nu=1}^{k} b_\nu(t) \cdot z_n^{k-\nu},\ h(t,z_n) := \sum_{\nu=-1}^{-\infty} b_\nu(t) \cdot z_n^{\nu}$$

provides the representation

$$\phi = q \cdot f + r + h$$

with the polynomial $r \in R_{n-1}[z_n]$ of degree $< k$. The vanishing $h = 0$ follows from the uniqueness of the Laurent splitting

$$0 = (-\phi + q \cdot f + r) + h$$

□

Corollary 4.10 (Weierstrass theorem on module finiteness). *Each element $q \in R_{n-1}[z_n]$ distinguished in z_n of degree k induces a surjective morphism of R_{n-1}-modules*

$$\gamma : R_n \to R_{n-1}^k,\ f \mapsto (a_0, ..., a_{k-1}).$$

Here the tuple $(a_0, ..., a_{k-1})$ results from the division with rest

$$f = g \cdot q + r$$

with

$$g \in R_n,\ r = \sum_{j=0}^{k-1} a_j \cdot z_n^j.$$

The map γ induces an isomorphism of R_{n-1}-modules

$$R_n/(q \cdot R_n) \xrightarrow{\simeq} R_{n-1}^k$$

Proof. The claim follows from Theorem 4.9. □

While Corollary 4.10 considered two R_{n-1}-*module* structures, the following Corollary 4.11 compares two $\mathbb{C}$-*algebra* structures.

Corollary 4.11 (Weierstrass isomorphism of $\mathbb{C}$-algebras). *Consider an element $q \in R_n$ distinguished in z_n and denote by $\omega \in R_{n-1}[z_n]$ the well-defined Weierstrass polynomial with*

$$q = u \cdot \omega$$

and $u \in R_n$ a unit. The injection

$$R_{n-1}[z_n] \hookrightarrow R_n$$

induces an isomorphism of $\mathbb{C}$*-algebras*

$$R_{n-1}/(\omega \cdot R_{n-1}) \xrightarrow{\simeq} R_n/(q \cdot R_n)$$

Proof. Because $u \in R_n$ is a unit there holds the equality of ideals

$$q \cdot R_n = \omega \cdot R_n \subset R_n$$

The injection

$$R_{n-1}[z_n] \hookrightarrow R_n$$

induces a morphism of $\mathbb{C}$-algebras

$$\alpha : R_{n-1}[z_n] \to R_n/(\omega \cdot R_n) = R_n/(q \cdot R_n)$$

The latter morphism is surjective, because each $f \in R_n$ is congruent modulo q to a polynomial $r \in R_{n-1}[z_n]$ due to Theorem 4.9. One has

$$ker\ \alpha = (\omega \cdot R_n) \cap R_{n-1}[z_n] = \omega \cdot R_{n-1}.$$

Hence

$$R_{n-1}/(\omega \cdot R_{n-1}) \xrightarrow{\simeq} R_n/(q \cdot R_n).$$

□

Corollary 4.12 (Primality of a Weierstrass with respect to R_n respectively to $R_{n-1}[z_n]$). *A Weierstrass polynomial $f \in R_{n-1}[z_n]$ is a prime in $R_{n-1}[z_n]$ if and only if $f \in R_n$ is a prime in R_n.*

Proof. We use the well-known characterization of primes in a ring A: An element

$$a \in A,\ a \neq 0,$$

is prime in A if and only if the quotient $A/ < a >$ is an integral domain. Hence Corollary 4.11 proves the claim. □

Theorem 4.13 (Hilbert-Rückert ideal basis theorem). *The ring $R_n,\ n \in \mathbb{N}$, is Noetherian.*

Proof. The proof is by induction on $n \in \mathbb{N}$.

Induction start $n = 0$: The ring $R_0 = \mathbb{C}$ is a field and has no proper ideal different from $\{0\}$.

Induction step $n-1 \mapsto n$: Consider a proper ideal

$$\{0\} \neq \mathfrak{a} \subset R_n$$

Due to Lemma 4.2 we may assume w.l.o.g the existence of an element

$$0 \neq p \in \mathfrak{a}$$

which is distinguished in z_n. For each $f \in R_n$ Theorem 4.9 provides a representation

$$f = q \cdot p + r$$

with $r \in R_{n-1}[z_n]$ of an order $k < ord\ f$. Hence the ring

$$R := R_n / < p > \simeq\ R_{n-1}^k$$

is a free R_{n-1}-module of finite rank. Because the ring R_{n-1} is Noetherian by induction assumption the ring R is a Noetherian R_{n-1}-module. Therefore also the quotient ring $R_n/\mathfrak{a}$ is a Noetherian R_{n-1}-module, generated by finitely many residue classes

$$\overline{f}_1, ..., \overline{f}_r,\ f_j \in R_n,\ j = 1, ..., r.$$

As a consequence, the ideal $\mathfrak{a} \subset R_n$ is generated by the elements $(f, f_1, ..., f_r)$. □

Recall that an integral domain A is *factorial* if each non-unit $a \in A$ is the product of finitely many prime elements from A.

Proposition 4.14 (Factoriality). *The ring* $R_n,\ n \in \mathbb{N}$, *is factorial.*

Proof. The proof is by induction on n, see [13, Chap. I, §5, Satz 5].

Induction start $n = 0$: Because $R_0 = \mathbb{C}$ is a field, each non-zero element of R_0 is a unit.

Induction step $n-1 \mapsto n$: Assume that R_{n-1} is factorial. By Gauss' theorem also the ring $R_{n-1}[z_n]$ is factorial.

To factor a given non-unit $f \in R_n,\ f \neq 0$, as a product of primes, we may assume

$$f \in R_{n-1}[z_n]$$

a Weierstrass polynomial according to Theorem 4.7. In particular $f \in R_{n-1}[z_n]$ is not a unit in $R_{n-1}[z_n]$: By induction assumption

$$f = \prod_{j=1}^{r} f_j \in R_{n-1}[z_n]$$

with primes $f_j \in R_{n-1}[z_n],\ j = 1, ..., r$. Because f is a monic polynomial we may arrange that f_j monic for each $j = 1, ..., r$. Then the monic polynomial

$$f_j \in R_{n-1}[z_n],\ j = 1, ..., r,$$

is a Weierstrass polynomial, prime in $R_{n-1}[z_n]$. Corollary 4.12 implies that $f_j,\ j = 1, ..., r$, is also prime in R_n. Hence the given element $f \in R_n$ factorizes into a product of prime elements. □

Corollary 4.15 (Normality). *The ring $R_n,\ n \in \mathbb{N}$, is normal, i.e. it is closed in its quotient field with respect to integral extensions of R_n.*

Proof. The claim holds for each factorial ring: Denote by $Q := Q(R_n)$ the quotient field of the integral domain R_n. Consider an element $f \in Q$ which is integral over R_n, i.e. f satisfies an integral equation

$$f^k = \sum_{j=0}^{k-1} a_j \cdot f^j$$

with $a_j \in R_n$ for all $j = 1, ..., k-1$. The factoriality of R_n provides a representation

$$f = \frac{g}{h}$$

with $g,\ h \in R_n$ without common prime factor. Then

$$g^k = \sum_{j=0}^{k-1} a_j \cdot g^j \cdot h^{k-j} = h \cdot \left(\sum_{j=0}^{k-1} a_j \cdot g_j \cdot k^{k-1-j} \right) \in R_n$$

Hence h divides g^k, contradicting the fact that h and g have no common prime factor. □

Proposition 4.16 (Hensel's Lemma). *Consider a monic polynomial $f \in R_{n-1}[z_n]$, and assume that the restriction factorizing as*

$$f(0, ..., 0, z_n) = \prod_{j=1}^{m} (z_n - \alpha_j)^{b_j},\ b_j \in \mathbb{N} \text{ for } j = 1, ..., m,$$

with zeros $\alpha_1, ..., \alpha_j \in \mathbb{C}$. Then exist uniquely determined, pairwise coprime, monic polynomials

$$f_1, ..., f_m \in R_{n-1}[z_n]$$

satisfying

$$f_j(0, ..., 0, z_n) = (z_n - \alpha_j)^{b_j},\ j = 1, ..., m,$$

such that f factorizes as

$$f = \prod_{j=1}^{m} f_j$$

Proof. i) *Existence*: The proof is by induction on m. The induction start $m = 1$ is obvious.

Induction step $m-1 \mapsto m$: The monic polynomial

$$g(t,z_n) := f(t,z_n+\alpha_1) \in R_{n-1}[z_n]$$

is distinguished in z_n of order b_1. Theorem 4.7 provides a representation

$$g = u \cdot p$$

with uniquely determined unit $u \in R_n$ and Weierstrass polynomial $p \in R_{n-1}[z_n]$ of order b_1. Lemma 4.8 ensures $u \in R_{n-1}[z_n]$. Define the two polynomials from $R_{n-1}[z_n]$

$$f_1(t,z_n) := p(t,z_n-\alpha_1) \text{ and } \tilde{f} := u(t,z_n-\alpha_1)$$

The polynomials $f_1,\ \tilde{f} \in R_{n-1}[z_n]$ are monic and satisfy

$$f_1(0,...,0,z_n) = (z_n-\alpha_1)^{b_1} \text{ and } \tilde{f}(0,...,0,z_n) = \prod_{j=2}^{m} (z_n-\alpha_j)^{b_j}$$

The induction assumptions applies to $\tilde{f}$ and provides monic polynomials

$$f_2,...,f_m \in R_{n-1}[z_n]$$

satisfying

$$f_j(0,...,0,z_n) = (z_n-\alpha_j)^{b_j},\ j = 2,...,m, \text{ and } \tilde{f} = \prod_{j=2}^{m} f_j$$

Hence

$$f = f_1 \cdot \tilde{f}$$

which terminates the induction step.

ii) *Uniqueness and coprimality*: The ring $R_{n-1}[z_n]$ is factorial according to Proposition 4.14 and Gauss' theorem. Consider the prime decomposition

$$f = \pi_1 \cdot ... \cdot \pi_s$$

Because f is monic we may arrange that the prime elements $\pi_i,\ i = 1,...,s$, are also monic polynomials. Then the factors are uniquely determined up to numbering. The restriction provides in the factorial ring $\mathbb{C}[z_n]$ the representation

$$\prod_{i=1}^{s} \pi_j(0,...,0,z_n) = \prod_{j=1}^{t} (z_n-\alpha_j)^{b_j} \in \mathbb{C}[z_n]$$

Because each monomial $(z_n - \alpha_j) \in \mathbb{C}[z_n]$ on the right-hand side is a prime element for each $i = 1, ..., s$ holds

$$\pi_i(0,...,0,z_n) = \prod_{j=1}^{t} (z_n - \alpha_j)^{b_{ij}} \text{ with } 0 \le b_{ij} \le b_j$$

If $b_{ij} \neq 0$ for at least two indices $j_1 \neq j_2$ then part i) of the proof would produce a reducible representation of $\pi_i \in R_{n-1}[z_n]$, contradicting the primality of π_i.
Hence $\pi_i(0,...,0,\alpha_j)$ vanishes for exactly one $j \in \{1,...,t\}$. Accordingly,
for $j = 1,...,t$ we define

$$F_j \in R_{n-1}[z_n] := \prod_{i \text{ with } \pi_i(0,...,0,\alpha_j)=0} \pi_i$$

The monic polynomials $F_1,...,F_t$ are pairwise coprime. Their product satisfies

$$\prod_{j=1}^{t} F_j = F = \prod_{j=1}^{t} f_j$$

Hence the monic prime elements on both sides coincide. As a consequence, for each $j = 1,...,t$ the prime factors of F_j andf_j on both sides coincide, which implies $F_j = f_j$. □

Corollary 4.17 (Irreducibility and Weierstrass polynomial). *Each normed, irreducible polynomial $f \in R_{n-1}[z_n]$ satisfying $f(0,...,0) = 0$ is a Weierstrass polynomial.*

Proof. The restriction fo the given polynomial $f \in R_{n-1}[z_n]$ factorizes as

$$f(0,...,0,z_n) = \prod_{j=1}^{m} (z_n - \alpha_j)^{b_j},\ b_j \in \mathbb{N} \text{ for } j = 1,...,m,$$

with pairwise distinct roots

$$\alpha_j \in \mathbb{C},\ j = 1,...,m,\ \alpha_1 = 0 \text{ and } b_1 \neq 0.$$

Proposition 4.16 implies the existence of normed polynomials

$$f_1,...,f_m \in R_{n-1}[z_n]$$

satisfying for $j = 1,...,m$

$$f_j(0,...,0,z_n) = (z_n - \alpha_j)^{b_j}$$

and

$$f = \prod_{j=1}^{m} f_j \in R_{n-1}[z_n]$$

The irreducibility of $f \in R_{n-1}[z_n]$ implies that for each $j = 2, ..., m$ the factor

$$f_j, \ j = 2, ..., m,$$

is a unit, hence $b_j = 0$. As a consequence $f = f_1$ is a Weierstrass polynomial. □

Corollary 4.18 (Parameter representation of the germ of hypersurface). *Consider an open set $U \subset \mathbb{C}^{n-1}$ and a holomorphic function $f \in \mathscr{O}(U)[z_n]$ of the form*

$$f : U \times \mathbb{C} \to \mathbb{C}, \ f(z', z_n) = z_n^k + \sum_{j=1}^{k} a_j(z') \cdot z_n^{k-j}, \ a_j \in \mathscr{O}(U), j = 1, ..., k,$$

Assume the existence of a point $c \in U$ such that the restricted polynomial

$$f(c, -) \in R_1$$

has pairwise distinct zeros $\alpha_j \in \mathbb{C}, \ j = 1, ..., k$. Then exists an open neighbourhood $V \subset U$ of c and holomorphic functions

$$\phi_j : V \to \mathbb{C}, \ j = 1, ..., k, \ \textit{satisfying} \ \phi_j(c) = \alpha_j, \ j = 1, ..., k,$$

such that for all $(z', z_n) \in U \times \mathbb{C}$ holds

$$f(z', z_n) = \prod_{j=1,...,k} (z_n - \phi_j(z')).$$

Figure 4.2 visualizes geometrically Corollary 4.18 as the representation of a hypersurface in non-singular points.

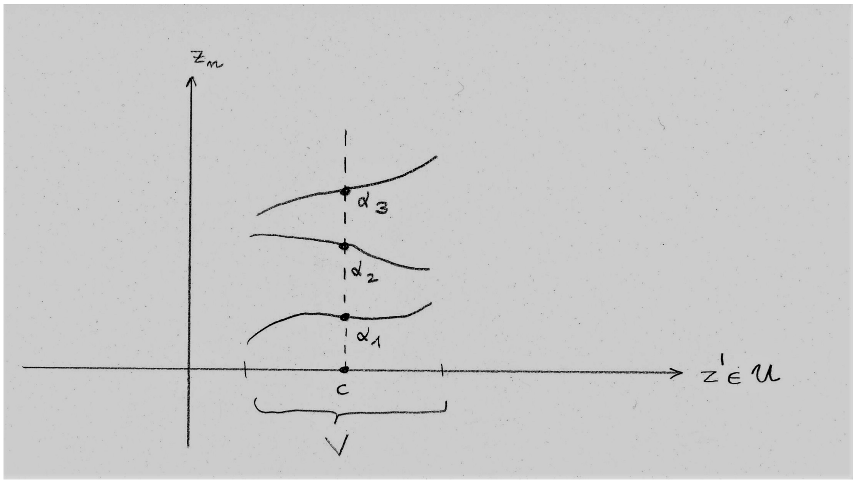


Fig. 4.2 The germ of a non-singular hypersurface as a non-branched covering

Proof. We assume $c = 0 \in \mathbb{C}^{n-1}$. Then f restricts to

$$f(0,...,0,z_n) = \prod_{j=1}^{k} (z_n - \alpha_j)$$

Proposition 4.16 provides normed polynomial

$$f_1,...,f_k \in R_{n-1}[z_n]$$

which factorize

$$f = \prod_{j=1}^{k} f_j \text{ satisfying } f_j(0,...,0,z_n) = z_n - \alpha_j,\ j = 1,...,k,$$

Hence for each $j = 1,...,k$, the polynomial f_j has degree $= 1$ and therefore the form

$$f_j = z_n - \phi_j \text{ with } \phi_j \in R_{n-1}$$

Then $V \subset U$ of c may be choosen as a neighbourhood if c where all functions $f_j \in R_{n-1},\ j = 1,...,k$ are represented by holomorphic functions. □

Proposition 4.19 (Nakayama lemma for local rings). *If a finitely generated R-module L satisfies*

$$L = \mathfrak{m} \cdot L$$

then $L = 0$.

Proof. The proof is indirect. Let

$$\{f_1,...,f_k\} \subset L$$

be a minimal set of generators of L, and assume $k \geq 1$. Then

$$f_k = \sum_{j=1}^{k} m_j \cdot f_j$$

for suitable coefficients

$$m_j \in \mathfrak{m},\ j = 1,...,k.$$

Hence

$$(1 - m_k) \cdot f_k = \sum_{j=1}^{k-1} m_j \cdot f_j$$

The element

$$1 - m_k \in R$$

is a unit because $1 - m_k \notin \mathfrak{m}$. Hence

$$f_k = \sum_{j=1}^{k-1} \frac{m_j}{1 - m_k} \cdot f_j,$$

a contradiction to the minimality. □

Corollary 4.20 (Implications of the Nakayama lemma). *If M is a finitely generated R-module and $N \subset M$ a submodule with*

$$M = \mathfrak{m} \cdot M + N$$

then

$$M = N.$$

Proof. By assumption

$$M/N = (\mathfrak{m} \cdot M + N)/N = \mathfrak{m} \cdot (M/N)$$

Hence Proposition 4.19 implies

$$M/N = 0, \text{ i.e. } M = N.$$

□

Proposition 4.21 (Krull Lemma). *For each finitely generated submodule*

$$M \subset R_n^p,\ p \in \mathbb{N},$$

holds:

$$M = \bigcap_{s \geq 1} (M + \mathfrak{m}^s \cdot R_n^p)$$

For a proof of Proposition 4.21 see [13, Anhang, §2, Satz 2 Folgerung].

From local homological algebra of finitely generated R_n-modules we need the following results, which will be later generalized to certain modules over the structure sheaf of a complex manifold.

Definition 4.22 (Homological dimension of finitely generated modules). Consider a finitely generated R-moduls M.

1. The *homological dimension* $hd_R\ M$ of M is defined as the minimal lenght d of a finite resolution of M by finitely generated free R-modules

$$0 \to R^{k_d} \xrightarrow{\phi_d} ... \to R^{k_1} \xrightarrow{\phi_1} R^{k_0} \xrightarrow{\phi_0} M \to 0$$

2. Each such resolution of M of lenght $= hd_R M$ is named a *Hilbert resolution* of M.

3. The R-modules

$$S_i := im\ [R^{k_{i+1}} \xrightarrow{\phi_{i+1}} R^{k_i}] \subset R^{k_i},\ i \in \{0, ..., d\}$$

from a Hilbert resolution of M are named i-th *syzygy modules* of M.

Proposition 4.23 (Homological dimension in exact sequences). *Consider an exact sequence*

$$0 \to K \to F \to M \to 0$$

of R-modules with a free R-module F of finite rank, and assume $hd_R\ M \geq 1$. Then

$$hd_R\ K = (hd_R\ M) - 1.$$

Note that in Proposition 4.23 both R-modules M and F are finitely generated.

For a proof of Proposition 4.23 see [13, Kap. III, §5, Satz 4 u. HS 5]. In order to apply HS 5 of the reference note the relation between homological dimension and profondeur of an R-module:

$$prof_R\ M = prof\ R - hd_R\ M \text{ and } prof_R\ K = prof\ R - hd_R\ K$$

which implies

$$hd_R\ M \geq 1 = hd_R\ F \iff prof_R\ M \leq (prof_R\ F) - 1.$$

Theorem 4.24 shows that each finitely generated R-module M has finite homological dimension.

Theorem 4.24 (Hilbert's syzygy theorem for finitely generated R-modules). *Each finitely generated R-module M has finite homological dimension*

$$hd_R M \leq n.$$

For a proof of Theorem 4.24 see [13, Kap. III, §5, Satz 6] or [17, Chap. II, Sect. C, Theor. 2].

4.2 Oka's coherence theorem for the structure sheaf

We now leave the field of sheaves of Abelian groups and focus on sheaves of rings and sheaves of modules over these of rings. On a complex manifold the most important sheaf of rings is the structure sheaf $\mathscr{O}$.

Definition 4.25 (Module sheaf). Consider a topological space X and a sheaf $\mathscr{R}$ of rings on X. A sheaf $\mathscr{F}$ on X is an *$\mathscr{R}$-module sheaf* or just an *$\mathscr{R}$-module* if

- for each open $U \subset X$ the set $\mathscr{F}(U)$ is a module over the ring $\mathscr{R}(U)$ due to a map
$$\mathscr{R}(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$$
- such that for each open $V \subset U$ the resulting diagram with vertical restrictions
$$\begin{array}{ccc} \mathscr{R}(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow \\ \mathscr{R}(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \end{array}$$
commutes.

In complex analysis the most important examples of $\mathscr{O}$-module sheaves are the coherent $\mathscr{O}$-modules, see Definition 4.26.

Definition 4.26 (Coherence). Consider a complex manifold X.

1. *Finite type*: An $\mathscr{O}$-module $\mathscr{F}$ is of *finite type* iff each $x \in X$ has a neighbourhood $U \subset X$ and finitely many sections
$$f_1, ..., f_k \in \mathscr{F}(U)$$
such that for all $y \in U$ the canonical map on the level of stalks
$$\mathscr{O}_y^k \to \mathscr{F}_y,\ (s_{1,y}, ..., s_{k,y}) \mapsto \sum_{j=1}^{k} s_{j,y} \cdot f_{j,y}$$
is surjective.

2. *Sheaf of relations*: Consider an open set $Y \subset X$ and finitely many sections
$$f_1, ..., f_k \in \mathscr{F}(Y)$$
of an $\mathscr{O}_Y$-module $\mathscr{F}$. Their *sheaf of relations*
$$\mathscr{R}(f_1, ..., f_k) \subset \mathscr{O}^k$$
is defined as the sheaf on Y
$$\mathscr{R}(f_1, ..., f_k)(U) := \left\{ (s_1, ..., s_k) \in \mathscr{O}^k(U) : \sum_{j=1}^{k} s_j \cdot (f_j|U) = 0 \right\},\ U \subset Y \text{ open},$$
with the canonical restrictions.

3. *Coherent*: An $\mathscr{O}$-module $\mathscr{F}$ of finite type is *coherent* if it is also *relation-finite*, i.e. for each open subset $V \subset X$ and finitely many section
$$f_1, ..., f_k \in \mathscr{F}(V)$$
their sheaf of relations $\mathscr{R}(f_1, ..., f_k)$ is an $\mathscr{O}$-module of finite type on V.

Remark 4.27 translates Definition 4.26 into a statement about exact sequences of $\mathscr{O}$-module sheaf morphisms over open subsets.

Remark 4.27 (Coherent $\mathscr{O}$-module). Consider a complex manifold X.

1. An $\mathscr{O}$-module $\mathscr{F}$ is of finite type iff each $x \in X$ has an open neighbourhood $U \subset X$ and over U an exact sequence of $\mathscr{O}$-module sheaf morphisms
$$\mathscr{O}^k \xrightarrow{\alpha} \mathscr{F} \to 0$$
for a suitable $k \in \mathbb{N}$.

2. An $\mathscr{O}$-module $\mathscr{F}$ of finite type is coherent iff each open $U \subset X$ and each sheaf morphisms of $\mathscr{O}$-modules over U
$$\mathscr{O}^k \xrightarrow{\alpha} \mathscr{F}$$
satisfies: For each $x \in U$ exists an open neighbourhood V of x in U and over V an exact sequence of $\mathscr{O}$-module sheaf morphisms
$$\mathscr{O}^m \xrightarrow{\beta} \mathscr{O}^k \xrightarrow{\alpha} \mathscr{F}$$
for a suitable $m \in \mathbb{N}$.

3. A submodule $\mathscr{G} \subset \mathscr{F}$ of a coherent $\mathscr{O}$-module $\mathscr{F}$ is coherent iff $\mathscr{G}$ is of finite type.

Theorem 4.28 (Oka's coherence theorem). *The structure sheaf $\mathscr{O}$ of a complex manifold X is coherent.*

Proof. Coherence is a local property. Therefore we may assume that the complex manifold is an open set $X \subset \mathbb{C}^n$.
Apparently the structure sheaf $\mathscr{O}$ is of finite type. Hence it remains to show its relation-finiteness. For technical reasons during the proof we show more general: For each $m \in \mathbb{N}^*$ the $\mathscr{O}$-module sheaf $\mathscr{O}^m$ is relation-finite.

For the proof we consider an arbitrary open subset $D \subset X$ and finitely many sections
$$F_1, ..., F_k \in \mathscr{O}^m(D)$$
We prove that the sheaf of relations
$$\mathscr{R}(f_1, ..., f_k)$$
is of finite type. The proof will be given by induction on $(n,m) \in \mathbb{N} \times \mathbb{N}^*$ according to the following scheme:
$$(n, \leq m) \implies (n, m+1)$$
and
$$\forall_m \; (n-1, m) \implies (n, 1)$$
i) *Induction start* $n = 0$: We have
$$X = \mathbb{C}^0 = \{0\}.$$
Hence $\mathscr{O}$ is concentrated on the point $0 \in \{0\}$ with stalk the field $\mathbb{C}$.

ii) *Induction step* $(n, \leq m) \implies (n, m+1)$: The induction step deals with increasing the rank of the $\mathscr{O}$-module in the image of the morphism

$$\mathscr{O}^k \to \mathscr{O}^m$$

under consideration. Consider given sections $F_1, ..., F_k \in \mathscr{O}^{m+1}(D)$. They define on D a sheaf morphism

$$F : \mathscr{O}^k \to \mathscr{O}^{m+1},\ (s_1, ..., s_k) \mapsto \sum_{j=1}^{k} s_j \cdot F_j,$$

We employ the exact sequence of $\mathscr{O}$-module morphisms

$$0 \to \mathscr{O} \xrightarrow{i} \mathscr{O}^{m+1} \xrightarrow{p} \mathscr{O}^m \to 0$$

with injection

$$i : \mathscr{O} \xrightarrow{i} \mathscr{O}^{m+1},\ i(s) := (s, 0, ..., 0)$$

and projection

$$p(s_1, ..., s_{m+1}) = (s_2, ..., s_m).$$

Define

$$F' := p \circ F : \mathscr{O}^k \to \mathscr{O}^m$$

with the commutative square at the right-hand side of the following diagram

$$\begin{array}{ccccccccc}
 & & \mathscr{O}^s & \xrightarrow{id} & \mathscr{O}^s & & & & \\
 & & \downarrow K & & \downarrow G \circ K & & & & \\
 & & \mathscr{O}^r & \xrightarrow{G} & \mathscr{O}^k & \xrightarrow{F'} & \mathscr{O}^m & & \\
 & & \downarrow H & & \downarrow F & & \downarrow id & & \\
0 & \longrightarrow & \mathscr{O} & \xrightarrow{i} & \mathscr{O}^{m+1} & \xrightarrow{p} & \mathscr{O}^m & \longrightarrow & 0
\end{array}$$

The induction assumption for the exponent $m \in \mathbb{N}^*$ applies to the morphism F': Each given point $a \in D$ has an open neighbourhood U in D and on U an $\mathscr{O}$-module morphism

$$G : \mathscr{O}^r \to \mathscr{O}^k$$

which provides an exact sequence

$$\mathscr{O}^r \xrightarrow{G} \mathscr{O}^k \xrightarrow{F'} \mathscr{O}^m$$

The composition $F \circ G$ satisfies

$$p \circ (F \circ G) = 0.$$

Because the lower row of the above diagram is exact, one obtains an $\mathscr{O}$-module morphism

$$H : \mathscr{O}^r \to \mathscr{O}$$

such that the left-hand square in the above diagram commutes. The induction assumption for the exponent $m = 1$ applies to the morphism H: For the given point $a \in D$ and possibly after restricting the neighbourhood U there exists on U an $\mathscr{O}$-module morphism

$$K : \mathscr{O}^s \to \mathscr{O}^r$$

which provides an exact sequence

$$\mathscr{O}^s \xrightarrow{K} \mathscr{O}^r \xrightarrow{H} \mathscr{O}$$

The upper square in the upper left part of the diagram is commutative. A final diagram chasing proves the exactness of the sequence on U

$$\mathscr{O}^s \xrightarrow{G \circ K} \mathscr{O}^k \xrightarrow{F} \mathscr{O}^{m+1}$$

and terminates the induction step.

iii) *Induction step* $\forall_m \ (n-1, m) \implies (n, 1)$: The induction step increases the dimension of the underlying domain of definition of the sheaves. One has to consider the relation sheaf of a morphism on an n-dimensional domain under the assumption that the relation sheaves of morphisms on $n-1$-dimensional domains are of finite type. The idea is to choose a coordinate transformation, which makes all sections $F_1, ..., F_k$ distinguished in z_n, and then to replace them by Weierstrass polynomials. The Weierstrass division theorem reduces modulo F_1 all occurring statements about R_n to statements about $R_{n-1}[z_n]$. For the reduction it is necessary to bound the degree of all occurrent polynomials. Hence F_1 is chosen as a polynomial with maximal degree.

iii,1) *Replacing the original sections by Weierstrass polynomials*: In case of an index $j \in \{1, ..., k\}$ with $F_j = 0$ the sheaf of relations satisfies

$$\mathscr{R}(F_1, ..., F_k) = \mathscr{R}(F_1, ..., \hat{F}_j, ..., F_k) \oplus \mathscr{O}, (\text{ skip } F_j)$$

Hence we may assume $F_j \neq 0$ for all $j = 1, ..., k$. Moreover:

- Because w.l.o.g $a = 0$ and $F_j \in R_n$ is distinguished in z_n, see Lemma 4.2, we may assume: For all $j = 1, ..., k$ the holomorphic function $F_j \in R_{n-1}[z_n]$ is polynomial and distinguished in z_n.

- The Weierstrass preparation Theorem 4.7 provides open zero-neighbourhoods

  $$U' \subset \mathbb{C}^{n-1}, \ U_0 \subset \mathbb{C}$$

  and in $U := U' \times U_0 \subset D$ representations

$$F_j = u_j \cdot \Phi_j,\ j = 1,...,k,$$

with invertible holomorphic functions $u_j \in \mathscr{O}(U)$ and Weierstrass polynomials

$$\Phi_j \in \mathscr{O}(U')[z_n].$$

We obtain on U an isomorphism of $\mathscr{O}$-modules

$$\mathscr{R}(F_1,...,F_k) \xrightarrow{\simeq} \mathscr{R}(\Phi_1,...,\Phi_k),\ (f_1,...,f_k) \mapsto (u_1 \cdot f_1,...,u_k \cdot f_k)$$

Hence w.l.o.g. the original functions from $\mathscr{O}(D)$ are Weierstrass polynomials

$$F_1,...,F_k \in \mathscr{O}(U')[z_n],\ j = 1,...,k,$$

with F_1 a polynomial of highest degree, which is denoted by d.

Having reduced the original *sections* to Weierstrass *polynomials* we now prove the induction step by showing that their *holomorphic* relations are already generated by *polynomial* relations. This will be done in two steps:

iii,2) *Reduction to germs of polynomial relations*: Consider an arbitrary but fixed point

$$x = (x', x_n) \in U = U' \times U_0,\ \text{w.l.o.g. } x = (0,0) \in U' \times U_0.$$

We show that the stalk of the relation sheaf

$$\mathscr{R}_x := \mathscr{R}(F_1,...,F_k)_x$$

contains a finite family of polynomial relations, which generate $\mathscr{R}_x$ as $\mathscr{O}_x$-module, and the degree of all involved polynomials is bounded by d.

Apparently, the $\mathscr{O}(U)$-module

$$\mathscr{R}(F_1,...,F_k)(U)$$

contains the relations

$$\mathrm{v}_2 := (-F_2, F_1, 0, ..., 0)$$

$$\vdots$$

$$\mathrm{v}_j = (-F_j, 0, ..., 0, F_1, 0, ...0),\ (F_1 \text{ at position } j)$$

$$\vdots$$

$$\mathrm{v}_k = (-F_k, 0, ..., 0, F_1).$$

These relations are polynomial, because each $F_j,\ j = 1,...,k$, is a Weierstrass polynomial due to part iii,1). Consider now an arbitrary but fixed relation

$$\phi = (\phi_1, ..., \phi_k) \in \mathscr{R}(F_1, ..., F_k)_x \subset \mathscr{O}_x^k$$

The Weierstrass preparation theorem 4.7 splits

$$F_1 = e \cdot P$$

with a unit $e \in \mathscr{O}_x$ and a Weierstrass polynomial

$$P \in \mathbb{C}\{z'\}[z_n]$$

of order $\leq d$. Lemma 4.8 shows that also

$$e \in \mathbb{C}\{z'\}[z_n]$$

is a polynomial. The Weierstrass division theorem, Theorem 4.9, reduces modulo F_1 the holomorphic components of the relation Φ as

$$\phi_j = \alpha_j \cdot F_1 + r_j,\ j = 1, ..., k,$$

with $\alpha_j \in \mathscr{O}_x$ and the rest

$$r_j \in \mathbb{C}\{z'\}[z_n]$$

a polynomial of $deg\ r_j < d$.

We verify that the difference between the two relations

$$\Delta := \phi - \sum_{j=2}^{k} \alpha_j \cdot \mathrm{v}_j \in \mathscr{R}_x$$

is polynomial: If the first component of Δ is denoted

$$\beta := \phi_1 - \sum_{j=2}^{k} \alpha_j \cdot (-F_j) \in \mathscr{O}_x$$

then

$$\Delta = (\beta, r_2, ..., r_k),$$

and we are left to show that β splits as

$$\beta = e^{-1} \cdot (e \cdot \beta)$$

with the unit $e^{-1} \in \mathscr{O}_x$ and

$$e \cdot \beta \in \mathbb{C}\{z'\}[z_n]$$

a polynomial: Note that

$$\beta \cdot F_1 = -\sum_{j=2}^{k} r_j \cdot F_j \in \mathbb{C}\{z'\}[z_n] \subset \mathscr{O}_x.$$

is polynomial, and that

$$\beta \cdot F_1 = (\beta \cdot e) \cdot P \in \mathscr{O}_x$$

with the Weierstrass polynomial $P \in \mathbb{C}\{z'\}[z_n]$. Lemma 4.8 shows

$$\beta \cdot e \in \mathbb{C}\{z'\}[z_n],$$

hence $\beta \cdot e$ polynomial. Summing up:

$$\phi = \left(\sum_{j=2}^{k} \alpha_j \cdot \mathrm{v}_j\right) + \Delta = \left(\sum_{j=2}^{k} \alpha_j \cdot \mathrm{v}_j\right) + e^{-1} \cdot (e \cdot \beta, e \cdot r_2, ..., e \cdot r_k) \in \mathscr{R}_x$$

represents ϕ as an $\mathscr{O}_x$ -linear combination of polynomial relations from $\mathscr{R}_x$, because

- the relations $\mathrm{v}_j,\ j = 2, ..., k$,
- the element $e \cdot \beta$,
- and the elements $e \cdot r_j,\ j = 2, ..., k$,

are polynomial. Each involved polynomial has degree $\leq d$ because

- $$deg\ \mathrm{v}_j \leq d,\ j = 2, ..., k,$$
- and
  $$deg(e \cdot \beta) = deg(\beta \cdot F_1) - deg\ P < deg\ P + d - deg\ P = d$$
  because $P \cdot (e \cdot \beta) = F_1 \cdot \beta$, and $deg\ (\beta \cdot F_1) < d + d$,
- and for $j = 2, ..., k$
  $$deg(e \cdot r_j) < deg\ e + \deg\ P = deg(e \cdot P) = deg\ F_1 = d$$

iii,3) *Finitely generated polynomial relations of bounded degree*: We show the existence of an open zero-neighbourhood $V \subset U$ and finitely many polynomial relations

$$s_1, ..., s_l \in \mathscr{R}(F_1, ..., F_k)(V)$$

such that for each point $x \in V$ each germ of a polynomial relation of degree $\leq d$

$$\sigma \in \mathscr{R}(F_1, ..., F_k)_x$$

is an $\mathscr{O}_x$-linear combination of the germs $(s_{1,x}, ..., s_{l,x})$.

In order to apply the induction hypothesis, which is valid on $\mathbb{C}^{n-1}$, we derive from the given sections

$$F_1, ..., F_k \in \mathscr{O}(D),\ D \subset \mathbb{C}^n,$$

a morphism between free sheaves on $\mathbb{C}^{n-1}$: For each $s \in \mathbb{N}$ consider the sheaf $\mathscr{P}_s$ on $\mathbb{C}^{n-1}$ defined as

$$\mathscr{P}_s(V') := \left\{ \sum_{j=0}^{s} \psi_j \cdot z_n^j \in \mathscr{O}_{\mathbb{C}^{n-1}}(V')[z_n] \right\},\ V' \subset U',$$

of polynomials of degree bounded by s with holomorphic coefficients from $\mathscr{O}_{\mathbb{C}^{n-1}}(V')$. These sheaves are free and isomorphic to $\mathscr{O}_{\mathbb{C}^{n-1}}^{s+1}$ via

$$\mathscr{O}_{\mathbb{C}^{n-1}}^{s+1} \xrightarrow{\simeq} \mathscr{P}_s,\ (\psi_0, ..., \psi_d) \mapsto \sum_{j=0}^{d} \psi_j \cdot z_n^j.$$

On $U' \subset \mathbb{C}^{n-1}$ we consider the morphism of $\mathscr{O}_{\mathbb{C}^{n-1}}$-modules

$$F' : \mathscr{P}_d^k \to \mathscr{P}_{2d},\ (\phi_1, ..., \phi_k) \mapsto \sum_{j=1}^{k} F_j \cdot \phi_j$$

To F' applies the induction hypothesis $(n-1, 2d+1)$: There exists an open zero-neighbourhood

$$V' \subset U' \subset \mathbb{C}^{n-1}$$

and finitely many sections

$$s_1, ..., s_l \in ker\ F'(V')$$

which define on V' an exact sequence of $\mathscr{O}_{V'}$-modules

$$\mathscr{O}_{\mathbb{C}^{n-1}}^l \to \mathscr{P}_d^k \xrightarrow{F'} \mathscr{P}_{2d}$$

For suitable $\varepsilon > 0$ and

$$V := V' \times \{z_n \in \mathbb{C} :\ |z_n| < \varepsilon\} \subset U$$

the migration from

$$V' \subset \mathbb{C}^{n-1} \text{ to } V \subset \mathbb{C}^n$$

is provided by a canonical injection of functions

$$\mathscr{P}_d^k(V') \hookrightarrow \mathscr{O}_{\mathbb{C}^n}^k(V)$$

For $x = (x', x_n) \in V$ each polynomial relation of degree $\leq d$

$$\sigma \in \mathscr{R}(F_1, ..., F_k)_x$$

can be considered an element $\sigma \in \ker\ F'_{x'}$, hence can be represented due to part iii,2) in the form

$$\sigma = \sum_{\lambda=1}^{l} g_\lambda \cdot s_\lambda$$

with coefficients

$$g_\lambda \in \mathscr{O}_{\mathbb{C}^{n-1},x'} \subset \mathscr{O}_{\mathbb{C}^n,x}$$

The representation of σ finishes the proof of the claim from part iii,3).

The results of the parts iii,1-3) finish the proof of the induction step

$$\forall_m \ (n-1,m) \implies (n,1)$$

□

4.3 Coherent $\mathscr{O}$-modules

Proposition 4.29 (Coherence of the ideal sheaf of an analytic submanifold). *Consider an open subset $U \subset \mathbb{C}^n$. Each analytic submanifold $A \subset U$ has a coherent ideal sheaf*

$$\mathscr{I}_A \subset \mathscr{O}_U$$

Here $\mathscr{I}_A$ is defined as the sheafification of the presheaf

$$V \mapsto \{f \in \mathscr{O}_U(V) : \ f|A \cap V = 0\}, \ V \subset U \textit{ open}.$$

Proof. Due to Oka's Theorem 4.28 the structure sheaf $\mathscr{O}_U$ is coherent. Hence it remains to prove that the subsheaf $\mathscr{I}_A \subset \mathscr{O}_U$ is of finite type. Consider a given point $x \in U$.

- *Case $x \in A$*: According to Theorem 2.8 we may assume an open set $W \subset \mathbb{C}^n$ and $x = 0 \in W$ satisfying

  $$A = W \cap E_k \text{ with } E_k = \{(z_1, ..., z_n) \in W : \ z_{k+1} = ... = z_n = 0\}$$

  We claim that the functions

  $$z_{k+1}, ..., z_n \in \mathscr{I}_A(W)$$

  generate $\mathscr{I}_A|W$: Assume

  $$y \in W \text{ and } f \in (\mathscr{I}_A)_y.$$

  i) If $y \in W \setminus E_k$ then exists index $j \in k+1, ..., n$ with $z_j(y) \neq 0$. Hence

  $$z_{j,y} \neq 0 \in (\mathscr{O}_U)_y \text{ and } \frac{f}{z_{j,y}} \in (\mathscr{O}_U)_y$$

which implies

$$f = \frac{f}{z_{j,y}} \cdot z_{j,y}$$

ii) If $y \in W \cap E_k$ then $y = (y_1, ..., y_k, 0, ..., 0)$. The function f is defined by a convergent power series

$$f(z) = \sum_{I=(i_1,...,i_n)\in\mathbb{N}^n} c_I \cdot (z_1 - y_1)^{i_1} \cdot ... \cdot (z_k - y_k)^{i_k} \cdot z_{k+1}^{i_{k+1}} \cdot ... \cdot z_n^{i_n}$$

The vanishing $f|E_k = 0$ implies that each summand of the power series contains a variable z_j for at least one index $j \in k+1, ..., n$. Hence in a neighbourhood of y

$$f = \sum_{j=k+1}^{n} s_j \cdot z_j$$

with holomorphic coefficients s_j.

- *Case* $x \notin A$: Because $A \subset U$ is closed we have $(\mathscr{I}_A)_x = (\mathscr{O}_U)_x$ which implies

  $$(\mathscr{I}_A)_y = (\mathscr{O}_U)_y$$

  for y in an open neighbourhood of x. Hence the constant 1 generates $\mathscr{I}_A$ in a neighbourhood of x.

□

The result of Proposition 4.29 holds more general also for analytic subsets: The ideal sheaf $\mathscr{I}_A$ of an analytic subset A in an open subset $U \subset \mathbb{C}^n$ is a coherent $\mathscr{O}_U$-module, see [7, §16, Satz 1] and [15, Chap. IV, §2, Fund. Theor.].

Coherence of an $\mathscr{R}$-module allows to extend properties of a given stalk to all stalks in a neighbourhood.

Proposition 4.30 (Stalks of a coherent $\mathscr{O}$-module). *Consider a coherent sheaf $\mathscr{F}$ on a complex manifold X. For each point*

$$x \in X \textit{ with } \mathscr{F}_x = 0$$

exists an open neighbourhood V of x with

$$\mathscr{F}_y = 0 \textit{ for all } y \in V.$$

Proof. Because $\mathscr{F}$ is of finite type there exists an open neighbourhood U of x and over U an exact sequence of $\mathscr{O}|U$-modules

$$\mathscr{O}^p \to \mathscr{F} \to 0$$

for a suitable $p \in \mathbb{N}$. The canonical base $(e_i)_{i=1,...,p}$ of the $\mathscr{O}(U)$-module $\mathscr{O}^p(U)$ provides a family of sections

$$f_1, ..., f_p \in \mathscr{F}(U)$$

such that for each $y \in U$ the germs

$$f_{1,y}, ..., f_{p,y} \in \mathscr{F}_y$$

generate the stalk $\mathscr{F}_y$ as $\mathscr{O}_y$-module. The assumption

$$\mathscr{F}_x = 0$$

implies according to the definition of stalks: In a common neighbourhood $V \subset U$ of x each section $f_1, ..., f_p$ restricts to zero, as a consequence its germ

$$f_{j,y} \in \mathscr{F}_y, \ y \in V,$$

vanishes. Hence $\mathscr{F}_y = 0$ for all $y \in V$. □

Proposition 4.31 (Kernel and cokernel of morphisms between coherent $\mathscr{O}$-modules). *Consider a complex manifold X and a morphism*

$$\alpha : \mathscr{F} \to \mathscr{G}$$

between two coherent $\mathscr{O}$-modules. Then the $\mathscr{O}$-modules on X

$$ker\ \alpha \text{ and } coker\ \alpha$$

are also coherent.

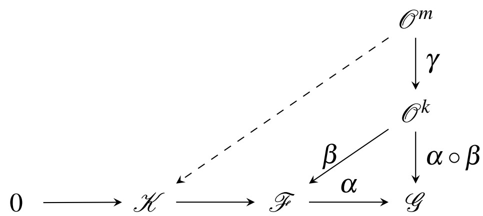
Proof. i) *Coherence of ker* α: The $\mathscr{O}$-module

$$\mathscr{K} := ker\ \alpha \subset \mathscr{F}$$

is a subsheaf of the coherent sheaf $\mathscr{F}$. Hence it suffices to show that $\mathscr{K}$ is of finite type, i.e. each $x \in X$ has a neighbourhood U_1 and over U_1 a sheaf epimorphism

$$\mathscr{O}^m \xrightarrow{\beta} \mathscr{K}$$

We construct the following commutative diagram with an exact row:



Coherence of $\mathscr{F}$ provides an open neighbourhood U of x with a sheaf epimorphism

$$\mathscr{O}^k \xrightarrow{\beta} \mathscr{F}.$$

The sheaf of relations

$$\mathscr{R} := ker\ [\mathscr{O}^k \xrightarrow{\alpha\circ\beta} \mathscr{G}] \subset \mathscr{O}^k$$

is of finite type because $\mathscr{G}$ is coherent. Hence over an open neighbourhood $U_2 \subset U_1$ of x exists a sheaf morphism

$$\mathscr{O}^m \xrightarrow{\gamma} \mathscr{O}^k$$

with an exact sequence

$$\mathscr{O}^m \xrightarrow{\gamma} \mathscr{O}^k \xrightarrow{\alpha\circ\beta} \mathscr{G}$$

The exact sequence

$$\mathscr{O}^m \xrightarrow{\beta\circ\gamma} \mathscr{F} \xrightarrow{\alpha} \mathscr{G}$$

induces the exact sequence over U_2

$$\mathscr{O}^m \xrightarrow{\beta\circ\gamma} \mathscr{K} \to 0$$

which proves that $\mathscr{K}$ is of finite type.

ii) *Coherence of coker* α: First, the induced sequence

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} coker\ \alpha \to 0$$

is exact. Because $\mathscr{G}$ is of finite type, also *coker* α is of finite type. Secondly, we have to show that for each morphism over an open set $U \subset X$

$$\mathscr{O}^p \xrightarrow{\gamma} coker\ \alpha$$

the sheaf of relations is of finite type:

First, the sheaf $\mathscr{O}^p$ is a free $\mathscr{O}$-module. On the level of sections for each open $V \subset X$ the $\mathscr{O}(V)$-module $\mathscr{O}^p(V)$ is free. Denote by

$$(e_i)_{i=1,\ldots,p}$$

its canonical base. Due to the surjectivity of β, for given $x \in X$ exist on the level of stalks germs

$$g_{i,x} \in \mathscr{G}_x,\ i = 1, ..., p,$$

with

$$\beta_x(g_{i,x}) = \gamma_x(e_{i,x}) \in (coker\ \alpha)_x$$

There exists a common open neighbourhood $V \subset X$ of x and for $i = 1, ..., p$ sections

$$g_i \in \mathscr{G}(V)$$

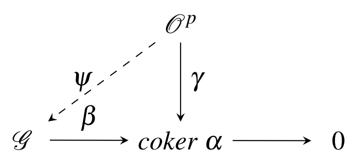
which represent the germ $g_{i,x} \in \mathscr{G}_x$. Setting

$$\psi(e_i) := g_i,\ i = 1, ..., p,$$

defines a morphism of $\mathscr{O}$-modules over V

$$\psi : \mathscr{O}^p \to \mathscr{G}$$

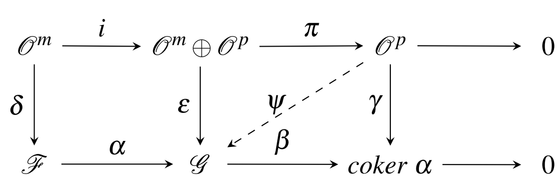
such that the following diagram of sheaf morphisms over V commutes



Secondly, by assumption $\mathscr{F}$ is of finite type. Hence each $x \in U$ has an open neighbourhood $W \subset V$ and over W an epimorphism

$$\mathscr{O}^m \xrightarrow{\delta} \mathscr{F}$$

Combining the latter two results provides over W the diagram of sheaf morphisms



Here

$$\varepsilon : \mathscr{O}^m \oplus \mathscr{O}^p \to \mathscr{G}$$

is defined as

$$\varepsilon(a,b) := \alpha(\delta(a)) + \psi(b)$$

and

$$\mathscr{O}^m \xrightarrow{i} \mathscr{O}^m \oplus \mathscr{O}^p \text{ and } \mathscr{O}^m \oplus \mathscr{O}^p \simeq \mathscr{O}^m \times \mathscr{O}^p \xrightarrow{\pi} \mathscr{O}^p$$

are the canonical injection and projection. The diagram is commutative with respect to its horizontal and vertical morphisms. Moreove by construction

$$\gamma = \beta \circ \psi$$

Because

$$\beta \circ \alpha = 0$$

the restriction

$$\pi | \ker\ \varepsilon : \ker\ \varepsilon \to ker\ \gamma$$

is well-defined. The restriction is surjective: Consider a point $x \in W$ and a germ $c \in \mathscr{O}_x^p$ with

$$0 = \gamma(c) = \beta(\psi(c)).$$

Then exist a germ $a \in \mathscr{F}_x$ satisfying

$$\psi(c) = \alpha(a),$$

and a germ

$$d \in \mathscr{O}_x^m$$

satisfying

$$\delta(d) = a.$$

Then $(-d,c) \in (ker\ \varepsilon)_x$ because

$$\varepsilon(-d,c) = -\alpha(\delta(a)) + \psi(c) = -\psi(c) + \psi(c) = 0$$

and

$$\pi(-d,c) = c,$$

which finishes the proof of the surjectivity of

$$\ker\ \varepsilon \to ker\ \gamma.$$

Because $ker\ \varepsilon$ is of finite type, also $ker\ \gamma$ is of finite type, which proves that the relation sheaf of γ is of finite type, and finishes the proof. □

Corollary 4.32 (Morphisms between coherent $\mathscr{O}$-modules). *Consider a morphism*

$$\phi : \mathscr{F} \to \mathscr{G}$$

between two coherent sheaves on a complex manifold X. The subsets

$$Inj(\phi) := \{x \in X :\ \phi_x : \mathscr{F}_x \to \mathscr{G}_x \text{ injective}\}$$

and

$$Surj(\phi) := \{x \in X : \ \phi_x : \mathscr{F}_x \to \mathscr{G}_x \ surjective\}$$

and

$$Iso(\phi) := \{x \in X : \ \phi_x : \mathscr{F}_x \to \mathscr{G}_x \ isomorphism\}$$

are open subsets of X.

Proof. According to Proposition 4.31 the $\mathscr{O}$-modules *ker* ϕ and *coker* ϕ are coherent. The injectivity of ϕ_x is equivalent to $(\ker\phi)_x = 0$ and the surjectivity of ϕ_x is equivalent to $(coker\ \phi)_x = 0$. Hence the claim follows from Proposition 4.30. □

Proposition 4.33 (Coherence in short exact sequences). *Consider a complex manifold X and an exact sequence of* $\mathscr{O}$*-modules on X*

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

If two of the sheaves are coherent, then also the third sheaf is coherent.

Proof. The proof follows from Proposition 4.31:

i) *Assume* $\mathscr{G}$ *and* $\mathscr{H}$ *coherent*: Then

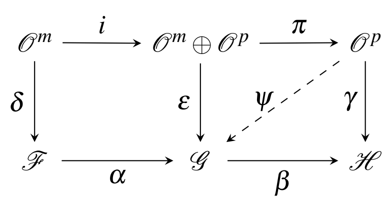
$$\mathscr{F} \simeq ker\ \beta$$

ii) *Assume* $\mathscr{F}$ *and* $\mathscr{G}$ *coherent*: Then

$$\mathscr{H} \simeq \mathscr{G}/ker\ \beta \simeq \mathscr{G}/im\ \alpha = coker\ \alpha$$

iii) *Assume* $\mathscr{F}$ *and* $\mathscr{H}$ *coherent*:

- *Finite type*: Alike to the proof from Proposition 4.31 we consider in a suitable neighbourhood of a given point $x \in X$ the diagram of sheaf morphisms with exact rows



The existence of the epimorphisms δ and γ follows because $\mathscr{F}$ and $\mathscr{H}$ are finitely generated. The morphism i and π are the canonical injection respectively projection. The diagram is commutative with respect to its horizontal and vertical morphisms. The morphism γ induces a morphism ψ with

$$\beta \circ \psi = \gamma$$

because $\mathscr{O}^p$ is a free sheaf. Eventually we define

$$\varepsilon : \mathscr{O}^m \oplus \mathscr{O}^p \to \mathscr{G},\ \varepsilon(d,b) := \alpha(\delta(d)) + \psi(b)$$

The morphism ε is surjective: For a given germ $g \in \mathscr{G}_x$ first choose a germ $b \in \mathscr{O}_x^p$ with

$$\beta(g) = \gamma(b)$$

Then

$$\beta(g - \psi(b)) = \beta(g) - \beta(\psi(b)) = \beta(g) - \gamma(b) = \gamma(b) - \gamma(b) = 0$$

Hence there exists a germ

$$d \in \mathscr{O}_x^m$$

such that $a := \delta(d)$ satisfies

$$\alpha(a) = g - \psi(b)$$

Then

$$\varepsilon(d,b) = \alpha(\delta(d)) + \psi(b) = \alpha(a) + \psi(b) = g - \psi(b) + \psi(b) = g,$$

which shows the surjectivity of ε and proves that $\mathscr{G}$ is of finite type.

- *Relation-finite*: Consider a given morphism

$$g : \mathscr{O}^k \to \mathscr{G}$$

over an open set $U \subset X$ and a given point $x \in U$. The morphism extends to a morphism

$$\beta \circ g : \mathscr{O}^k \to \mathscr{H}.$$

Then exists over a suitable open neighbourhood $V \subset U$ of x a commutative diagram of sheaf morphisms over V with exact rows

$$\begin{array}{ccccccccc}
 & & \mathscr{O}^m & \overset{\gamma}{\dashrightarrow} & \mathscr{O}^k & \xrightarrow{\beta \circ g} & \mathscr{H} & & \\
 & & \psi \downarrow & & g \downarrow & & \downarrow id & & \\
0 & \longrightarrow & \mathscr{F} & \xrightarrow[\alpha]{} & \mathscr{G} & \xrightarrow[\beta]{} & \mathscr{H} & \longrightarrow & 0
\end{array}$$

Here the morphism γ is induced because $\mathscr{H}$ is relation-finite. And the morphism ψ is induced because

$$\beta \circ (g \circ \gamma) = 0 \text{ and } ker\ \beta = im\ \alpha,$$

and because the sheaf $\mathscr{O}^m|U$ is free. The restriction

$$\gamma|ker\ \psi : ker\ \psi \to ker\ g$$

is well-defined and surjective. The coherence of $\mathscr{F}$ implies that *ker* ψ is of finite type. Hence also *ker g* is of finite type, which finishes the proof of the coherence of $\mathscr{G}$.
□

As a consequence of Theorem 4.28 and Proposition 4.33, besides the structure sheaf $\mathscr{O}$ of a complex manifold X also finite direct sums $\mathscr{O}^p,\ p \in \mathbb{N}$, and locally free $\mathscr{O}$-modules of finite rank are coherent.

The concept of the direct image from Definition 2.25 carries over to cohomology groups of higher order.

Definition 4.34 (Higher direct image). Consider a continuus map

$$f : X \to Y$$

between topological spaces and a sheaf $\mathscr{F}$ on X. For each $q \geq 1$ define the presheaf on Y

$$V \mapsto H^q(f^{-1}(V), \mathscr{F}),\ V \subset Y \text{ open},$$

Its sheafification, denoted $R^q f_* \mathscr{F}$, is named the *q-th direct image sheaf* of $\mathscr{F}$; see [3, Chap. III, § 1].

Remark 4.35 (Grauert's coherence theorem). Consider a holomorphic map

$$f : X \to Y$$

between complex manifolds and a coherent $\mathscr{O}_X$-module $\mathscr{F}$ on X. Grauert's coherence theorem states: Under the additional topological assumption that f is a proper map, i.e. the inverse image of compact sets is compact, all direct image sheaves

$$R^q f_* \mathscr{F},\ q \in \mathbb{N},$$

are coherent $\mathscr{O}_Y$-modules.

Grauert's coherence theorem is a highlight of complex analysis. Besides Grauert's original proof there exist separate proofs by Forster-Knorr and Kiehl-Verdier. Grauert's theorem is considered one of the deepest theorems of complex analysis, see also [3, Chap. III, §2].

Chapter 5
Cartan's lemma for holomorphic matrices

5.1 Proof of Cartan's lemma

Cartan's lemma for holomorphic matrices is a result about the multiplicative splitting of invertible matrices of bounded holomorphic functions which are defined in a neighbourhood of the intersection of two adjacent product domains. It generalizes Proposition 2.34, which can be considered the particular case of holomorphic functions with values in $GL(1,\mathbb{C}) = \mathbb{C}^*$.

In order to prove Cartan's lemma about the splitting of matrices we have to generalize the analysis used so far. Until now we considered maps which depend on a finite set of complex variables and have their values in $\mathbb{C}^n$. These maps are assumed to be smooth or holomorphic. The present section considers maps with arguments depending on infinitely many parameters. More precisely, the argument varies in Banach spaces of bounded holomorphic functions. Also the image of the maps are Banach spaces. As a consequence we use some results from functional analysis.

Differentiability of such a map f at a point in an open set is defined via the usual differential quotient, i.e. by approximating f in a neighbourhood of the point by a linear map. Definition 5.3 formalizes a condition which is slightly stricter concerning the precision of the approximation.

Notation 5.1 (Banach norms). *For an open set $U \subset \mathbb{C}^n$ denote by $\mathscr{B}(U)$ the Banach algebra of bounded holomorphic functions $f \in \mathscr{O}(U)$ with the norm*

$$\|f\|_U := \sup\{|f(z)| : \ z \in U\}$$

Consider two open sets $U_j \subset \mathbb{C}^n, \ j = 1,2$ with intersection

$$U_{12} := U_1 \cap U_2$$

1. For $k \in \mathbb{N}$ and $j = 1,2$ denote by

$$E_j := M(k \times k, \mathscr{B}(U_j))$$

the vector space of all $k \times k$-matrices with entries in $\mathscr{B}(U_j)$ and by

$$E_j^* := GL(k, \mathscr{B}(U_j)) \subset E_j$$

the subgroup of invertible matrices.

2. *We consider the matrices $A \in E_j$ as linear maps between the Banach spaces $\mathscr{B}(U_j)^{\oplus k}$ with respect to the following norms: For*

$$f = (f_1, ..., f_k) \in \mathscr{B}(U_j)^{\oplus k}$$

set

$$\|f\| := \|f\|_{U_j} := sup\left\{\|f_\kappa\|_{U_j} : \ \kappa = 1, ..., k\right\}.$$

We recall that a linear map $A \in E_j,\ j = 1, 2,$ is continuous *iff the operator norm*

$$\|A\| := sup\{\|Af\| : \ f \in \mathscr{B}(U_j)^{\oplus k} \text{ and } \|f\| \leq 1\}$$

is finite. For a pair of continuous linear maps $(A, B) \in E_1 \times E_2$ define the operator norm

$$\|(A, B)\| := \sup\{\|A\|, \|B\|\}$$

We will derive Cartans's lemma, Theorem 5.8, about the splitting of holomorphic matrices from the following results:

- Additive splitting of bounded holomorphic functions (Prop. 5.2)
- Openness of strictly differentiable maps between Banach spaces at points with surjective tangent map (Proposition 5.4)
- Approximation of holomorphic matrix functions on product domains by global holomorphic matrix functions (Cor. 5.7)

We first extend Cartan's lemma for holomorphic functions, Proposition 2.34, to the corresponding statement for bounded holomorphic functions.

Proposition 5.2 (Additive splitting of bounded holomorphic functions). *Consider two compact adjacent product domains*

$$Q_1,\ Q_2 \subset \mathbb{C}^n,$$

an open set $U \subset \mathbb{C}^n$ with

$$(Q_1 \cap Q_2) \subset U$$

together with a bounded holomorphic function $f \in \mathscr{B}(U)$. Then exist open product domains

$$U_1,\ U_2 \textit{ with } Q_j \subset U_j,\ j=1,2, \textit{ and } U_0 := (U_1 \cap U_2) \subset\subset U,$$

together with a pair of bounded holomorphic functions

$$(f_1, f_2) \in \mathscr{B}(U_1) \times \mathscr{B}(U_2)$$

satisfying on U_0

$$f = f_1 - f_2.$$

Proof. Under the assumptions of the proposition above, Proposition 2.34 provides a pair of holomorphic functions

$$(f_1, f_2) \in \mathscr{O}(U_1) \times \mathscr{O}(U_2)$$

satisfying on U_0

$$f = f_1 - f_2.$$

We now restrict the solution to a pair of relatively compact, open product domains

$$Q_j \subset U'_j \subset\subset U_j,\ j=1,2.$$

Then the restrictions are bounded

$$f'_j := f_j | U'_j \in \mathscr{B}(U'_j),\ j=1,2,$$

and satisfy on

$$U'_0 := U'_1 \cap U'_1$$

the splitting with bounded holomorphic functions

$$f|U_0 = f'_1 - f'_2.$$

□

Next we need a kind of implicit function theorem to conclude that a differentiable map with surjective tangent map at a point is locally an open map. The maps in question are maps between Banach spaces, specifically maps with arguments in an open subset of a Banach space of bounded holomorphic functions. We first define the concept of strict differentiability, see Definition 5.3. Then we prove a criterion for the local openness of strictly differentiable maps between Banach spaces, see Lemma 5.4.

Definition 5.3 (Strict differentiability). Consider two Banach spaces E and F, an open set $U \subset E$ and a point $a \in U$. A map

$$f : U \to F$$

is *differentiable in the strict sense* at the point a if there exists a linear continuous map

$$\phi : E \to F$$

satisfying

$$\lim_{\substack{u,\text{v}\to a\\ u\neq \text{v}}} \frac{\|f(u)-f(\text{v})-\phi(u-\text{v})\|}{\|u-\text{v}\|} = 0$$

The tangent map

$$\phi =: f'(a)$$

is named the *derivation* of f at a.

The notion of *strict differentiability* is stricter than the notion of differentiability. Strict differentiability considers the limit with two arguments u and v varying independently around a when approximating f near a by a continous linear map ϕ with

$$f(u)-f(\text{v}) = \phi(u-\text{v}) + o(\|u-\text{v}\|)$$

Lemma 5.4 proves the local openness criterion for strictly differentiable maps with surjective tangent map at the distinguished point.

Proposition 5.4 (Local openness criterion). *Consider two Banach spaces E and F, an open set $U \subset E$, and a map*

$$f : U \to F,$$

which is strict differentiable at a point $a \in U$ and has a surjective tangent map at a

$$f'(a) : E \to F$$

Then f is open at a, i.e. the image $f(U) \subset F$ is a neighbourhood of $f(a)$.

Proof. i) *Openness of the linear tangent map*: W.l.o.g. we may assume

$$a = 0,\ f(a) = 0.$$

According to Banach's theorem the tangent map, the surjective linear continuous map,

$$\phi := f'(a) : E \to F$$

is open, i.e. for each zero-neighbourhood $V \subset E$ exists a zero-neighbourhood $W \subset F$ with

$$W \subset \phi(V).$$

When expressing this property by the Banach norms it suffices to consider only the specific zero-neighbourhood

$$V := \{x \in E : \|x\|_E \leq 1\}$$

The condition on openness states: There exists $\varepsilon > 0$ such that

$$W := \{y \in F : \|y\|_F \leq \varepsilon\}$$

satisfies

$$W \subset \phi(V).$$

In particular, for each $y \in F$ with $\|y\|_F = \varepsilon$ exists

$$x \in \phi^{-1}(y) \cap V.$$

Then

$$\|x\|_E \leq 1 = \frac{1}{\varepsilon} \cdot \|y\|_F = C \cdot \|y\|_F,\ C := \frac{1}{\varepsilon}$$

We may assume $C = 1$ after rescaling $\|\|_F$ by the factor $1/\varepsilon$. Then each $y \in F$ has an inverse image $x \in E$ with

$$\|x\|_E \leq \|y\|_F.$$

ii) *Local openness of f at $a = 0$*: We derive from the openness of the tangent map ϕ at a that the original map f is open at a. Strict differentiability of f at a means for pairs $(u, \mathrm{v}) \in E \times E$

$$\|f(u) - f(\mathrm{v}) - \phi(u - \mathrm{v})\|_F = o(\|u - \mathrm{v}\|_E)$$

In particular, there exists $\delta > 0$ such that for all pairs from the product of the closed δ-neighbourhoods of the origin $(0,0) \in E \times E$

$$(u, \mathrm{v}) \in \overline{K}_\delta(0) \times \overline{K}_\delta(0) \subset E \times E$$

holds

$$\|f(u) - f(\mathrm{v}) - \phi(u - \mathrm{v})\|_F \leq (1/2) \cdot \|u - \mathrm{v}\|_E$$

At this point, referring to pairs, we use that the differentiability of f holds in the strict sense.

We choose a radius

$$r \in \mathbb{R}^*_+ \text{ with } 0 < r \leq \delta \text{ and } \overline{K}_r(0) \subset U$$

and claim

$$V := \overline{K}_{r/2} \subset f(U).$$

For each given $y_0 \in V$ we have to find an element $\tilde{x} \in U$ with

$$f(\tilde{x}) = y_0.$$

By part i) there exists an element $x_0 \in E$ with

$$\phi(x_0) = y_0 \text{ and } \|x_0\|_E \leq \|y_0\|_F$$

Because not necessarily $f(x_0) = y_0$ we construct a sequence $(x_n)_{n\in\mathbb{N}}$ in $K_r(0)$ which converges against a solution $\tilde{x}$ with

$$f(\tilde{x}) = y_0.$$

The stepwise construction relies on the map

$$g : U \to F,\ g(x) := y_0 + \phi(x) - f(x)$$

The construction is by induction on $n \in \mathbb{N}$. The induction step assumes the existence of the elements $x_0, ..., x_n$ from the sequence and constructs the next element $x_{n+1} \in E$ as an inverse image satisfying Equation (1)

$$g(x_n) = \phi(x_{n+1})$$

and Equation (2):

$$\|x_{n+1} - x_n\|_E \leq (1/2)^n \cdot (r/4)$$

As a consequence, x_{n+1} satisfies also

$$\|x_{n+1}\|_E \leq \|x_n\|_E + \|x_{n+1} - x_n\|_E \leq r \cdot \left((3/4) + (1/4) \cdot \sum_{k=1}^{n-1} (1/2)^k\right) + \frac{r}{4} \cdot (1/2)^n$$

$$\leq r \cdot \left((3/4) + (1/4) \cdot \sum_{k=1}^{n} (1/2)^k\right) < r$$

- Induction start $n = 0$: Due to

  $$\|x_0\|_E \leq \|y_0\|_F \leq \frac{r}{2}$$

  the estimate from part i), applied to the pair

  $$(x_0, 0) \in \overline{K}_\delta(0) \times \overline{K}_\delta(0)$$

  shows

  $$\|\phi(x_0) - f(x_0)\|_F \leq (1/2) \cdot \|x_0\|_E \leq (1/2) \cdot \frac{r}{2}$$

  Hence taking the inverse image with respect to ϕ provides an element $z \in E$ satisfying

  $$\phi(z) = \phi(x_0) - f(x_0) \text{ and } \|z\|_E \leq \|\phi(x_0) - f(x_0)\|_F \leq \frac{r}{4}$$

  Define

  $$x_1 := z + x_0$$

The pair (x_0, x_1) satisfies Equation (1)

$$\phi(x_1) = \phi(z) + \phi(x_0) = (\phi(x_0) - f(x_0)) + \phi(x_0) = y_0 - f(x_0) + \phi(x_0) = g(x_0),$$

and Equation (2)

$$\|x_1 - x_0\|_E = \|z\|_E \le \frac{r}{4}$$

- Induction step $n \mapsto n+1$: Assume the existence of the elements $x_0, ..., x_n$. Considering inverse images with respect to ϕ provides an element $z \in E$ satisfying

$$\phi(z) = g(x_n) - g(x_{n-1})$$

The element $z \in E$ improves the approximation of the sequence: Define

$$x_{n+1} := z + x_n$$

The pair (x_n, x_{n+1}) satisfies equation (1)

$$\phi(x_{n+1}) = \phi(z + x_n) = \phi(z) + \phi(x_n) = (g(x_n) - g(x_{n-1})) + \phi(x_n) =$$

$$= g(x_n) - \phi(x_n) + \phi(x_n) = g(x_n)$$

using the induction assumption $g(x_{n-1}) = \phi(x_n)$, and equation (2)

$$\|x_{n+1} - x_n\|_E = \|z\|_E \le \|g(x_n) - g(x_{n-1})\|_F = \|\phi(x_n) - \phi(x_{n-1}) - (f(x_n) - f(x_{n-1}))\|_F =$$

$$= \|f(x_n) - f(x_{n-1}) - (\phi(x_n) - \phi(x_{n-1}))\|_F \le (1/2) \cdot \|x_n - x_{n-1}\|_E \le$$

$$\le (1/2) \cdot (1/2)^{n-1} \cdot \frac{r}{4} = (1/2)^n \cdot \frac{r}{4}$$

using the openness of the tangent map from part i) and the induction asumption.

Due to the set of equations (2) the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in E. Hence there exists the limit

$$\tilde{x} := \lim_{n \to \infty} x_n \in \overline{K}_r(0) \subset U$$

In order to show $f(\tilde{x}) = y_0$ we first note

$$f(\tilde{x}) = \lim_{n \to \infty} f(x_n)$$

due to the continuity of f. Secondly we consider the equality due to equation (1)

$$y_0 - f(x_n) = (y_0 + \phi(x_n) - f(x_n)) - \phi(x_n) = g(x_n) - \phi(x_n) =$$

$$= \phi(x_{n+1}) - \phi(x_n) = \phi(x_{n+1} - x_n)$$

Hence

$$\lim_{n \to \infty} \|y_0 - f(x_n)\|_F = \lim_{n \to \infty} \|\phi(x_{n+1} - x_n)\|_F = 0$$

which completes the proof. □

Proposition 5.5 (Holomorphic approximation: additive case). *For an open product domain $Q \subset \mathbb{C}$ the restriction*

$$\mathscr{O}(\mathbb{C}) \to \mathscr{O}(Q),\ f \mapsto f|Q,$$

has dense image with respect to the canonical Fréchet topologies.

Proposition 5.5 is a specific case of the Runge approximation theorem for the 1-dimensional case, see [24, Kap. 12, §2, Approximationssatz]. We give an elementary proof for product domains. It relies on Cauchy's integral formula. First, the integral is approximated by a suitable Riemann sum. Secondly, the integrand gets evaluated at finitely many points. At each point a suitable Taylor approximation provides a global holomorphic approximation of the integrand.

Proof. Consider a given function $f \in \mathscr{O}(Q)$, a compact $K \subset Q$, and a number $\varepsilon > 0$. W.l.o.g. K is a compact product domain itself.

i) *Approximation by Riemann sums*:

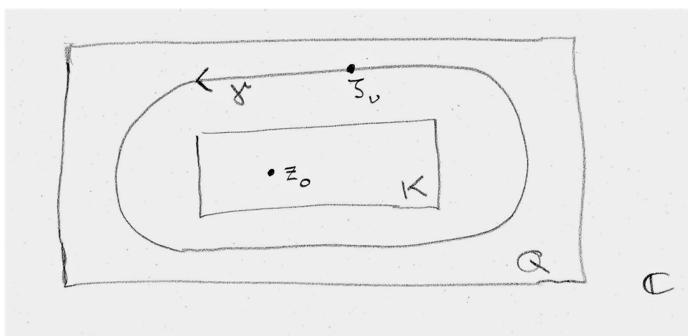


Fig. 5.1 Uniform approximation over K by global functions

We choose a closed path

$$\gamma \subset Q \setminus K$$

around K with positive orientation. For each point $z_0 \in K$ the Cauchy integral formula from complex analysis in the plane states

$$f(z_0) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

Approximating the integral by its Riemann sums provides a finite set of intermediate points $\eta_\nu \in \gamma,\ \nu = 1, \ldots, m,$ such that

$$\left| f(z_0) - \frac{1}{2\pi i} \cdot \sum_{\nu=1}^{m-1} \frac{f(\eta_\nu)}{\eta_\nu - z_0} \cdot (\eta_{\nu+1} - \eta_\nu) \right| < \varepsilon$$

The estimate holds for each refinement of intermediate points, and generalizes also to each point $z \in K$ varying in a suitable neighbourhood of z_0: Due to the compactness of K there exist finitely many intermediate points

$$\zeta_\nu \in \gamma, \ \nu = 1, ..., N$$

and a family

$$(c_1, ..., c_N) \in \mathbb{C}^N$$

such that for all $z \in K$

$$\sup_{z \in K} \left| f(z) - \sum_{\nu=1}^{N} \frac{c_\nu}{\zeta_\nu - z} \right| < \varepsilon/2$$

ii) *Approximation by Taylor polynomials*: For each index $\nu = 1, ..., N$ the point

$$\zeta_\nu \in \gamma \subset Q \setminus K$$

and K can be separated by a circle, i.e. there exists a center $m_\nu \in \mathbb{C}$ and a radius $r_\nu > 0$ such that

$$K \subset \Delta_{r_\nu}(m_\nu) \text{ but } \zeta_\nu \notin \overline{\Delta}_{r_\nu}(m_\nu),$$

see Figure 5.1. Here we use that K is a compact product domain.

For each $\nu = 1, ..., N$ the summand

$$\frac{c_\nu}{\zeta_\nu - z}$$

from the estimate of part i) is holomorphic in the disc $\Delta_{R_\nu}(m_\nu)$ when considered as a function of z. Therefore the summand can be approximated by one of its Taylor polynomials $g_\nu \in \mathscr{O}(\mathbb{C})$ with precision

$$\sup_{z \in K} \left| \frac{c_\nu}{\zeta_\nu - z} - g_\nu(z) \right| < \frac{\varepsilon}{2 \cdot N}$$

As a consequence

$$\sup_{z \in K} \left| f(z) - \sum_{\nu=1}^{N} g_\nu(z) \right| \le \sup_{z \in K} \left| f(z) - \sum_{\nu=1}^{N} \frac{c_\nu}{\zeta_\nu - z} \right| + \sum_{\nu=1}^{N} \sup_{z \in K} \left| \frac{c_\nu}{\zeta_\nu - z} - g_\nu(z) \right| \le$$

$$\le (\varepsilon/2) + N \cdot \varepsilon/(2N) = \varepsilon$$

Hence the global holomorphic function

$$g := \sum_{\nu=1}^{N} g_\nu \in \mathscr{O}(\mathbb{C})$$

approximates f with the given precision $\varepsilon > 0$ on K. $\square$

Lemma 5.6 (Path connectedness of $GL(k,\mathbb{C})$). *The topological space*

$$GL(k,\mathbb{C}) \subset \mathbb{C}^{k^2}$$

is path connected.

Proof. For each non-zero polynomial $P(t) \in \mathbb{C}[t]$ its zero set

$$\{t \in \mathbb{C} : P(t) = 0\}$$

is finite, hence the complement in $\mathbb{C}$

$$\{t \in \mathbb{C} : P(t) \neq 0\}$$

is path connected. For a given matrix $A \in GL(k,\mathbb{C})$ the specific polynomial

$$P(t) := det(A + t \cdot (\mathbb{1} - A))$$

satisfies

$$P(0) = det\ A \neq 0 \text{ and } P(1) = det\ \mathbb{1} = 1 \neq 0,$$

and in particular $P \neq 0$. Hence there exists a path in $\mathbb{C}$ from 0 to 1

$$\gamma : [0,1] \to \mathbb{C}$$

which avoids the zero set of $P(t)$, i.e. for all $t \in [0,1]$

$$P(\gamma(t)) \neq 0.$$

Therefore the path

$$[0,1] \to GL(k,\mathbb{C}),\ t \mapsto A + \gamma(t) \cdot (\mathbb{1} - A),$$

is well-defined and joins A and $\mathbb{1}$. As a consequence $GL(k,\mathbb{C})$ is path connected. $\square$

Corollary 5.7 (Holomorphic approximation: multiplicative case). *For an open product domain*

$$Q := \prod_{j=1}^{n} Q_j \subset \mathbb{C}^n$$

with product domains $Q_j \subset\subset \mathbb{C},\ j = 1,...,n$, *the restriction map*

$$GL(k, \mathscr{O}(\mathbb{C}^n)) \to GL(k, \mathscr{O}(Q)),\ F \mapsto F|Q,$$

has dense image with respect to the canonical Fréchet topologies.

The proof derives the claim as a corollary of Proposition 5.5. The reduction relies on the exponential series of matrices and its logarithm

$$log(1+X) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{X^n}{n},\ X \in M(k \times k, \mathbb{C}) \text{ with } \|X\| < 1$$

which mediates between the additive and the multiplicative context.

Proof. i) *Approximation in the additive case*: By Riemann's mapping theorem each product domain $Q_j,\ j = 1,...,n$, is biholomorphic equivalent to the unit disc $\Delta \subset \mathbb{C}$, i.e. there exists a biholomorphic map

$$\phi_j : Q_j \xrightarrow{\simeq} \Delta.$$

Define the biholomorphic map

$$\phi := \prod_{j=1}^{n} \phi_j : Q \xrightarrow{\simeq} \Delta^n \subset \mathbb{C}^n.$$

For each holomorphic function $f \in \mathscr{O}(Q)$ and compact set $K \subset Q$ the holomorphic function

$$f \circ \phi^{-1} \in \mathscr{O}(\Delta^n)$$

expands into a convergent Taylor series. Hence it can be approximated on the compact set $\phi(K) \subset \Delta^n$ with arbitrary precision by a suitable Taylor polynomial

$$(f \circ \phi^{-1})(z_1,...,z_n) = \sum_{k_1,...,k_n=0}^{N} c_{k_1,...,k_n} \cdot z^{k_1} \cdot ... \cdot z^{k_n}$$

Proposition 5.5 - applied to the holomorphic functions $\phi_j,\ j = 1,...,n$, and their powers - implies: The finite sum

$$\sum_{k_1,...,k_n=0}^{N} c_{k_1,...,k_n} \cdot \phi^{k_1} \cdot ... \cdot \phi^{k_n} \in \mathscr{O}(Q)$$

can be approximated on $K \subset Q$ with arbitrary precision by a global holomorphic function from $\mathscr{O}(\mathbb{C}^n)$, which provides an approximation of f over K.

ii) *Approximation in the multiplicative case near the unit matrix*: Consider a given compact set $K \subset Q$, w.l.o.g. K a compact product domain, and $\varepsilon > 0$. We choose an open product domain $K' \subset \mathbb{C}^n$ with

$$K \subset K' \subset\subset Q$$

For each matrix function $A \in GL(k, \mathscr{O}(Q))$ with

$$\|\mathbb{1} - A\|_{K'} < 1$$

the logarithm

$$G := log(A|K') := -\sum_{n=1}^{\infty} \frac{(\mathbb{1} - A|K')^n}{n} \in M(k \times k, \mathscr{O}(K'))$$

is well-defined, and satisfies on K'

$$e^G = (A|K'),$$

see [31, Chap. 1.3]. Part i) applied to the matrix function $G \in M(k \times k, \mathscr{O}(K'))$ provides a global matrix function

$$\tilde{G} \in M(k \times k, \mathscr{O}(\mathbb{C}^n))$$

which approximates G on K by such precision, that

$$e^{\tilde{G}} \text{ approximates } A = e^G \text{ on } K$$

with precision $\varepsilon > 0$. Here we use the continuity of the exponential series of matrices.

iii) *Starlikeness of the product domain*: The product domain $Q \subset \mathbb{C}^n$ is starlike, w.l.o.g. $0 \in Q$. For each $F \in GL(k, \mathscr{O}(Q))$ we construct a continuous map

$$\alpha : [0,1] \to GL(k, \mathscr{O}(Q))$$

with

$$\alpha(0) = F \text{ and } \alpha(1) = \mathbb{1} :$$

First, we define the continuous map

$$[0,1/2] \to GL(k, \mathscr{O}(Q)),\ t \mapsto F_t,$$

with

$$F_t(z) := F((1-2t) \cdot z),\ z \in Q.$$

Secondly, Lemma 5.6 provides a path

$$\beta : [1/2, 1] \to GL(k, \mathbb{C})$$

satisfying

$$\beta(1/2) = F(0) \text{ and } \beta(1) = \mathbb{1}.$$

Combining both maps provides the continuous map

$$\alpha : [0,1] \to GL(k, \mathscr{O}(Q)),\ \alpha(t) := \begin{cases} F_t & t \in [0,1/2] \\ \beta(t) & t \in [1/2,1] \end{cases}$$

which in $GL(k,\mathscr{O}(Q))$ joins F and the unit matrix $\mathbb{1}$.

iv) *Approximation in the multiplicative case: general case*. In order to approximate a general matrix function $F \in GL(k,\mathscr{O}(Q))$ consider the neighbourhood

$$U := \{A \in GL(k,\mathscr{O}(Q)) :\ \|\mathbb{1} - A\|_{K'} < 1\}$$

of $\mathbb{1}$ in $GL(k,\mathscr{O}(Q))$ where the logarithm series from part ii) is defined. Note that F does not necessarily belong to U. Part iii) provides a path

$$\alpha : [0,1] \to GL(k,\mathscr{O}(Q))$$

from F to $\mathbb{1}$, see Figure 5.2. The family $(G \cdot U)_{G \in \alpha([0,1])}$ is an open covering of the compact set $\alpha([0,1])$. Hence there exists a finite set of matrices

$$G_j \in \alpha([0,1]),\ j = 0,...,s,\ \text{ with } G_0 = F \text{ and } G_s = \mathbb{1}$$

with products

$$G_j \cdot G_{j+1}^{-1} \in U \text{ for } j = 0,...,s-1.$$

By construction

$$F = \prod_{j=0}^{s-1} G_j \cdot G_{j+1}^{-1}$$

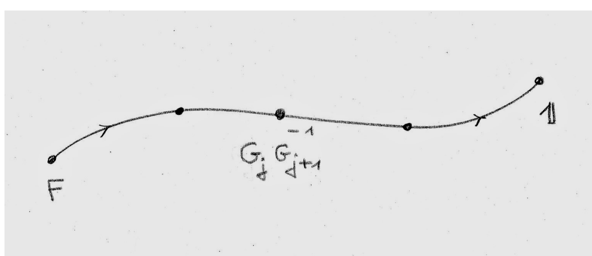


Fig. 5.2 Product splitting of F

Part ii) provides for each $j = 0,...,s-1$ a global approximation of the factor

$$G_j \cdot G_{j+1}^{-1} \in U$$

with the necessary precision on K such that their product provides a global approximation of F with the given precision $\varepsilon > 0$ on K. $\square$

Theorem 5.8 (Cartan's lemma for holomorphic matrices). *Consider two adjacent product domains*

$$Q_1, Q_2 \subset \mathscr{Q}(\mathbb{C}^n),$$

an open set $U \subset \mathbb{C}^n$ with

$$(Q_1 \cap Q_2) \subset U$$

and a holomorphic matrix

$$A \in GL(k, \mathscr{O}(U)),$$

see Figure 5.3. Then for $j = 1, 2$ exist open neighbourhoods U_j of Q_j with

$$(U_1 \cap U_2) \subset\subset U$$

and two matrix functions

$$A_j \in GL(k, \mathscr{O}(U_j)),\ j = 1, 2,$$

satisfying on $U_1 \cap U_2$

$$A = A_1 \cdot A_2^{-1}.$$

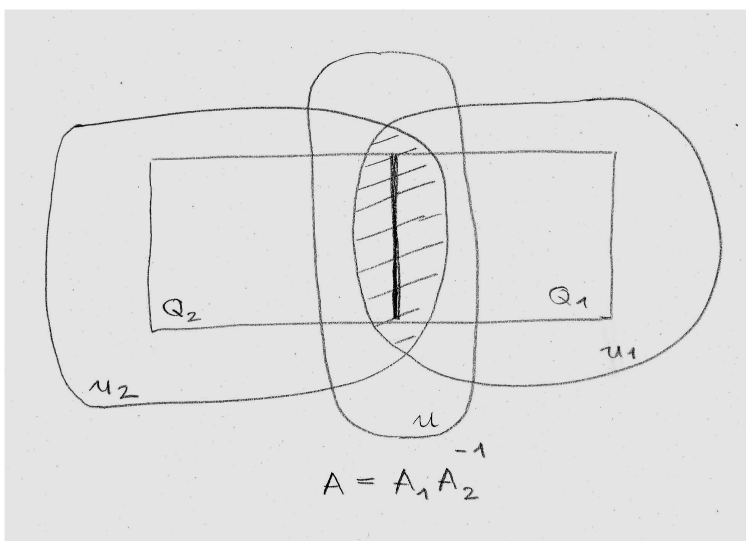


Fig. 5.3 Cartan's lemma for invertible matrices

Proof. We choose $U_j, j = 1,2$, as open product domains with

$$U_0 := (U_1 \cap U_2) \subset\subset U.$$

Due to the relative compactness the matrix function

$$A \in GL(k, \mathscr{O}(U))$$

restricts to a bounded matrix function

$$A|U_0 \in GL(k, \mathscr{B}(U_0))$$

For $j = 0,1,2$ consider the Banach spaces

$$E_j := M(k \times k, \mathscr{B}(U_j)),\ j = 0,1,2$$

and the open subsets

$$E_j^* = GL(k, \mathscr{B}(U_j)) \subset E_j.$$

The idea of the proof is to reformulate the lemma as a claim about the surjectivity of the map

$$f : E_1^* \times E_2^* \to E_0^*,\ (A,B) \mapsto A|U_0 \cdot B^{-1}|U_0.$$

i) *Strict differentiability*: The map f is strictly differentiable at the point

$$a := (\mathbb{1}, \mathbb{1}) \in E_1^* \times E_2^*.$$

For the proof consider the linear map

$$\phi : E_1 \times E_2 \to E_0, (A,B) \mapsto (A-B)|U_0$$

For two arbitrary pairs $(A,B),\ (C,D) \in E_1^* \times E_2^*$ factoring out implies

$$f(A,B) - f(C,D) - \phi((A,B)-(C,D)) = f(A,B) - f(C,D) - ((A-B)-(C-D)) =$$

$$= A \cdot B^{-1} - C \cdot D^{-1} - ((A-C)-(B-D)) =$$

$$= (C-B) \cdot D^{-1} \cdot (D-B) \cdot B^{-1} + (A-C) \cdot (B^{-1} - \mathbb{1}) - (B-D)(D^{-1} - \mathbb{1})$$

The norm definition

$$\|(A,B)-(C,D)\| = \|(A-C, B-D)\| = \sup\{\|A-C\|,\ \|B-D\|\}$$

provides the estimate

$$\lim_{(A,B),(C,D)\to a} \frac{\|f(A,B) - f(C,D) - \phi((A,B)-(C,D))\|}{\|(A,B)-(C,D)\|} \leq$$

$$\leq \lim_{(A,B),(C,D)\to a} \left(\frac{\|A-C\| \cdot \|B^{-1}-\mathbb{1}\|}{\|A-C\|} + \frac{\|B-D\| \cdot \|D^{-1}-\mathbb{1}\|}{\|B-D\|} + \frac{\|C-B\| \cdot \|D^{-1}\| \cdot \|B-D\| \cdot \|B^{-1}\|}{\|B-D\|} \right) =$$

$$= \lim_{(A,B),(C,D)\to a} \left(\|B^{-1}-\mathbb{1}\| + \|D^{-1}-\mathbb{1}\| + \|C-B\| \cdot \|D^{-1}\| \cdot \|B^{-1}\| \right) = 0$$

The result also holds for $A = C$ or $B = D$ as far as $(A,B) \neq (C,D)$. As a consequence f is strictly differentiable at a with tangent map

$$f'(a) = \phi.$$

ii) *Local openness of* f: Due to Proposition 5.2 the map ϕ is surjective. Hence Lemma 5.4 implies that f is open at a: There exists a neighbourhood $V \subset E_0^*$ of $f(a) = \mathbb{1} \in E_0^*$ such that for each $G \in V$ exists a pair

$$(G_1, G_2) \in E_1^* \times E_2^*$$

with

$$G = G_1|U_0 \cdot G_2^{-1}|U_0$$

iii) *Holomorphic approximation of matrix functions*: Due to Corollary 5.7 the matrix function

$$A^{-1} \in GL(k, \mathscr{O}(U))$$

can be approximated on $\overline{U}_0$ with arbitrary precision by global elements from $GL(k, \mathscr{O}(\mathbb{C}^n))$: For the distinguished neighbourhood V of $\mathbb{1} \in GL(k, \mathscr{O}(U_0))$ from part ii) exists

$$B \in GL(k, \mathscr{O}(\mathbb{C}^n))$$

with

$$(A \cdot B)|U_0 \in V$$

The product splits due to part ii): There exist matrix functions

$$A_1 \in GL(k, \mathscr{O}(U_1)) \text{ and } \tilde{A}_2 \in GL(k, \mathscr{O}(U_2))$$

which satisfy on U_0

$$A \cdot B = A_1 \cdot \tilde{A}_2^{-1}$$

Set

$$A_2 := B \cdot \tilde{A}_2 \in GL(k, \mathscr{O}(U_2)).$$

Then on U_0

$$A = A_1 \cdot (\tilde{A}_2^{-1} \cdot B^{-1}) = A_1 \cdot A_2^{-1}$$

which finishes the proof. □

5.2 Hilbert's syzygy theorem for coherent $\mathscr{O}$-modules

The present section proves for coherent sheaves the global form of Hilbert's syzygy Theorem 4.24 after shrinking to a relatively-compact, open polydisc Δ. As a consequence coherent sheaves are acyclic on Δ.

Proposition 5.9 (Acyclicity of $\mathscr{O}$-modules with a finite free resolution on a polydisc). *If an $\mathscr{O}$-module $\mathscr{F}$ on a polydisc $\Delta \subset \mathbb{C}^n$ has a finite resolution by free sheaves of finite rank*

$$0 \to \mathscr{O}^{p_d} \to \mathscr{O}^{p_{d-1}} \to ... \to \mathscr{O}^{p_0} \to \mathscr{F} \to 0,$$

then $\mathscr{F}$ is acyclic.

Proof. The proof is by induction on the length $d \in \mathbb{N}$ of the resolution.

i) *Induction start $d = 0$*: Then $\mathscr{F} \simeq \mathscr{O}^p$ and

$$H^j(\Delta, \mathscr{F}) = H^j(\Delta, \mathscr{O})^p$$

The claim follows because the theorem holds for the structure sheaf $\mathscr{O}$ due to Corollary 3.23.

ii) *Induction step $d-1 \mapsto d$*: We set

$$\mathscr{K} := ker\ [\mathscr{O}^{p_0} \to \mathscr{F}]$$

and obtain the free resolution of the coherent sheaf $\mathscr{K}$ of length $d-1$

$$0 \to \mathscr{O}^{p_d} \to ... \to \mathscr{O}^{p_1} \to \mathscr{K} \to 0$$

By induction assumption for all $j \geq 1$

$$H^j(\Delta, \mathscr{K}) = 0$$

The exact sequence

$$0 \to \mathscr{K} \to \mathscr{O}^{p_0} \to \mathscr{F} \to 0$$

provides for each $j \geq 1$ the section of the long exact cohomology sequence

$$0 = H^j(\Delta, \mathscr{O}^{p_0}) \to H^j(\Delta, \mathscr{F}) \to H^{j+1}(\Delta, \mathscr{K}) = 0,$$

which proves the induction step and finishes the proof of the proposition. □

Definition 5.10 (Locally free sheaf of finite rank). On a complex manifold X an $\mathscr{O}$-module $\mathscr{L}$ is *locally-free* of finite rank, if for each point $x \in X$ exist an open neighbourhood $U \subset X$ and an isomorphism of $\mathscr{O}_U$-modules onto a free $\mathscr{O}$-module of finite rank

$$\mathscr{L}|U \xrightarrow{\simeq} (\mathscr{O}|U)^k, \ k \in \mathbb{N}.$$

If $k = 1$ then $\mathscr{L}$ is named an *invertible* sheaf.

Remark 5.11 (Invertible sheaves and line bundles). Consider a complex manifold X, an $\mathscr{O}$-module $\mathscr{L}$ and an open covering $\mathscr{U} := (U_j)_{j \in I}$ of X together with a family of isomorphisms

$$\phi_j : \mathscr{L}|U_j \xrightarrow{\simeq} (\mathscr{O}|U_j), \ j \in I.$$

Then the family

$$\left(\phi_j|U_{ij} \circ \phi_i^{-1}|U_{ij}\right)_{i,j} \in Z^1(\mathscr{U}, \mathscr{O}^*), \ U_{ij} := U_i \cap U_j,$$

defines a holomorphic *line bundle* ξ on X with $\mathscr{L}$ the sheaf of holomorphic sections in ξ. In the reverse direction, each holomoprhic line bundle ξ on X defines a 1-cocycle with values in $\mathscr{O}^*$ over a suitable open covering of X. Two holomorphic line bundles on X are isomorphic iff their 1-cocylces are cohomologous: The isomorphy classes of holomorphic line bundles on X correspond bijectively to the elements of the cohomology group $H^1(X, \mathscr{O}^*)$.

Theorem 5.12 (Locally free sheaves after shrinking to a relatively compact polydisc). *Consider an open set $U \subset \mathbb{C}^n$ and a relatively compact polydisc*

$$\Delta \subset\subset U.$$

Then each locally free $\mathscr{O}_U$-module $\mathscr{L}$ of finite rank is free on Δ.

Proof. We may assume U connected. Denote by $k \in \mathbb{N}$ the rank of $\mathscr{L}$.

i) *Transformation to product domains*: We choose an open polydisc $\Delta' \subset U$ with

$$\Delta \subset\subset \Delta' \subset U$$

Riemann's mapping theorem provides an open product domain

$$Q' \subset \mathbb{C}^n$$

and a biholomorphic map

$$f : \Delta' \xrightarrow{\simeq} Q'.$$

Consider on Q' the locally free sheaf

$$\mathscr{F} := f_*(\mathscr{L}|\Delta')$$

There exists a compact product set $\hat{Q} \subset \mathbb{C}^n$ satisfying

$$f(\Delta) \subset \hat{Q} \subset\subset Q',$$

see the following figure.

$$\begin{array}{ccccc} \Delta & \subset\subset & \Delta' & \subset & U \\ \Big\downarrow f|\Delta & & \Big\downarrow f & & \\ f(\Delta) & \subset \hat{Q} \subset\subset & Q' & & \end{array}$$

We show that $\mathscr{F}$ is free in an open neighbourhood of $\hat{Q}$.

ii) *Cousin's principle of induction*: To apply Proposition 2.33 we consider the following function

$$A : \mathscr{Q}(Q') \to \{TRUE, FALSE\},$$

which is defined for each product domain $Q \in \mathscr{Q}(Q')$ as

$$A(Q) := \begin{cases} TRUE & \text{if } \mathscr{F} \simeq \mathscr{O}^k \text{ in an open neighbourhood of } Q \\ FALSE & \text{otherwise} \end{cases}$$

The function A satisfies the assumptions of Proposition 2.33:

- Because $\mathscr{F}$ is locally free, each point $x \in Q'$ has an open neighbourhood V such that $A(Q) = TRUE$ for all product domains $Q \in \mathscr{Q}(V)$.

- Consider two adjacent product domains

  $$Q_1, Q_2 \in \mathscr{Q}(Q')$$

  satisfying

  $$A(Q_1) = A(Q_2) = TRUE.$$

  For $j = 1, 2$ exist by assumption open neighbourhoods U'_j of Q_j such that $\mathscr{F}$ is free on U'_j: There exist isomorphisms over U'_j

  $$\phi_j : \mathscr{O}^k \xrightarrow{\simeq} \mathscr{F}$$

  Over $U'_1 \cap U'_2$ the transition function

  $$\phi := \phi_1^{-1} \circ \phi_2 : \mathscr{O}^k \xrightarrow{\simeq} \mathscr{O}^k$$

  is represented by a holomorphic matrix function

$$\tilde{\phi} \in GL(k, \mathscr{O}(U_1' \cap U_2'))$$

Cartan's lemma for holomorphic matrix functions, Theorem 5.8, provides for $j = 1, 2$ open neighbourhoods U_j of Q_j with

$$Q_j \subset U_j \subset U_j'$$

and matrix functions $\tilde{\Phi}_j \in GL(k, \mathscr{O})(U_j)$ which split $\tilde{\phi}$ on $U_1 \cap U_2$ as

$$\tilde{\phi} = \tilde{\Phi}_1 \circ \tilde{\Phi}_2^{-1}.$$

The corresponding isomorphisms on U_j

$$\phi_j \circ \tilde{\Phi}_j : \mathscr{O}^k \to \mathscr{F}$$

satisfy on $U_1 \cap U_2$

$$\phi_1 \circ \tilde{\Phi}_1 = \phi_2 \circ \tilde{\Phi}_2$$

because

$$\phi_1^{-1} \circ \phi_2 = \tilde{\Phi}_1 \circ \tilde{\Phi}_2^{-1}.$$

Hence the family

$$(\phi_j \circ \tilde{\Phi}_j)_{j=1,2}$$

defines on the open neighbourhood $U_1 \cup U_2$ of $Q_1 \cup Q_2$ an isomorphism

$$\mathscr{O}^k \xrightarrow{\simeq} \mathscr{F},$$

i.e. $A(Q_1 \cup Q_2) = TRUE$.

Proposition 2.33 implies

$$A(Q) = TRUE$$

for each $Q \in \mathscr{Q}(Q')$. In particular

$$A(\hat{Q}) = TRUE.$$

Hence the restriction $\mathscr{F}|f(\Delta)$ is free, which implies that $\mathscr{L}$ is free because

$$f|\Delta : \Delta \xrightarrow{\simeq} f(\Delta)$$

is a holomorphic isomorphism. □

Definition 5.13 (Homological dimension of coherent $\mathscr{O}$-modules). Consider a complex manifold X and a coherent $\mathscr{O}$-module $\mathscr{F}$. The *homological dimension* of $\mathscr{F}$ is defined as the supremum of the homological dimension of its stalks in the sense of Definition 4.22

$$hd_{\mathscr{O}}\mathscr{F} := sup\{hd_{\mathscr{O}_x}\, \mathscr{F}_x : \ x \in X\}.$$

Theorem 4.24 implies that on an n-dimensional complex manifold each coherent $\mathcal{O}$-module $\mathcal{F}$ has finite homological dimension

$$hd_{\mathcal{O}}\mathcal{F} \le n.$$

Theorem 5.14 (Hilbert's syzygy theorem for a coherent $\mathcal{O}$-module after shrinking to a polydisc). *Consider an open set $U \subset \mathbb{C}^n$ and a relatively compact, open polydisc*

$$\Delta \subset\subset U.$$

Each coherent $\mathcal{O}$-module $\mathcal{F}$ on U with homological dimension

$$d := hd_{\mathcal{O}}\mathcal{F}$$

has over Δ a resolution of length d by free $\mathcal{O}$-modules of finite rank

$$0 \to \mathcal{O}^{k_d} \to \mathcal{O}^{k_{d-1}} \to ... \to \mathcal{O}^{k_0} \to \mathcal{F} \to 0.$$

Proof. The claim is proved by induction on d.

i) *Induction start for $d = 0$*: Due to Theorem 5.12 it suffices to prove that $\mathcal{F}$ is locally free on Δ. Consider an arbitrary point $x \in U$. Because

$$hd_{\mathcal{O}_x} \mathcal{F}_x = 0$$

there exists a $k \in \mathbb{N}$ and an isomorphism of stalks

$$\alpha_x : \mathcal{O}_x^k \xrightarrow{\simeq} \mathcal{F}_x$$

We denote by

$$(e_j \in \mathcal{O}^k(U))_{j=1,...,k}$$

the canonical $\mathcal{O}(U)$-basis of $\mathcal{O}^k(U)$ and consider the germs

$$f_{j,x} := \alpha_x(e_{j,x}) \in \mathcal{F}_x$$

There exists a neighbourhood $V \subset U$ of x and sections

$$f_j \in \mathcal{F}(V), j = 1, ..., k,$$

representing $f_{j,x}$. Those sections define an $\mathcal{O}$-module morphisms over V

$$\alpha : \mathcal{O}^k|V \to \mathcal{F}|V,\ \alpha(e_j) := f_j,\ j = 1, ..., k, \text{ on sections,}$$

which extends the primary isomorphism on the stalks at x

$$\alpha_x : \mathcal{O}_x^k \xrightarrow{\simeq} \mathcal{F}_x$$

The coherence of $\mathscr{F}$ implies due to Corollary 4.32 the existence of a neighbourhood $W \subset V$ of x such that the restriction

$$\alpha|W : \mathscr{O}^k|W \to \mathscr{F}|W$$

is an isomorphism of sheaves, i.e. it induces an isomorphism on the stalks for all points $y \in W$.

ii) *Preparing the induction step*: For the induction step in part iii) we need an argument to carry over the surjectivity of a sheaf morphism over a polydisc Y

$$\alpha : \mathscr{O}^k \to \mathscr{F},$$

i.e. the surjectivity on the level of stalks, to the surjectivity of the induced morphism of sections over Y

$$\alpha_Y : \mathscr{O}(Y)^k \to \mathscr{F}(Y).$$

The argument will be the vanishing of a first cohomology group:

Assume: The claim of the theorem holds for $d-1$. Consider a relatively compact, open polydisc $Y \subset\subset U$ and over Y an epimorphism of sheaves

$$\alpha : \mathscr{O}^k \to \mathscr{F}.$$

If $hd_{\mathscr{O}_U} \mathscr{F} = d$ then the $\mathscr{O}_U$-module

$$\mathscr{K} := ker\,[\mathscr{O}^k \xrightarrow{\alpha} \mathscr{F}]$$

fits into the short exact sequence

$$0 \to \mathscr{K} \to \mathscr{O}^k \xrightarrow{\alpha} \mathscr{F} \to 0$$

and has homomological dimension

$$hd_{\mathscr{O}_U} \mathscr{K} < d$$

as a consequence of Proposition 4.23. Hence by induction assumption $\mathscr{K}$ has a free resolution on Y. Then Proposition 5.9 implies the vanishing

$$H^1(Y, \mathscr{K}) = 0$$

The segment from the long exact cohomology sequence

$$\mathscr{O}^k(Y) \xrightarrow{\alpha_Y} \mathscr{F}(Y) \to H^1(Y, \mathscr{K}) = 0$$

implies the surjectivity of the morphism of sections

$$\alpha_Y : \mathscr{O}^k(Y) \to \mathscr{F}(Y).$$

iii) *Induction step* $d-1 \mapsto d$: Consider a coherent $\mathscr{O}_U$-module $\mathscr{F}$ with

$$hd_{\mathscr{O}_U}\mathscr{F} = d \geq 1$$

We choose a polydisc Δ' with

$$\Delta \subset\subset \Delta' \subset\subset U$$

and construct an $\mathscr{O}$-module epimorphism over Δ'

$$\mathscr{O}^k|\Delta' \to \mathscr{F}|\Delta' \to 0:$$

The construction uses Cousin's principle of induction. We consider the function

$$A : \mathscr{Q}(U) \to \{TRUE, FALSE\}$$

which is defined for $Q \in \mathscr{Q}(U)$ as

$$A(Q) := \begin{cases} TRUE & \text{There exists an epimorphism } \mathscr{O}^k \to \mathscr{F} \text{ in an open neighbourhood of } Q \\ FALSE & \text{otherwise} \end{cases}$$

The function A satisfies the assumptions of Proposition 2.33:

- The coherence of $\mathscr{F}$ implies that each point $x \in U$ has an open neighbourhood V and for each $Q \in \mathscr{Q}(V)$ an epimorphism over V

$$\mathscr{O}^k|V \to \mathscr{F}|V.$$

- For $j = 1, 2$ consider two compact adjacent product domains $Q_j \in \mathscr{Q}(U)$ with

$$A(Q_j) = TRUE.$$

  Hence for $j = 1, 2$ exist open neighbourhoods $U_j \supset Q_j$ with epimorphisms of sheaves over U_j

$$\mathscr{O}^{k_j} \xrightarrow{\alpha_j} \mathscr{F}$$

On the level of sections we denote for $j = 1$ by

$$f_1 := (f_{1,1}, ..., f_{1,k_1}) \in \mathscr{F}^{k_1}(U_1)$$

the family of images under α_1 of the elements of the canonical base of $\mathscr{O}^{k_1}(U_1)$, and analogously for $j = 2$ the family

$$f_2 := (f_{2,1}, ..., f_{2,k_2}) \in \mathscr{F}^{k_2}(U_2)$$

For each

$$x \in U_1 \cap U_2$$

the germs of the components of f_1 as well as the germs of the components of f_2 generate the stalk $\mathscr{F}_x$ over $\mathscr{O}_x$. Part ii) implies the existence of an open polydisc

$$V \subset\subset (U_1 \cap U_2)$$

such that for $j = 1,2$ each sheaf morphism α_j induces an epimorphism of sections

$$\alpha_{j,V} : \mathscr{O}^{k_j}(V) \to \mathscr{F}(V)$$

We obtain two holomorphic matrix functions

$$A_1 \in M(k_2 \times k_1, \mathscr{O}(V)) \text{ and } A_2 \in M(k_1 \times k_2, \mathscr{O}(V))$$

satisfying

$$f_2 \cdot A_1 = f_1 \text{ and } f_1 \cdot A_2 = f_2.$$

The two matrix equations combine to one single equation of $(k_1 + k_2)$-tuples

$$(0, f_2) = (f_1, 0) \cdot C$$

with the matrix function

$$C := \begin{pmatrix} \mathbb{1} - A_2 \cdot A_1 & A_2 \\ & \\ -A_1 & \mathbb{1} \end{pmatrix} \in M((k_1 + k_2) \times (k_1 + k_2), \mathscr{O}(V)).$$

The matrix is invertible

$$C \in GL(k_1 + k_2, \mathscr{O}(V))$$

with inverse

$$C^{-1} = \begin{pmatrix} \mathbb{1} & -A_2 \\ & \\ A_1 & \mathbb{1} - A_1 \cdot A_2 \end{pmatrix}$$

Cartan's lemma, Theorem 5.8, applies to the matrix function C and provides for $j = 1,2$ two open neighbourhoods

$$V_j \subset U_j \text{ of } Q_j \text{ with } V_1 \cap V_2 \subset V$$

and corresponding matrix functions

$$C_j \in GL(k_1 + k_2, \mathscr{O}(V_j)),\ j = 1,2,$$

which satisfy on $V_1 \cap V_2$

$$C = \begin{pmatrix} \mathbb{1} - A_2 \cdot A_1 & A_2 \\ & \\ -A_1 & \mathbb{1} \end{pmatrix} = C_1 \cdot C_2^{-1}$$

Hence over $V_1 \cap V_2$

$$(0, f_2) = (f_1, 0) \cdot C = (f_1, 0) \cdot C_1 \cdot C_2^{-1} \implies (0, f_2) \cdot C_2 = (f_1, 0) \cdot C_1$$

or

$$f_2 \cdot C_2' = f_1 \cdot C_1'$$

with each matrix $C_j',\ j = 1,2$, derived from two blocks of the corresponding matrix C_j. On the union $V_1 \cup V_2$ both families

$$f_1 \cdot C_1' \text{ and } f_2 \cdot C_2'$$

combine to a family of sections

$$F := (F_1, ..., F_{k_1+k_2}) \in \mathscr{F}^{k_1+k_2}(V_1 \cup V_2)$$

satisfying

$$(0, f_2) \cdot C_2 = F \text{ over } V_2 \text{ and } (f_1, 0) \cdot C_1 = F \text{ over } V_1$$

Each matrix function $C_j,\ j = 1,2$ is invertible. Hence the germs of F generate the stalk $\mathscr{F}_x$ for each point $x \in V_1 \cup V_2$: The family provides over $V_1 \cup V_2$ the epimorphism of sheaves

$$\mathscr{O}^{k_1+k_2} \xrightarrow{F} \mathscr{F}.$$

Hence

$$A(Q_1 \cup Q_2) = TRUE.$$

Cousin's principle of induction ensures that each product domain $Q \in \mathscr{Q}(U)$ has an epimorphism of $\mathscr{O}$-modules

$$\mathscr{O}^k \to \mathscr{F}$$

over a suitable open neighbourhood of Q. The $\mathscr{O}$-module

$$\mathscr{K} := ker\,[\mathscr{O}^k \to \mathscr{F}]$$

satisfies

$$hd\ \mathscr{K} = (hd\ \mathscr{F}) - 1$$

due to Proposition 4.23. Hence the induction assumption applied to $\mathscr{K}$ finishes the induction step and terminates the proof of the theorem. □

Corollary 5.15 shows the consequences of Hilbert's syzygy theorem for the cohomology of coherent sheaves on polydiscs. The corollary will play an important role in Chapter 6 for the proof of Theorem B.

Corollary 5.15 (Acyclicity of coherent sheaves after shrinking to a relatively compact, open polydisc). *Consider a relatively compact polydisc*

$$\Delta \subset\subset \mathbb{C}^n$$

and a coherent sheaf $\mathscr{F}$ defined in an open neighbourhood of $\overline{\Delta}$. Then the restriction $\mathscr{F}|\Delta$ is acyclic.

Proof. The corollary follows from Theorem 5.14 and Proposition 5.9. □

5.3 Fréchet topology in the context of cohomology

Definition 5.16 (Fréchet sheaf). A sheaf of complex vector spaces $\mathscr{F}$ on a topological space X is a *Fréchet sheaf* if for each open $U \subset X$ the vector space of sections $\mathscr{F}(U)$ is a Fréchet space and the restrictions

$$\mathscr{F}(U) \to \mathscr{F}(V),\ V \subset U \text{ open},$$

are continuous linear maps.

The structure sheaf $\mathscr{O}$ of a complex manifold X is a Fréchet sheaf when providing the vector spaces $\mathscr{O}(U)$, $U \subset X$ open, with the topology of compact convergence. The result follows from Proposition 1.19.

Proposition 5.17 (Coherent sheaves as Fréchet sheaves). *On a complex manifold X each coherent $\mathscr{O}$-module $\mathscr{F}$ is a Fréchet sheaf.*

Proof. i) *Distinguished basis of the topology*: We choose a basis $\mathscr{B}$ of the topology of X by open sets $U \subset X$ with the following property: For each $U \in \mathscr{B}$ exists an open set

$$U' \subset X \text{ with } U \subset\subset U'$$

and a biholomorphic map

$$\alpha : U' \xrightarrow{\simeq} \Delta'$$

onto a polydisc Δ' such that

$$\Delta := \alpha(U) \subset\subset \Delta'$$

is a relatively-compact, concentric polydisc. The family $\mathscr{B}$ can be obtained by choosing first a locally finite covering $(U_i')_{i \in I}$ by basis elements, and then selecting the family $(U_i)_{i \in I}$ as a suitable shrinking, see Remark 2.3.

Theorem 5.14 provides over U an epimorphism

$$\mathscr{O}^k \xrightarrow{\phi} \mathscr{F} \to 0.$$

The kernel
$$\mathscr{K} := ker\ \phi$$
is coherent according to Proposition 4.31. Corollary 5.15 implies
$$H^1(U,\mathscr{F}) = 0.$$
The exact sequence of vector spaces
$$0 \to \mathscr{K}(U) \to \mathscr{O}^k(U) \to \mathscr{F}(U) \to 0$$
shows the algebraic isomorphy of vector spaces
$$\mathscr{F}(U) \simeq \mathscr{O}^k(U)/\mathscr{K}(U).$$
ii) *Closedness of* $\mathscr{K}(U) \subset \mathscr{O}^k(U)$: One has to show: For a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of k-tuples of sections
$$f_\nu \in \mathscr{K}(U) \subset \mathscr{O}^k(U),\ \nu \in \mathbb{N},$$
with limit
$$f := \lim_{\nu\to\infty} f_\nu \in \mathscr{O}^k(U)$$
holds
$$f \in \mathscr{K}(U).$$
For a section $f \in \mathscr{O}^k(U)$ holds the equivalence
$$f \in \mathscr{K}(U) \iff f_x \in \mathscr{K}_x \text{ for all } x \in U$$
because $\mathscr{K}$ is a sheaf. For given $x \in U$ set
$$R := \mathscr{O}_x,\ \mathfrak{m} \subset R \text{ the maximal ideal},\ M := \mathscr{K}_x$$
Due to the Krull lemma, see Proposition 4.21,
$$M = \bigcap_{s\in\mathbb{N}^*} (M + \mathfrak{m}^s \cdot R^k)$$
Hence it suffices to show: For each $s \in \mathbb{N}^*$
$$f_x \in M + \mathfrak{m}^s \cdot R^k$$
For given $s \in \mathbb{N}^*$ consider the residue map
$$\pi_s : R^k \to R^k/\mathfrak{m}^s \cdot R^k$$
with
$$R^k/\mathfrak{m}^s \cdot R^k \simeq (P_{s-1})^k \text{ with } P_{s-1} \subset \mathbb{C}[z_1,...,z_n]$$

the vector space of all polynomials of degree $\leq s-1$. The vector space is finite-dimensional. Therefore it has a unique structure of a Hausdorff topological vector space, and its subspace

$$\pi_s(M) \subset (P_{s-1})^k$$

is closed. The Cauchy integral formula, Corollary 1.7, implies: For arbitrary but fixed $I \in \mathbb{N}^k$ and $j \in \{1,...,k\}$ the sequence of I-th coefficents of the power series of the j-th components of the sequence $(f_{\nu,x})_\nu$ are convergent towards the I-th coefficent of the j-th component of f_x. Hence

$$\pi_s(f_x) = \lim_{\nu\to\infty} \pi_s(f_{\nu,x}) \in \pi_s(M)$$

which implies

$$f_x \in M + \mathfrak{m}^s \cdot R^k$$

Because $s \in \mathbb{N}^*$ was arbitrary, the result finishes the proof of the claim $f_x \in M$. As a consequence

$$\mathscr{K}(U) \subset \mathscr{O}^k(U)$$

is closed, and the quotient

$$\mathscr{F}(U) = \mathscr{O}^k(U)/\mathscr{K}(U)$$

provided with the quotient topology becomes a Fréchet space.

iii) *Independence of the topology*: We have to show that the topology on $\mathscr{F}(U)$ does not depend on the choice of a resolution of $\mathscr{F}$, i.e. on the choice of generators $f_1,...,f_k \in \mathscr{F}(U)$. Hence we assume a second choice $f_{k+1},...,f_m \in \mathscr{F}(U)$ of generators. We show that the two families

$$(f_1,...,f_k) \text{ and } (f_1,...,f_k,f_{k+1},...,f_m)$$

induce the same topology: The injection

$$\mathscr{O}^k(U) \hookrightarrow \mathscr{O}^m(U),\ e_j \mapsto e_j,\ j = 1,...,k,$$

induces the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{K}(U) & \longrightarrow & \mathscr{O}^k(U) & \longrightarrow & \mathscr{F}(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \mathscr{K}'(U) & \longrightarrow & \mathscr{O}^m(U) & \longrightarrow & \mathscr{F}(U)' & \longrightarrow & 0
\end{array}$$

Here $\mathscr{F}(U)'$ denotes the quotient topology on the vector space $\mathscr{F}(U)$ induced from the lower row. The map

$$id : \mathscr{F}(U) \to \mathscr{F}(U)'$$

is continuous and surjective between Fréchet spaces, hence a homeomorphism due to Remark 1.18.

iv) *Continuity of the restriction maps*: Consider two open sets

$$U,\ V \in \mathscr{B} \text{ with } V \subset U$$

and let

$$\phi : \mathscr{F}(U) \to \mathscr{F}(V)$$

be the restriction. We obtain the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K}(U) & \longrightarrow & \mathscr{O}^k(U) & \longrightarrow & \mathscr{F}(U) & \longrightarrow & 0 \\ & & \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \\ 0 & \longrightarrow & \mathscr{K}(V) & \longrightarrow & \mathscr{O}^k(V) & \longrightarrow & \mathscr{F}(V) & \longrightarrow & 0 \end{array}$$

The map ϕ_2 is continuous because $\mathscr{O}$ is a Fréchet sheaf due to Proposition 1.19. Hence also the restriction ϕ_1 is continuous. Diagram chasing and the definition of the quotient topology on $\mathscr{F}(U)$ implies the continuity of ϕ_3.

v) *General open sets*: An arbitrary open set U can be represented as

$$U = \bigcup_{i \in I} U_i$$

with a countable index set I and open sets

$$U_i \in \mathscr{B},\ i \in I.$$

The canonical injection of vector spaces

$$\lambda : \mathscr{F}(U) \hookrightarrow \prod_{i \in I} \mathscr{F}(U_i),\ f \mapsto (f|U_i)_i$$

has a closed image: The idea is to represent the image as a kernel. For each given pair $(i, j) \in I \times I$ the open intersection

$$U_{ij} := U_i \cap U_j$$

can be represented as

$$U_{ij} = \bigcup_{U_{i,j,\nu} \in \mathscr{B}} U_{ij,\nu},\ U_{ij,\nu} := U_{ij} \cap U_\nu$$

The continuous map

$$\kappa : \prod_{i\in I} \mathscr{F}(U_i) \to \prod_{i,j,\nu} \mathscr{F}(U_{ij,\nu}),\ (f_i)_i \mapsto ((f_i - f_j)|U_{ij,\nu})_{i,j,\nu},$$

satisfies

$$ker\ \kappa = im\ \lambda$$

Hence

$$\mathscr{F}(U) = im\ \lambda = ker\ \kappa \subset \prod_{i\in I} \mathscr{F}(U_i)$$

is a closed subspace, and therefore a Fréchet space itself with respect to the induced topology. □

Corollary 5.18 (Fréchet topology on cochains and cocycles). *Consider a complex manifold X and a coherent $\mathscr{O}$-module $\mathscr{F}$. For each countable open covering $\mathscr{U} = (U_i)_{i\in I}$ of X and each $q \in \mathbb{N}$ the group of Čech cochains*

$$C^q(\mathscr{U},\mathscr{F})$$

are Fréchet spaces and the coboundary maps are continuous. The subgroup of cocycles

$$Z^q(\mathscr{U},\mathscr{F}) \subset C^q(\mathscr{U},\mathscr{F})$$

is a closed subspace, hence a Fréchet space too.

Note that the subspace of coboundaries

$$B^q(\mathscr{U},\mathscr{F}) \subset C^q(\mathscr{U},\mathscr{F})$$

is not necessarily closed. As a consequence the quotient $H^q(\mathscr{U},\mathscr{F})$ is not necessarily a Fréchet space; see [14, Kap. V, §6.5 Bem.]

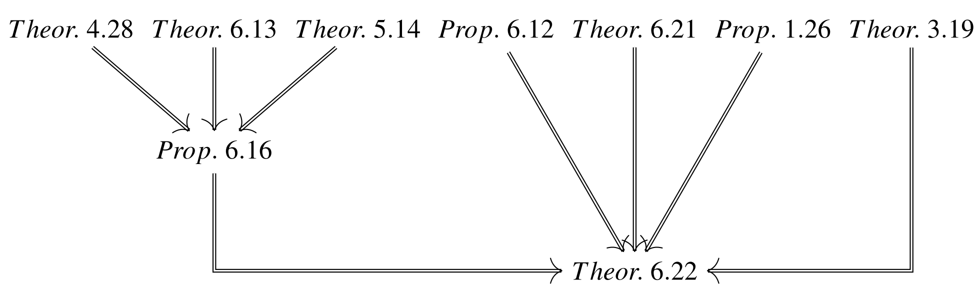
Chapter 6
Theorem *B* and Theorem *A* on Stein manifolds

The present chapter defines the concept of a *Stein manifold*, Definition 6.10 and proves the main result on Stein manifolds, Theorem B, see Theorem 6.22, as well as some of its fundamental consequences. The theorem will be proved in the following steps:

- Theor. 4.28 (Oka's coherence theorem)
- Theor. 6.13 (Embedding analytic polyhedra of a Stein manifold)
- Theor. 5.14 (Hilbert's syzygy theorem)
- Prop. 6.16 (Theorem *B* for analytic polyhedra)

- Prop. 6.12 (Exhaustion by analytic polyhedra)
- Theor. 6.21 (Runge approximation for coherent sheaves)
- Prop. 1.26 (Mittag-Leffler principle for projective limes)
- Theor. 3.19 (Leray's theorem)

The following diagram shows always for a given result the sufficient combination of prerequisites.



6.1 Holomorphic convexity and analytic polyhedra

We start our investigation with two fundamental concepts from complex analysis:

- *Domain of holomorphy*: Characterize those domains $G \subset \mathbb{C}^n$ which are a domain of holomorphy, i.e. the domain of a holomorphic function which does not extend across any boundary point. During the first half of the 20th century this problem was the main challenge for complex analysis in several variables.

- *Holomorphic approximation*: Find conditions for a pair (X,Y) with a complex manifold X and an open $Y \subset X$ which ensure: The canonical restriction of Fréchet spaces
$$\mathscr{O}(X) \to \mathscr{O}(Y)$$
has dense image.

We first investigate domains of holomorphy. We consider holomorphic convexity as a tool, to test a domain for being a domain of holomorphy.

Definition 6.1 (Domain of holomorphy and holomorphic convexity).

1. A domain $G \subset \mathbb{C}^n$ is a *domain of holomorphy* if there exists a holomorphic function $f \in \mathscr{O}(G)$ which has no holomorphic extension over any boundary point $x \in \partial G$.

Consider a complex manifold X.

2. The *holomorphically convex hull* of a compact set $K \subset X$ is the set
$$\hat{K} := \hat{K}_X := \bigcap_{f \in \mathscr{O}(X)} \{x \in X : \ |f(x)| \le \|f\|_K\}.$$

3. The manifold X is *holomorphically convex* if each compact set $K \subset X$ has a compact holomorphically convex hull $\hat{K}$.

4. An open subset $Y \subset X$ is *relatively-holomorphically convex with respect to X* if for each compact $K \subset Y$ the relatively-holomorphically convex hull
$$\hat{K}_{X,Y} := \bigcap_{f \in \mathscr{O}(X)} \{y \in Y : \ |f(y)| \le \|f\|_K\}$$
is compact.

For a relatively-holomorphically convex open subset $Y \subset X$ one builds the holomorphically convex hull $\hat{K}_{X,Y}$ only with those global functions from $\mathscr{O}(Y)$ which are restrictions of holomoprhic functions on X.

Remark 6.2 (Holomorphically convex).

1. The holomorphic convex hull $\hat{K}$ of a compact set $K \subset X$ is the subset of X such for each global holomorphic function $f \in \mathscr{O}(X)$ the modulus $|f|$ on $\hat{K}$ is bounded by its maximum on K.

2. Due to the continuity of holomorphic functions, for a compact $K \subset X$ the holomorphically convex hull $\hat{K} \subset X$ is closed. The fact that in locally compact topological spaces each closed subset of a compact set is compact itself implies: If an open $Y \subset X$ is relatively-holomorphically convex with respect to X then Y is even holomorphically convex.

Proposition 6.3 proves a useful criterion for holomophic convexity.

Proposition 6.3 (Holomorphic convexity). *A complex manifold X is holomorphically convex if it satisfies the following property: For each discrete infinite closed subset $D \subset X$ exists a holomorphic function $f \in \mathscr{O}(X)$ which is unbounded on D, i.e.*

$$sup\ \{|f(x)|:\ x \in D\} = \infty.$$

Proof. We have to show that under the assumptions of the proposition holds

$$K \subset X \text{ compact } \implies \hat{K} \subset X \text{ compact.}$$

Compactness of $K \subset X$ implies the finiteness

$$\|f\|_K < \infty$$

To verify compactness of $\hat{K}$ it is enough to show that $\hat{K}$ is sequentially compact, because X has a second countable topology. Therefore we prove that $\hat{K}$ does not contain a closed, infinite subset which is discrete. The proof is by contradiction: For a given discrete infinite closed subset $D \subset X$ exists by assumption a holomorphic function $f \in \mathscr{O}(X)$ with

$$sup\ \{x \in D:\ |f(x)|\} = \infty > \|f\|_K$$

Hence for a suitable element $x_0 \in D$

$$|f(x_0)| > \|f\|_K$$

and therefore

$$x_0 \notin \{x \in X:\ |f(x)| \leq \|f\|_K\}$$

In particular

$$x_0 \notin \bigcap_{g \in \mathscr{O}(X)} \{x \in X:\ |g(x)| \leq \|g\|_K\} = \hat{K}.$$

□

The sufficient condition for holomorphic convexity from Proposition 6.3 is also necessary: For each discrete infinite subset $D \subset X$ of a holomorphically complex manifold X exists a holomorphic function $f \in \mathscr{O}(X)$ which is unbounded on D, see [14, Kap. IV, §2, Satz 12] and for the affine case [16, Satz 5.6].

The existence of domains in $\mathbb{C}^n$, $n \geq 2$, which are not holomorphically convex, distinguishes complex analysis in several variables from the 1-dimensional theory.

Example 6.4 (Holomorphic convexity).

1. Each domain $X \subset \mathbb{C}$ is holomorphically convex.

   For an indirect proof assume the existence of a compact set $K \subset X$ with $\hat{K}$ not compact. The set $\hat{K}$ is closed. There exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\hat{K}$ with pointwise distinct elements without accumulation point.

   - If the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded then the holomorphic function

     $$f \in \mathscr{O}(X) \text{ defined as } f(z) := z,\ z \in X,$$

     is unbounded, a contradiction to

     $$\infty = sup\ \{|f(x_n)| :\ n \in \mathbb{N}\} \leq sup\ \{|f(x)| :\ x \in \hat{K}\} \leq \|f\|_K < \infty.$$

   - If the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded then it has an accumulation point $a \in \partial X$ due to the Heine-Borel theorem. The openness of X implies $a \notin X$.The holomorphic function

     $$f \in \mathscr{O}(X) \text{ defined as } f(z) := \frac{1}{z-a},\ z \in X,$$

     is unbounded which implies an analoguous contradiction.

2. Each polydisc $\Delta \subset \mathbb{C}^n$ is holomorphically convex: Denote by

   $$r = (r_1, ..., r_n)$$

   the polyradius of Δ. We apply Proposition 6.3: Consider a discrete infinite closed sequence $(a_\nu)_{\nu\in\mathbb{N}}$ in Δ. W.l.o.g. $(a_\nu)_{\nu\in\mathbb{N}}$ is convergent towards a point

   $$a = (a_1, ..., a_n) \in \overline{\Delta}$$

   because $\overline{\Delta}$ is compact. Because the sequence forms a discrete set and is closed in Δ, the limit satisfies $a \notin \Delta$. Hence for a suitable $j \in \{1, ..., n\}$ holds $a_j = r_j$. Then

   $$f \in \mathscr{O}(\Delta),\ f(z_1, ..., z_n) := \frac{1}{z_j - r_j}$$

is unbounded on Δ.

3. The Hartogs figure from Figure 1.3

$$H := U \times S \cup G \times R \subset \mathbb{C}^2$$

with its radii $r_1 < r_2$ is not a domain of holomorphy due to Theorem 1.14: Each holomorphic function $f \in \mathscr{O}(H)$ extends to a holomorphic function

$$\tilde{f} \in \mathscr{O}(G \times S).$$

Note: The Hartogs figure $H \subset \mathbb{C}$ is not holomorphically convex: Using the annotation from Figure 1.3 choose $a \in G \setminus U$ and $\rho > 0$ satisfying

$$r_1 < \rho < r_2,$$

and consider the compact subset of H

$$K := H \cap \{(a,w) \in \mathbb{C}^2 : \ |w| = \rho\} \simeq \partial(\Delta(\rho)),$$

which is biholomorphic to the boundary of the circle of radius $= \rho$.

Theorem 1.14 implies

$$\mathscr{O}(H) = \mathscr{O}(G \times S)$$

Hence the restriction

$$\tilde{f}(a,-) : \Delta(r_2) \to \mathbb{C}$$

is a well-defined holomorphic function, satisfying

$$\|f(a,-)\|_{r_1<|w|\le\rho} = \|\tilde{f}(a,-)\|_{\Delta(\rho)} \le \|\tilde{f}(a,-)\|_{\partial(\Delta(\rho))} = \|f\|_K$$

due to the maximum principle of complex analysis of one variable. Choose a sequence $(w_\nu)_{\nu\in\mathbb{N}}$ of elements $w_\nu \in \Delta(\rho), \ \nu \in \mathbb{N}$, with

$$\lim_{\nu\to\infty} w_\nu = r_1$$

Then for all $\nu \in \mathbb{N}$

$$(a,w_\nu) \in \hat{K} \text{ but } \lim_{\nu\to\infty}(a,w_\nu) = (a,r_1) \notin H$$

in particular

$$\lim_{\nu\to\infty}(a,w_\nu) \notin \hat{K},$$

which proves that $\hat{K}$ is not compact. Hence H is not holomorphically convex.

The interest for holomorphically convex domains $X \subset \mathbb{C}^n$ results from the fact that these domains are exactly the domains of holomorphy in $\mathbb{C}^n$.

Remark 6.5 (Domain of holomorphy).

1. For a domain $X \subset \mathbb{C}^n$ are equivalent:
   - The set X is a domain of holomorphy.
   - The set X is holomorphically convex.
   - The set X satisfies the sufficent condition from Proposition 6.3.
   - The set X is Hartogs-pseudoconvex, see Figure 6.1 for a counter example.

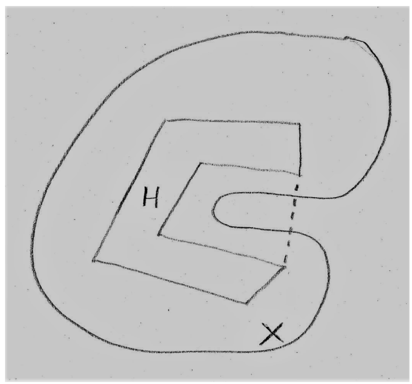


Fig. 6.1 Hartogs figure $H \subset X$ which hinders X being Hartogs-pseudoconvex.

2. A domain $X \subset \mathbb{C}^n$ with a boundary ∂X, locally defined by a C^2-function ϕ, is a domain of holomorphy iff the Levi form $L_x\phi$ is positive semi-definite at each boundary point $x \in \partial X$, see Figure 6.2.

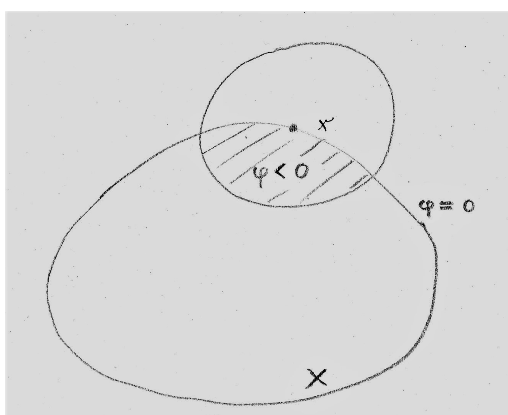


Fig. 6.2 Local description of the boundary

Concerning the proof of Remark 6.5: For part 1) see [16, Satz 6.2] and also [10, Sect. 2.1]. For part 2) cf. [10, Theor. 2.1.4].

Complex polydiscs are distinguished open subsets of the affine space $\mathbb{C}^n$. According to Corollary 3.23 the structure sheaf of a polydisc Δ is acyclic. On a general complex manifold X the concept of a polydisc is not well-defined because the concept refers to the coordinate functions of $\mathbb{C}^n$.

We will see in Section 6.2: On a Stein manifold there are enough global holomorphic functions to replace the concept of a polydisc by the concept of an *analytic polyhedron*, see also [20, 12.10 Exampl.]. An analytic polyhedron is a global concept. Its definition is independent from any choice of coordinates. Analytic polyhedra turn out as the main tool to prove Theorem B on Stein manifolds. We will show:

- Proposition 6.12: Each Stein manifold can be exhausted by a sequence of analytic polyhedra.

- Theorem 6.13: Analytic polyhedra of a Stein manifold embed as analytic submanifolds into polydiscs. Hence the vanishing result from the local theory for polydiscs, Corollary 5.15, carries over to analytic polyhedra.

- Theorem 6.21: For relatively-holomorphically convex subdomains Y of a Stein manifold X holds an approximation theorem even for sections of coherent sheaves.

Definition 6.6 (Analytic polyhedron). Consider a complex manifold X. A relatively compact, open subset

$$P \subset\subset X$$

is an *analytic polyhedron* in X if there exist an open neighbourhood U of $\overline{P}$ and finitely many holomorphic functions

$$f_1, ..., f_k \in \mathscr{O}(X)$$

such that

$$P = \bigcap_{j=1}^{k} \left\{ x \in U : \ |f_j(x)| < 1 \right\}$$

Note in Definition 6.6 that the functions $f_j \in \mathscr{O}(X)$ are defined on the whole manifold X.

Being an analytic polyhedron is a relative concept. Whether a set is an analytic polyhedron depends on the choice of the ambient manifold to consider.

Example 6.7 (Analytic polyhedron).

1. Each relatively compact, open polydisc

$$\Delta \subset\subset \mathbb{C}^n$$

is an analytic polyhedron in $\mathbb{C}^n$: If

$$\Delta = \left\{(z_1,...,z_n) \in \mathbb{C}^n : \ |z_j| < r_j \text{ for } j = 1,...,n\right\}$$

then consider

$$f_j(z) := \frac{z_j}{r_j} \in \mathscr{O}(\mathbb{C}^n), \ j = 1,...,n$$

As a consequence

$$\Delta = \bigcap_{j=1}^{n} \{z \in \mathbb{C}^n : \ |f_j(z)| < 1\},$$

one may choose $U = \mathbb{C}^n$.

2. The 1-dimensional annulus

$$P := \{z \in \mathbb{C} : \ 1 < |z| < 2\}$$

is an analytic polyhedron in $\mathbb{C}^*$ being the intersection of two analytic polyhedra in $\mathbb{C}^*$

$$P = \{z \in \mathbb{C}^* : \ |z| < 2\} \cap \{z \in \mathbb{C}^* : \ |1/z| < 1\}$$

But P is not an analytic polyhedron in $\mathbb{C}$ due to the maximum principle.

Proposition 6.8 (Analytic polyhedra are relatively-holomorphically convex). *Consider a complex manifold X and an analytic polyhedron $P \subset\subset X$. Then P is relatively-holomorphically convex with respect to X.*

Proof. By definition exists an open set $U \subset X$ and finitely many holomorphic functions

$$f_j \in \mathscr{O}(X), \ j = 1,...,k,$$

such that

$$P = \{x \in U : \ |f_j(x)| < 1 \text{ for all } j = 1,...,k\} \subset\subset U$$

Consider a given compact $K \subset P$. Because K is compact and P is open we have

$$K \cap \partial P = \emptyset.$$

i) If there exists an index $j \in \{1,...,k\}$ with

$$\|f_j\|_K = 1,$$

then compactness of K implies the existence of a point $x \in K \subset P$ with

$$|f_j(x)| = 1$$

Hence

$$x \in \partial P \cap K = \emptyset,$$

a contradiction. As a consequence for all $j = 1, ..., k$

$$\|f_j\|_K < 1$$

ii) We have to show: For each compact $K \subset P$ the hull

$$L := \hat{K}_{X,P} := \bigcap_{f \in \mathscr{O}(X)} \{x \in P : |f(x)| \le \|f\|_K\}$$

is compact. It suffices to show that each sequence $(x_\nu)_{\nu \in \mathbb{N}}$ in L has a subsequence, which is convergent towards a limit $x_0 \in L$.

The relative compactness

$$P \subset\subset U$$

implies the compactness of $\overline{P}$ and a forteriori the compactness of

$$\overline{L} \subset \overline{P}$$

Hence there exists a subsequence $(x_{\nu_k})_{k \in \mathbb{N}}$ in $L \subset P$ which is convergent towards a limit

$$x_0 \in \overline{L} \subset U.$$

Part i) implies for all $j = 1, ..., k$ in the limit

$$|f_j(x_0)| \le \|f_j\|_K < 1, \text{ hence } x_0 \in P.$$

Moreover for all $f \in \mathscr{O}(X)$ in the limit

$$|f(x_0)| \le \|f\|_K$$

As a consequence $x_0 \in L$. $\square$

Proposition 6.9 (Neighbourhood basis of analytic polyhedra). *Consider a complex manifold X.*

1. Holomorphically convex*: If X is holomorphically convex, $K \subset X$ compact and $U \subset X$ an open neighbourhood of $\hat{K}$, then exists an analytic polyhedron P in X with*

$$\hat{K} \subset P \subset\subset U.$$

2. Relatively-holomorphically convex*: If $Y \subset X$ is open and relatively-holomorphically convex with respect to X, $K \subset Y$ compact and $U \subset Y$ an open neighbourhood of $\hat{K}$, then exist an analytic polyhedron P in Y, defined in U by finitely many holomorphic functions from $\mathscr{O}(X)$, which satisfies*

$$\hat{K} \subset P \subset\subset U.$$

Proof. i) The set $\hat{K}$ is compact because X is holomorphically convex. The manifold X is locally compact. Hence we may assume $U \subset\subset X$ relatively compact.

$$\hat{K} \subset U \text{ and } U \text{ open } \implies \hat{K} \cap \partial U = \emptyset.$$

There exists for each $x \in \partial U$ a holomorphic function $f_x \in \mathscr{O}(X)$ with

$$|f_x(x)| > \|f_x\|_K, \text{ w.l.o.g. } |f_x(x)| > 1 \text{ and } \|f_x\|_K < 1$$

Then also

$$|f_x(y)| > 1$$

for all $y \in V_x$, a suitable neighbourhood of x, see Figure 6.3. The relative compactness of U implies the compactness of ∂U. Hence there exist finitely many open sets $V_{x_1}, ..., V_{x_k}$ with

$$\partial U \subset \bigcup_{j=1,...,k} V_{x_j}$$

- Define

  $$P := \{x \in U : |f_{x_j}(x)| < 1 \text{ for } j = 1, ..., k\}$$

  Then

  $$P \subset U \implies \overline{P} \subset \overline{U} \text{ compact}$$

  Hence $\overline{P}$ is compact.

- Moreover

  $$x \in \hat{K} \subset U \implies \text{For each } j = 1, ..., k : |f_{x_j}(x)| \le \|f_{x_j}\|_K < 1.$$

  Hence $\hat{K} \subset P$.

- Eventually

  $$\overline{P} \subset \{x \in U : |f_{x_j}(x)| \le 1 \text{ for } j = 1, ..., k\} \text{ and } \{x \in \partial U : |f_{x_j}(x)| \le 1 \text{ for } j = 1, ..., k\} = \emptyset,$$

  Hence $\overline{P} \subset U$.

As a consequence P is an analytic polyhedron with

$$\hat{K} \subset P \subset\subset U.$$

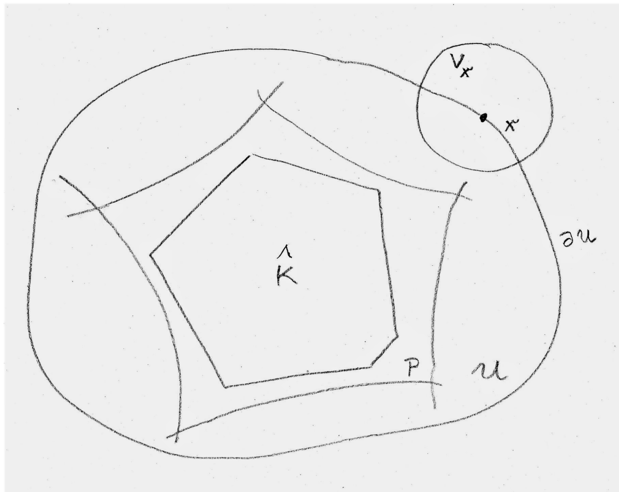


Fig. 6.3 Separating K and ∂U

ii) The proof for the relatively-holomorphically convex case is analogous. □

6.2 Stein manifolds and Theorem *B*

Definition 6.10 is the original definition, see [12, Einleitg.].

Definition 6.10 (Stein manifold). A *Stein manifold* is a complex manifold X being

1. *holomorphically convex*,
2. and *holomorphically separable*,

   i.e. for each pair of points $x \neq y \in X$ exists a global holomorphic function $f \in \mathscr{O}(X)$ with $f(x) \neq f(y)$,

3. and *locally uniformizable by global holomorphic functions*,

   i.e. for each $x \in X$ exist an open neighbourhood $U \subset X$ of x and global holomorphic functions
   $$f_1, ..., f_n \in \mathscr{O}(X)$$
   such that the restriction
   $$(f_1|U, ..., f_n|U) : U \to \mathbb{C}^n$$
   is a chart of X around x.

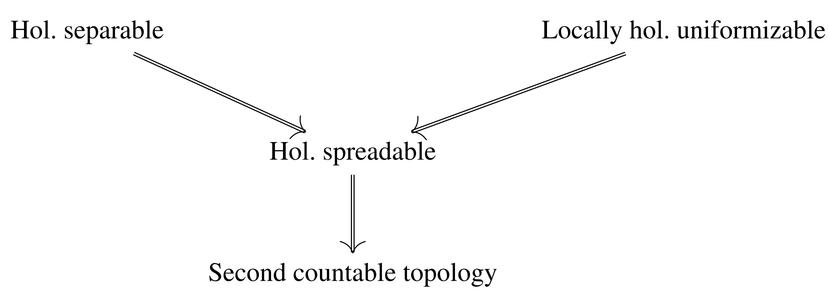
Each of the requirements from Definition 6.10 refers to *global* holomorphic functions resp. requires the existence of certain gobal holomorphic functions on X. Each subdomain $X \subset \mathbb{C}^n$ is holomorphically separable and locally uniformizable. But if $n \geq 2$ then X is not necessarily holomorphically convex, see Example 6.4. Hence the characteristic property of an open Stein submanifold $X \subset \mathbb{C}^n$ is its holomorphic convexity.

Remark 6.11 (Stein manifold). The conditions from Definition 6.10 which characterize a Stein manifold can be weakened considerably.

1. A complex manifold X is *holomorphically spreadable* if each point $x \in X$ has an open neighbourhood U and finitely many global holomorphic function
$$f_1, ..., f_k \in \mathscr{O}(X)$$
such that
$$\{z \in U : \ f_j(z) = f_j(x) \text{ for } j = 1, ..., k\} = \{x\}$$
Apparently, if X is locally uniformizable then X is also holomorphically spreadable.

2. Due to a result of Grauert for a complex manifold holds:
$$\text{Holomophically convex + holomorphic spreadable} \implies \text{Stein manifold}$$
Moreover, the fact that X is holomorphically spreadable implies that the topology of X is second countable, see [12, Satz 5 und 8] and also [20, 51. A.3].

The following diagram shows the fundamental implications between those concepts.



Proposition 6.12 sharpens Proposition 6.9 in case the ambient manifold X is holomorphically convex.

Proposition 6.12 (Exhaustion by analytic polyhedra). *Each holomorphically convex complex manifold X, notably each Stein manifold, has an exhaustion by a sequence of analytic polyhedra, i.e. there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of analytic polyhedra in X satisfying*

$$P_n \subset\subset P_{n+1},\ n \in \mathbb{N},\ \textit{with}\ X = \bigcup_{n\in\mathbb{N}} P_n$$

Proof. Because X is locally compact with second countable topology there exists an exhaustion of X by an ascending sequence of compact subsets $(K_\nu)_{\nu\in\mathbb{N}}$ with

$$K_\nu \subset K_{\nu+1}, \nu \in \mathbb{N}, \text{ and } X = \bigcup_{\nu\in\mathbb{N}} K_\nu$$

The assumption that X is holomorphically convex implies: For each compact $K \subset X$ its holomorphically convex hull $\hat{K} \subset X$ is also compact.

We construct the required sequence $(P_\nu)_{\nu\in\mathbb{N}}$ of analytic polyhedra by induction: Due to compactness of $\hat{K}_0$ Proposition 6.9 provides an analytic polyhedron P_0 in X with

$$\hat{K}_0 \subset P_0 \subset\subset X.$$

The set

$$L_0 := K_1 \cup \overline{P}_0$$

is compact, hence its convex hull $\hat{L}_0$ is compact too. As a consequence there exists an analytic polyhedron P_1 in X satisfying

$$\hat{L}_0 \subset P_1 \subset\subset X$$

The construction continues with the compact set

$$L_1 := K_2 \cup \overline{P}_1$$

and proceeds step by step in an analoguous way. □

Theorem 6.13 (Closed embedding of analytic polyhedra of a Stein manifold). *Each analytic polyhedron*

$$P \subset\subset X$$

in a Stein manifold X embeds by global holomorphic functions as analytic submanifold of a polydisc, i.e. there exist finitely many holomorphic functions

$$\phi_1, ..., \phi_m \in \mathscr{O}(X)$$

such that the restriction

$$\phi := (\phi_1, ..., \phi_m)|P : P \to \Delta \subset \mathbb{C}^m$$

is a closed embedding.

Proof. By definition of the analytic polyhedron there exist an open set $U \subset X$ and holomorphic functions

$$f_1, ..., f_r \in \mathscr{O}(X)$$

such that

$$P = \{z \in U : |f_j(z)| < 1 \text{ for } j = 1, ...r\} \subset\subset U$$

Because $\overline{P}$ is compact, we may assume after shrinking that also $\overline{U}$ is compact.

The proof will extend charts (part i)) to a global injective immersion (part ii)), a proper map (part iii)) and finally to a closed embedding of P.

i) *Immersion due to local uniformizability*: Set $n := dim\, X$. The local uniformization of the Stein manifold X by global holomorphic functions, see Definition 6.10, part 3, implies that each point $a \in \overline{P}$ has an open neighbourhood U_a and an n-tuple of holomorphic functions

$$g_a := (g_{a,1}, ..., g_{a,n}) \in \mathscr{O}(X)^{\oplus n}$$

such that the restriction

$$g_a|U_a : U_a \to \mathbb{C}^n$$

is a chart of U_a around a. Compactness of $\overline{P}$ implies that $\overline{P}$ is covered by finitely such open neighbourhoods

$$U_{a_1}, ..., U_{a_k} \text{ with corresponding } n\text{-tuples } g_{a_1}, ..., g_{a_k} \in \mathscr{O}(X)^{\oplus n}.$$

Denote by

$$G := (g_1, ..., g_s) \in \mathscr{O}(X)^{\oplus s},\ s = k \cdot n,$$

the family of all component functions of these n-tuples. The map

$$G|P : P \to \mathbb{C}^s$$

is an immersion of the analytic polyhedron P.

ii) *Injectivity due to holomorphic separability*: Consider the diagonal

$$Diag := \{(x, y) \in \overline{P} \times \overline{P} :\ x = y\}$$

The union of product sets

$$W := (\overline{P} \times \overline{P}) \cap \bigcup_{j=1,...,k} (U_{a_j} \times U_{a_j})$$

is an open neighbourhood of $Diag$.

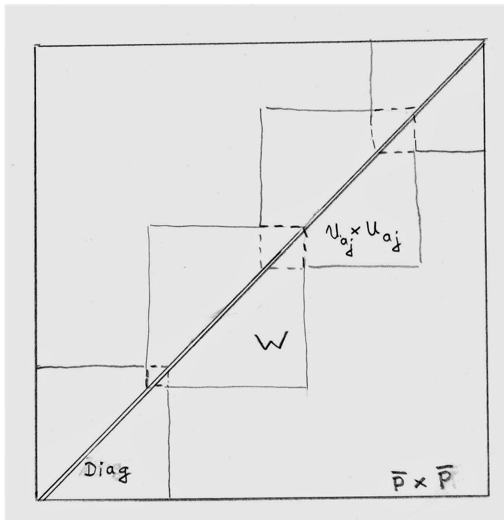


Fig. 6.4 *Diag* $\subset W \subset \overline{P} \times \overline{P}$.

For a pair

$$(x,y) \in \overline{P} \times \overline{P}$$

we distinguish two cases:

- If

$$(x,y) \in W$$

  then exists an index $j \in \{1, ..., k\}$ with $(x,y) \in U_{a_j} \times U_{a_j}$. If $x \neq y$ then part i) provides a component function $g \in \mathscr{O}(X)$ of G with

$$g(x) \neq g(y)$$

- If

$$(x,y) \in (\overline{P} \times \overline{P}) \setminus W$$

  then $x \neq y$ and due to the holomorphic separability of X there exists $h = h_{(x,y)} \in \mathscr{O}(X)$ with

$$h(x) \neq h(y).$$

  Due to continuity the function h separates also points which form a pair in a neighbourhood $W_{(x,y)}$ of (x,y). The set

$$(\overline{P} \times \overline{P}) \setminus W$$

  is a closed subset of the compact

$$\overline{P} \times \overline{P}$$

and therefore compact itself. As a compact set it is covered by finitely many global $W_1, ..., W_t$ with separating functions $h_1, ..., h_t \in \mathscr{O}(X)$.

By combining the families $G = (g_1, ..., g_s)$ and $H := (h_1, ..., h_t)$ we obtain a finite family $(G,H) \in \mathscr{O}(X)^{s+t}$ which provides an injective map

$$(G,H)|\overline{P} : \overline{P} \to \mathbb{C}^{s+t}.$$

We may normalize the components such that

$$\|g_j\|_{\overline{P}} < 1,\ j = 1, ..., s \text{ and } \|h_j\|_{\overline{P}} < 1,\ j = 1, ..., t.$$

The map

$$(G,H)|P : P \to \Delta \subset \mathbb{C}^{s+t}$$

is an injective immersion.

iii) *Closed embedding due to relative compactness*: To obtain a proper map we now add the family of defining functions of P

$$F := (f_1, ..., f_r) \in \mathscr{O}(X)^{\oplus r}$$

to the map from part ii). Set

$$m := r + s + t$$

and consider the polydisc

$$\Delta \subset \mathbb{C}^m$$

and the map

$$\Phi : P \to \Delta$$

with

$$\Phi(x) := (F,G,H)(x) = (f_1(x), ..., f_r(x), g_1(x), ..., g_s(x), h_1(x), ..., h_t(x)).$$

In order to show the properness of Φ consider a compact $K \subset \Delta$. For a suitable radius $r < 1$

$$K \subset \Delta_r := \{w = (w_1, ..., w_m) \in \Delta :\ |w_\nu| \le r,\ \nu = 1, ..., m\}$$

Then $\Phi^{-1}(K) \subset \Phi^{-1}(\Delta_r)$ is closed. Moreover

$$\Phi^{-1}(\Delta_r) \subset \{x \in P :\ |f_\nu(x)| \le r,\ \nu = 1, ..., r\},$$

and

$$\{x \in P :\ |f_\nu(x)| \le r,\ \nu = 1, ..., r\} = \{x \in \overline{P} :\ |f_\nu(x)| \le r,\ \nu = 1, ..., r\}$$

because each point

$$x \in \overline{P} \setminus P \text{ with } |f_j(x)| \leq r \text{ for } j = 1, ..., r$$

had to satisfy $|f_j(x)| = 1$ for at least one index $j \in \{1, ..., r\}$, which is impossible.

iv) *Closed embedding*: Due to part i) - iii) the map

$$\Phi : P \to \Delta \subset \mathbb{C}^m$$

is an injective immersion, which is a proper map. Hence Φ is a closed embedding.
□

Lemma 6.14 (Coherence of the direct image under closed embeddings). *Consider a complex manifold X, an open set $U \subset \mathbb{C}^n$, and a closed holomorphic embedding*

$$j : X \to U$$

For each coherent $\mathscr{O}_X$-module $\mathscr{F}$ the direct image $j_*\mathscr{F}$ is a coherent $\mathscr{O}_U$-module.

Proof. Denote by $A := j(X) \subset U$ the analytic submanifold.

i) *Structure sheaf*: First we consider the specific case of the structure sheaf $\mathscr{F} = \mathscr{O}_X$. We have

$$j_*\mathscr{O}_X = \mathscr{O}_U / \mathscr{I}_A$$

with $\mathscr{I}_A \subset \mathscr{O}_U$ the ideal sheaf of $A \subset U$. The sheaf $\mathscr{I}_A \subset \mathscr{O}_U$ is coherent due to Proposition 4.29. Hence

$$\mathscr{O}_A = \mathscr{O}_U / \mathscr{I}_A = coker\ [\mathscr{I}_A \to \mathscr{O}_U]$$

is coherent due to Proposition 4.31.

ii) *General case*: A general coherent $\mathscr{O}_X$-module $\mathscr{F}$ has locally in X a resolution by free sheaves of finite rank

$$\mathscr{O}_X^q \to \mathscr{O}_X^p \to \mathscr{F} \to 0$$

The induced sequence of sheaf morphisms

$$j_*\mathscr{O}_X^q \to j_*\mathscr{O}_X^p \to j_*\mathscr{F} \to 0$$

is also exact, because $j_*\mathscr{F}$ has the stalks

$$(j_*\mathscr{F})_y = \begin{cases} \mathscr{F}_x & \text{if } y = j(x),\ x \in X \\ 0 & \text{otherwise} \end{cases}$$

Due to part i) the $\mathscr{O}_U$-modules $j_*\mathscr{O}_X^q$ and $j_*\mathscr{O}_X^p$ are coherent. Hence Proposition 4.31 implies the coherence of

$$j_*\mathscr{F} = coker\ [j_*\mathscr{O}_X^q \to j_*\mathscr{O}_X^p]$$

□

Apparently Lemma 6.14 is a simple example of Grauert's general coherence theorem, see Remark 4.35.

Definition 6.15 (Concentric polyhedra). Two analytic polyhedra

$$P \subset P'$$

of a given complex manifold are *concentric* if there exists a closed embedding

$$j : P' \to \Delta'$$

into a polydisc $\Delta' \subset \mathbb{C}^n$, and a concentric polydisc $\Delta \subset\subset \Delta'$ such that the restriction

$$j|P : P \to \Delta'$$

embeds P as an analytic submanifold of Δ.

Note: Two concentric polyhedra satisfy $P \subset\subset P'$.

Proposition 6.16 (Theorem B for concentric analytic polyhedra in Stein manifolds). *Consider a pair $P \subset\subset P'$ of concentric analytical polyhedra of a Stein manifold X. For each coherent $\mathscr{O}_{P'}$-module $\mathscr{F}$ the restriction $\mathscr{F}|P$ is acyclic.*

The proof reduces the claim to the analogue for polydiscs in Corollary 5.15, which follows from Hilbert's syzygy theorem for coherent sheaves on polydiscs, Theorem 5.14.

Proof. i) *Consequences of Hilbert's syzygy theorem*: Due to Definition 6.15 there exists an embedding

$$j : P' \to \Delta'$$

with $A := j(P')$ an analytic submanifold of a polydisc Δ' and

$$P = j^{-1}(\Delta) \subset\subset P'$$

for a concentric polydisc

$$\Delta \subset\subset \Delta'.$$

Lemma 6.14 implies that $j_*\mathscr{F}$ is a coherent $\mathscr{O}_{\Delta'}$-module. Corollary 5.15 of Hilbert's syzygy theorem after shrinking to a polydisc implies that

$$j_*\mathscr{F}|\Delta$$

is acyclic, i.e.

$$H^q(\Delta, j_*\mathscr{F}) = 0 \text{ for all } q \geq 1.$$

ii) *Comparing cohomology by comparing coverings*: We claim

$$H^q(P,\mathscr{F}) = H^q(\Delta, j_*\mathscr{F}).$$

On the right-hand side of the equation: For each open covering $\mathscr{U}$ of Δ holds by definition of the direct image

$$H^q(\mathscr{U}, j_*\mathscr{F}) = H^q(j^{-1}(\mathscr{U}),\mathscr{F}) \text{ with } j^{-1}(\mathscr{U}) := (j^{-1}(U_i))_{i\in I}$$

We have

$$H^q(\Delta, j_*\mathscr{F}) = \varinjlim_{\mathscr{U}} H^q(\mathscr{U}, j_*\mathscr{F})$$

There is a bijective correspondence between open coverings $\mathscr{U}$ of Δ for the right-hand side and open coverings $\mathscr{V}$ of P for the left-hand side: Each open covering $\mathscr{U} = (U_i)_{i\in I}$ of Δ defines the open covering

$$\mathscr{V} := j^{-1}(\mathscr{U})$$

of P. For the opposite direction consider an open covering

$$\mathscr{V} = (V_j)_{j\in J}$$

of P. For each $j \in J$ the set $j(V_j) \subset j(P)$ is open. Hence there exist an open $U_j \subset \Delta$ with

$$j(V_j) = U_j \cap j(P).$$

Because $j(P) \subset \Delta$ is closed, the additional set

$$U := \Delta \setminus j(P) \subset \Delta$$

is also open. Then

$$\mathscr{U} := (U_j)_{j\in J} \cup \{U\}$$

is an open covering of Δ with

$$j^{-1}(\mathscr{U}) = \mathscr{V}.$$

Hence the vanishing of $H^q(\Delta, j_*\mathscr{F})$ from part i) implies: For all $q \geq 1$

$$0 = \varinjlim_{\mathscr{U}} H^q(\mathscr{U}, j_*\mathscr{F}) = \varinjlim_{\mathscr{U}} H^q(j^{-1}(\mathscr{U}),\mathscr{F}) = \varinjlim_{\mathscr{V}} H^q(\mathscr{V},\mathscr{F}) = H^q(P,\mathscr{F})$$

□

Proposition 6.16 provides a local step to obtain Theorem B on a Stein manifold X. In order to extend the solution to all of X we prove a result about holomorphic approximation by global functions and sections. Therefore we introduce the concept of *Runge pairs*.

Definition 6.17 (Runge pair). A *Runge pair* (X,Y) is a complex manifold X and an open $Y \subset X$ such that the restriction map

$$\mathscr{O}(X) \to \mathscr{O}(Y),\ f \mapsto f|Y,$$

has dense image with respect to the Fréchet topologies on both vector spaces.

We recall the 1-dimensional situation in $\mathbb{C}$.

Remark 6.18 (Runge pair).

1. Each holomorphic function on the disc $\Delta \subset \mathbb{C}$ is the compact limit of a sequence of holomorphic functions on $\mathbb{C}$, e.g., of its Taylor polynomials. Hence the canonical restriction
$$\mathscr{O}(\mathbb{C}) \to \mathscr{O}(\Delta),\ f \mapsto f|\Delta,$$
has dense image with respect to the Fréchet topology of compact convergence on each of the two spaces. As a consequence the pair $(\mathbb{C},\Delta)$ is a Runge pair.
Note: Taylor approximation also applies to holomorphic functions in higher-dimensional polydiscs Δ^n. Hence each pair $(\mathbb{C}^n,\Delta^n)$ is a Runge pair.

2. For a general domain $G \subset \mathbb{C}$ the pair $(\mathbb{C},G)$ is not necessarily a Runge pair. A counterexample is the domain $G := \mathbb{C}^*$: Consider
$$f : G \to \mathbb{C},\ f(z) := \frac{1}{z}$$
and assume
$$f = \lim_{n\to\infty} f_n,\ f_n \in \mathscr{O}(\mathbb{C}).$$
The residue theorem and the compact convergence imply
$$2\pi i = \int_{|z|=1} f(z)\ dz = \int_{|z|=1} (\lim_{n\to\infty} f_n(z))\ dz = \lim_{n\to\infty}\int_{|z|=1} f_n(z)\ dz = 0$$
The contradiction shows that $(\mathbb{C},\mathbb{C}^*)$ is not a Runge pair.

3. For a domain $G \subset \mathbb{C}$ holds the Runge approximation theorem: $(\mathbb{C},G)$ is a Runge pair iff the complement $\mathbb{C}\setminus G$ has no relatively compact holes, i.e. iff
$$G = h_{\mathbb{C}}(G) := G \cup \Big(\bigcup_{\substack{C \subset\subset \mathbb{C}\setminus G \\ \text{component}}} C \Big),\ \text{the Runge hull of } G \text{ with respect to } \mathbb{C}.$$

See [24, Kap. 12, §2, Approximationssatz] and note that each entire function on $\mathbb{C}$ is the compact limit of poynomials. For a proof of the more general case of an open Riemann surface see [32, Chap. 14.2 and Theor. 14.14].

Theorem 6.19 (Runge approximation for the structure sheaf). *Consider a Stein manifold X and an open, relatively-holomorphically convex subset $Y \subset X$. Then (X,Y) is a Runge pair.*

Proof. Consider a holomorphic function $f \in \mathscr{O}(Y)$ and a compact $K \subset Y$. Because Y is relatively-holomorphically convex with respect to X, the hull

$$\hat{K}_{X,Y} \subset Y$$

is compact. Hence we may assume $K = \hat{K}_{X,Y}$. Proposition 6.9 provides an analytic polyhedron $P \subset Y$

- which is defined in an open set

$$U \subset Y \text{ with } P \subset\subset U$$

by finitely many holomorphic functions $f_j,\ j = 1,...,k$, not only from $\mathscr{O}(Y)$ but even from $\mathscr{O}(X)$, as

$$P := \{y \in U :\ |f_j(y)| < 1 \text{ for all } j = 1,...,k\}$$

- and satisfies

$$K \subset P \subset\subset U$$

i) *Reduction to a polydisc*: Because X is a Stein manifold, Theorem 6.13 provides a closed affine embedding of P: There exists finitely many holomorphic functions

$$\phi_1,...,\phi_n \in \mathscr{O}(X)$$

such that

$$\phi := (\phi_1|P,...,\phi_n|P) : P \to \Delta$$

embeds P into the unit polydisc $\Delta \subset \mathbb{C}^n$ as an analytic submanifold $A \subset \Delta$. Consider the holomorphic function on A

$$g := f|P \circ \phi^{-1} \in \mathscr{O}(A).$$

Then

$$f|P = g \circ \phi.$$

ii) *Theorem B for polydiscs*: For each given polydisk

$$\Delta' \subset\subset \Delta$$

we extend the restriction $g|A\cap\Delta'$ to a holomorphic function on Δ', i.e. we construct a holomorphic function

$$G\in\mathscr{O}(\Delta') \text{ with } G|A\cap\Delta' = g|A\cap\Delta' :$$

The ideal sheaf $\mathscr{I}\subset\mathscr{O}_\Delta$ of the submanifold $A\subset\Delta$ fits into the exact sequence of sheaves on Δ

$$0\to\mathscr{I}_A\to\mathscr{O}_\Delta\to\mathscr{O}_A\to 0$$

which defines the structure sheaf $\mathscr{O}_A$. Proposition 4.29 implies that $\mathscr{I}_A$ is coherent, and Corollary 5.15 implies for the restriction

$$H^1(\Delta',\mathscr{I}_A|\Delta') = 0.$$

The segment of the long exact cohomology sequence

$$H^0(\Delta',\mathscr{O}_{\Delta'})\to H^0(\Delta',\mathscr{O}_A|\Delta')\to H^1(\Delta',\mathscr{I}_A|\Delta') = 0$$

implies the surjectivity of the restriction

$$\mathscr{O}(\Delta')\to\mathscr{O}_A(A\cap\Delta')$$

and the existence of a holomorphic function $G\in\mathscr{O}(\Delta')$ with

$$G|A\cap\Delta' = g|A\cap\Delta'.$$

iii) *Holomorphic approximation by Taylor polynomials*: We choose a polydisc Δ' with

$$\phi(K)\subset\Delta'\subset\subset\Delta.$$

For given precision $\varepsilon>0$ we choose a finite segment F of the n-dimensional Taylor expansion F of G on Δ'

$$F(z) = \sum_{|I|\le k} a_I\cdot z^I$$

such that

$$\|G-F\|_{\Phi(K)}\le\varepsilon$$

For $x\in K$

$$f(x) = g(\phi(x)) = G(\phi(x))$$

hence

$$\|f-(F\circ\phi)\|_K = \|G\circ\Phi - F\circ\Phi\|_K\le\varepsilon$$

with the global holomorphic approximation

$$F\circ\phi\in\mathscr{O}(X).$$

One has

$$(F \circ \phi)(x) = F(\phi(x)) = \sum_{i_1+\ldots+i_n \le k} a_{i_1 \ldots i_n} \cdot \phi_1(x)^{i_1} \cdot \ldots \cdot \phi_n(x)^{i_n}$$

□

Lemma 6.20 prepares the proof of Theorem 6.21. The theorem generalizes Theorem 6.19 to sections of coherent sheaves.

Lemma 6.20 (Runge approximation along analytic polyedra). *Consider a Stein manifold X and two pairs*

$$(P, P') \text{ and } (Q, Q')$$

of concentric analytic polyhedra in X with

$$P' \subset Q.$$

For each coherent $\mathscr{O}$-module $\mathscr{F}$ on X the restriction of Fréchet spaces

$$\mathscr{F}(Q) \to \mathscr{F}(P)$$

has dense image.

The idea is to apply Hilbert's syzygy theorem, Theorem 5.14, for a pair of analytic polyhedra, which embedd into the concentric pair (Δ, Δ') of polydiscs. Note that $P \subset Q$ is not required to be a pair of concentric polyhedra.

Proof. i) *Resolving $\mathscr{F}$ over Q*: Theorem 6.13 provides a closed embedding

$$\phi : Q' \to \Delta'$$

The direct image $\phi_* \mathscr{F}$ is a coherent $\mathscr{O}_{\Delta'}$-module due to Lemma 6.14. Theorem 5.14 provides over the shrinked polydisc

$$\Delta \subset\subset \Delta' \text{ with } \phi^{-1}(\Delta) = Q$$

an epimorphism

$$\mathscr{O}_\Delta^k \to \phi_* \mathscr{F} \to 0,$$

i.e. there exist sections

$$f_1, ..., f_k \in (\phi_* \mathscr{F})(\Delta) = \mathscr{F}(\phi^{-1}(\Delta)) = \mathscr{F}(Q)$$

which generate for each $y \in \Delta$ the stalk

$$(\phi_* \mathscr{F})_y.$$

The latter satisfies for each $y = \phi(x),\ x \in Q$,

$$(\phi_* \mathscr{F})_y = \mathscr{F}_x$$

The sections define an epimorphism of $\mathscr{O}$-modules over Q

$$\mathscr{O}_Q^k \xrightarrow{\beta} \mathscr{F},\ e_j \mapsto f_j,\ j = 1,...k,$$

with $(e_1,...,e_j) \in \mathscr{O}^p(Q)$ the canonical $\mathscr{O}(Q)$-base of $\mathscr{O}^p(Q)$. The epimorphism induces a short exact sequence of sheaves

$$0 \to \mathscr{K} \to \mathscr{O}^k \xrightarrow{\beta} \mathscr{F} \to 0$$

with the coherent $\mathscr{O}_Q$-module

$$\mathscr{K} := ker\,\beta.$$

ii) *Approximation of holomorphic functions*: The vanishing

$$H^1(P,\mathscr{K}) = 0$$

due to Theorem B, Proposition 6.16, applied to the concentric polydiscs

$$P \subset\subset P'$$

provides the epimorphism of sections

$$\mathscr{O}^k(P) \xrightarrow{\beta_P} \mathscr{F}(P)$$

A given section $f \in \mathscr{F}(P)$ can be represented as

$$f = \sum_{j=1}^{k} \alpha_j \cdot f_j$$

with holomorphic coefficients $\alpha_1,...,\alpha_k \in \mathscr{O}(P)$. Due to Theorem 6.19 each holomorphic function

$$\alpha_j,\ j = 1,...,k,$$

can be approximated with arbitrary precision on each compact subset of P by global holomorphic functions from $\mathscr{O}(X)$, in particular by holomorphic functions on Q. The module multiplication is continuous. Hence $f \in \mathscr{F}(P)$ can be approximated with arbitrary precision by sections from $\mathscr{F}(Q)$. □

Theorem 6.21 (Runge approximation for coherent sheaves). *Consider a Stein manifold X and a relatively-holomorphically convex open $Y \subset X$. For each coherent sheaf $\mathscr{F}$ on X the canonical restriction*

$$\mathscr{F}(X) \to \mathscr{F}(Y),\ f \mapsto f|Y,$$

has dense image with respect to the canonical Fréchet topologies. In particular, (X,Y) is a Runge pair.

Proof. i) *Exhausting Y by analytic polyhedra P_i*: Due to Remark 6.2 the subset $Y \subset X$ is holomorphically convex. Proposition 6.12 provides an exhaustion $(P_i)_{\in \mathbb{N}}$ of Y by analytic polyhedra in Y. W.l.o.g. for each index $i \in \mathbb{N}$ exists a concentric polyhedron P_i' in Y with

$$P_i \subset\subset P_i'.$$

Due to Proposition 6.9 we may assume that these polyhedra are defined by holomorphic functions from $\mathscr{O}(X)$.

ii) *Restriction $\mathscr{F}(X) \to \mathscr{F}(P_i)$*: We claim for each arbitrary, but fixed $i \in \mathbb{N}$: The restriction

$$\rho^X_{P_i} : \mathscr{F}(X) \to \mathscr{F}(P_i)$$

has dense image with respect to the canonical Fréchet topologies.

The Stein manifold X is holomorphically convex. Proposition 6.12 provides an exhaustion of X by analytic polyhedra in X. W.l.o.g. the exhaustion starts with a neighbourhood of the relatively compact $P_i' \subset\subset X$: Choose a sequence of pairs of concentric analytic polyhedra in X

$$(Q_\nu, Q_\nu')$$

satisfying

$$P_i' \subset Q_0,\ Q_\nu' \subset\subset Q_{\nu+1},\ X = \bigcup_{\nu \in \mathbb{N}} Q_\nu$$

Lemma 6.20 ensures: All restrictions

$$\mathscr{F}(Q_0) \to \mathscr{F}(P_i) \text{ and } \mathscr{F}(Q_{\nu+1}) \to \mathscr{F}(Q_\nu),\ \nu \in \mathbb{N},$$

have dense image.

For the given index $i \in \mathbb{N}$ consider an arbitrary but fixed section

$$f_i \in \mathscr{F}(P_i)$$

and a fixed closed neighbourhood

$$N_i \subset \mathscr{F}(P_i) \text{ of } f_i$$

in the Fréchet space $\mathscr{F}(P_i)$. If

$$\rho_\nu := \rho^{Q_\nu}_{P_i} : \mathscr{F}(Q_\nu) \to \mathscr{F}(P_i)$$

denotes the restriction, then the inverse images

$$M_\nu := \rho_\nu^{-1}(N_i) \subset \mathscr{F}(Q_\nu)\ \nu \in \mathbb{N},$$

are complete metric spaces. The family of continuous morphisms with dense image

$$\ldots \to M_3 \to M_2 \to M_1 \to M_0$$

satisfies the assumptions of the Mittag-Leffler principle of exhaustion:
Proposition 1.26 provides a family of compatible sections $(F_\nu \in \mathscr{F}(Q_\nu))_{\nu\in\mathbb{N}}$, i.e.

$$F_{\nu+1}|Q_\nu = F_\nu,\ \nu \in \mathbb{N}.$$

Hence the family defines a section

$$F \in \mathscr{F}(X)$$

satisfying for all $\nu \in \mathbb{N}$

$$\rho^X_{Q_\nu}(F) = F_\nu.$$

As a consequence $F \in \mathscr{F}(X)$ approximates $f_i \in \mathscr{F}(P_i)$ with the precision determined by the neighbourhood N_i.

iii) *Restriction* $\mathscr{F}(X) \to \mathscr{F}(Y)$: We prove that the restriction

$$\rho^X_Y : \mathscr{F}(X) \to \mathscr{F}(Y)$$

has dense image with respect to the canonical Fréchet topologies: For given

$$f \in \mathscr{F}(Y) \text{ and zero-neighbourhood } N \subset \mathscr{F}(Y)$$

one has to find a global section $F \in \mathscr{F}(X)$ such $F|Y \in \mathscr{F}(Y)$ approximates f up to N.

The proof relies on the following observation: Approximation with precision $N \subset \mathscr{F}(Y)$ can be verified by considering a single index i_0 and approximation of $f|P_{i_0} \in \mathscr{F}(P_{i_0})$ up to a certain zero-neighbourhhod $N_{i_0} \subset \mathscr{F}(P_{i_0})$. Because

$$\mathscr{F}(Y) \subset \prod_{i\in\mathbb{N}} \mathscr{F}(P_i)$$

is a Fréchet subspace and open zero-neighbourhoods in the product

$$\prod_{i\in\mathbb{N}} \mathscr{F}(P_i)$$

are determined by open zero-neighbourhoods from finitely many factors $\mathscr{F}(P_i)$. Because $(P_i)_{i\in\mathbb{N}}$ is an exhaustion of Y, those finitely many zero-neighbourhoods result from a zero neighbourhood N_{i_0} in a single factor $\mathscr{F}(P_{i_0})$.

Stated in a formal way: The Fréchet space $\mathscr{F}(Y)$ injects via the continuous map

$$\alpha : \mathscr{F}(Y) \hookrightarrow \prod_{j=0}^{\infty} \mathscr{F}(P_j),\ g \mapsto (g|P_i)_{i\in\mathbb{N}},$$

as a closed subspace K into a product of Fréchet spaces. The open mapping theorem for Fréchet spaces implies that

$$\alpha : \mathscr{F}(Y) \to K$$

is a homeomorphism. The given zero-neighborhood N maps homeomorphic to a zero-neighbourhood

$$\Psi := \alpha(N) \subset K.$$

Here Ψ restricts only finitely many factors. There exists an index $i \in \mathbb{N}$ such that Ψ contains an open zero-neighbourhood $\Psi' \subset \Psi$ of the form

$$\Psi' = \Phi \times \prod_{j=i+1}^{\infty} \mathscr{F}(P_j)$$

with

$$\Phi \subset \prod_{j=0}^{i} N_j,\ N_j \subset \mathscr{F}(P_j) \text{ open zero-neighbourhood.}$$

Because there are only finitely many factors to consider, one may assume that for $0 \leq j < i$ each restriction

$$\rho^{P_i}_{P_j} : \mathscr{F}(P_i) \to \mathscr{F}(P_j) \text{ satisfies } \rho^{P_i}_{P_j}(N_i) \subset N_j$$

- if necessary replace N_i by

$$N_i \cap \bigcap_{j=0}^{i-1} (\rho^{P_i}_{P_j})^{-1}(N_j)$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(Y) & \xrightarrow{\beta} & \prod_{j=0}^{i} \mathscr{F}(P_j) \\ \rho := \rho^{Y}_{P_i} \downarrow & \nearrow \gamma & \\ \mathscr{F}(P_i) & & \end{array}$$

with

$$\beta := [\mathscr{F}(Y) \xrightarrow{\alpha} \prod_{j=0}^{\infty} \mathscr{F}(P_j) \to \prod_{j=0}^{i} \mathscr{F}(P_j)]$$

the composition of α with the canonical projection, and

$$\gamma : \mathscr{F}(P_i) \to \prod_{j=0}^{i} \mathscr{F}(P_j),\ g \mapsto (g|P_0, ..., g|P_{i-1}, g|P_i).$$

The composition

$$\gamma \circ \rho = \beta$$

implies

$$\alpha^{-1}(\Psi') = \beta^{-1}(\Phi) = \rho^{-1}(\gamma^{-1}(\Phi))$$

The open set

$$M := \gamma^{-1}(\Phi) \subset \mathscr{F}(P_i)$$

is a zero-neighbourhood satisfying

$$\rho^{-1}(M) = \alpha^{-1}(\Psi') \subset \alpha^{-1}(\Psi) = N,$$

i.e. for each section $g \in \mathscr{F}(Y)$ holds

$$g|P_i \in M \subset \mathscr{F}(P_i) \implies g \in N \subset \mathscr{F}(Y)$$

Due to part ii) there exists $F \in \mathscr{F}(X)$ with restriction $F|P_i$ approximating $f|P_i$ up to M. As a consequence, the restriction

$$F|Y \in \mathscr{F}(Y)$$

approximates $f \in \mathscr{F}(Y)$ up to N. □

Theorem 6.22 (Theorem *B* for Stein manifolds). *On a Stein manifold $(X, \mathscr{O})$ each coherent $\mathscr{O}$-module $\mathscr{F}$ is acyclic.*

Proof. i) *Čech cohomology with respect to a Leray covering*: Proposition 6.12 provides an exhaustion of X

$$(P'_n)_{n \in \mathbb{N}}$$

by relatively compact analytic polyhedra. For each given $n \in \mathbb{N}$ exists due to Theorem 6.13 a closed embedding

$$\phi : P' := P'_n \to \Delta'$$

into a polydisc $\Delta' \subset \mathbb{C}^m$. The components of ϕ are the restrictions of global holomorphic functions on X. There exists a concentric polydisc $\Delta \subset\subset \Delta'$ with

$$\phi(P'_{n-1}) \subset \Delta \subset\subset \Delta',\ P'_{-1} := \emptyset.$$

Set

$$P_n := \phi^{-1}(\Delta).$$

Then

$$P_n \subset\subset P'_n$$

is a pair of concentric analytic polyhedra satisfying

$$P'_{n-1} \subset P_n \subset\subset P'_n$$

Therefore also the family

$$\mathscr{P} := (P_n)_{n\in\mathbb{N}}$$

is an exhaustion of X by analytic polyhedra. For each $n \in \mathbb{N}$ the pair of concentric polyhedra

$$P_n \subset\subset P'_n$$

satisfies the assumption of Proposition 6.16. Hence the restriction $\mathscr{F}|P_n$ is acyclic. Because the intersection of finitely many analytic polyhedra is again an analytic polyhedron, the family $\mathscr{P}$ is a Leray covering for $\mathscr{F}$: Leray's Theorem 3.19 implies that the cohomology of $\mathscr{F}$ on X can be computed as Čech cohomology with respect to $\mathscr{P}$:

$$H^\bullet(X,\mathscr{F}) = H^\bullet(\mathscr{P},\mathscr{F}).$$

ii) *Splitting of cocycles, local solution*: To prove the theorem we have to construct for each index $j \geq 1$ and for each cocyle

$$\xi = (\xi_n)_{n\in\mathbb{N}} \in Z^j(\mathscr{P},\mathscr{F})$$

a splitting cochain

$$\eta \in C^{j-1}(\mathscr{P},\mathscr{F}) \text{ with } \xi = \delta\eta.$$

The present step constructs a local splitting, i.e. for given $n \in \mathbb{N}$ a splitting of the family $\xi|P_n$ over the open subset $P_n \subset X$: The finite covering

$$\mathscr{P}^n := (P_i)_{i\leq n}$$

of P_n by analytic polyhedra is a Leray covering for the sheaf

$$\mathscr{F}_n := \mathscr{F}|P_n :$$

Leray's theorem shows

$$H^\bullet(P_n,\mathscr{F}_n) = H^\bullet(\mathscr{P}^n,\mathscr{F}_n),$$

in particular for all $j \geq 1$ due to Proposition 6.16

$$0 = H^j(P_n,\mathscr{F}_n) = H^j(\mathscr{P}^n,\mathscr{F}_n).$$

Due to

$$H^j(\mathscr{P}^n,\mathscr{F}_n) = 0$$

the affine space of splitting cochains is not empty

$$M_n := \left\{\eta \in C^{j-1}(\mathscr{P}^n, \mathscr{F}_n) : \ \delta\eta = \xi|P_n\right\} \neq \emptyset$$

iii) *Mittag-Leffler principle and global solution*: Due to part ii) for each $n \in \mathbb{N}$ we may choose an element $\eta_n \in M_n$. Then the affine space M_n has the form

$$M_n = \eta_n + Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n)$$

Due to Corollary 5.18 the vector space of cocycles

$$Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n) = ker\left[\delta : C^{j-1}(\mathscr{P}^n, \mathscr{F}_n) \to C^j(\mathscr{P}^n, \mathscr{F}_n)\right]$$

is a Fréchet space. The Fréchet topology on $Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n)$ induces a Fréchet topology on the spaces M_n of local solutions. The Fréchet topology is independent of the choice of η_n because switching between two translates is a topological isomorphism of the Fréchet space $C^{j-1}(\mathscr{P}^n, \mathscr{F}_n)$.

For each $n \in \mathbb{N}$ the restriction

$$M_{n+1} \to M_n$$

is linear and continuous and has dense image: Continuity follows from Proposition 5.17. To prove that the restriction has dense image, we may assume for a given pair $(n, n+1)$ that

$$\eta_n = \eta_{n+1}|U_n$$

Then we have to show: For all $j \geq 1$ the restriction

$$Z^{j-1}(\mathscr{P}^{n+1}, \mathscr{F}_{n+1}) \to Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n)$$

has dense image.

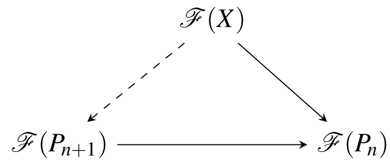
- $j = 1$: The horizontal maps of the following diagram are the canonical restrictions. The vertical maps result from the representation as affine spaces.

$$\begin{array}{ccc} Z^0(\mathscr{P}^{n+1}, \mathscr{F}_{n+1}) & \longrightarrow & Z^0(\mathscr{P}^n, \mathscr{F}_n) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathscr{F}(P_{n+1}) & \longrightarrow & \mathscr{F}(P_n) \end{array}$$

  The diagram commutes due to the choice of the pair (η_n, η_{n+1}). Due to Theorem 6.21 the restriction

$$\mathscr{F}(X) \to \mathscr{F}(P_n)$$

  has dense image. The following diagram of canonical restrictions is commutative



Hence also the restriction

$$\mathscr{F}(P_{n+1}) \to \mathscr{F}(P_n)$$

has dense image.

- $j \geq 2$: We show that the restriction

  $$Z^{j-1}(\mathscr{P}^{n+1}, \mathscr{F}_{n+1}) \to Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n)$$

  is even surjective: The vanishing

  $$H^{j-1}(\mathscr{P}^n, \mathscr{F}_n) = 0$$

  implies for a given cocycle $\zeta \in Z^{j-1}(\mathscr{P}^n, \mathscr{F}_n)$ the existence of a cochain

  $$\chi \in C^{j-2}(\mathscr{P}^n, \mathscr{F}_n) \text{ with } \delta\chi = \zeta$$

  The cochain χ extends by zero on the additional covering element P_{n+1} of $\mathscr{P}^{n+1}$ to a cochain

  $$\tilde{\chi} \in C^{j-2}(\mathscr{P}^{n+1}, \mathscr{F}_{n+1}) \text{ with } \tilde{\chi}|\mathscr{P}^n = \chi.$$

  Hence the coboundary, notably cocycle

  $$\tilde{\zeta} := \delta\tilde{\chi} \in Z^{j-1}(\mathscr{P}^{n+1}, \mathscr{F}_{n+1})$$

  restricts as

  $$\tilde{\zeta}|\mathscr{P}^n = \delta\tilde{\chi}|\mathscr{P}^n = \delta\chi = \zeta.$$

Applying to the family of restrictions

$$(M_{n+1} \to M_n)_{n\in\mathbb{N}}$$

the Mittag-Leffler principle, see Proposition 1.26, provides a compatible family of elements

$$\eta = (\tilde{\eta}_n \in M_n)_{n\in\mathbb{N}}$$

which split ξ, hereby finishing the proof. □

The proof of Theorem 6.22, part iii) makes use of the following trick: Extending a cochain by zero provides again a cochain, while the same does not hold for extending cocycles. In part i) of the proof each analytic polyhedron $P'_n,\ n \in \mathbb{N}$, provides an open neighbourhood of the closure of the polyhedron $\overline{P}_n$, which is necessary to apply Proposition 6.16.

The global result for Stein manifolds from Theorem 6.22 has been proved along the local results Corollary 5.15 for polydiscs and Proposition 6.16 for concentric analytic polyhedra.

6.3 Theorem A and further applications

The first fundamental result, Theorem A on Stein manifolds, is an easy consequence of Theorem B and of the Nakayama Lemma from Corollary 4.20.

Theorem 6.23 (Theorem A on Stein manifolds). *On a Stein manifold X with structure sheaf $\mathscr{O}$ each coherent $\mathscr{O}$-module sheaf $\mathscr{F}$ is* globally generated*, i.e. for each point $x \in X$ exist finitely many global sections*

$$f_1, ..., f_k \in \mathscr{F}(X)$$

such that their germs

$$f_{1,x}, ..., f_{k,x} \in \mathscr{F}_x$$

generate the stalk $\mathscr{F}_x$ as O_x-module.

Proof. Denote by

$$\mathscr{I} := \mathscr{I}_{\{x\}} \subset \mathscr{O}$$

the ideal sheaf of a given point $x \in X$:

$$\mathscr{I}(U) := \{f \in \mathscr{O}(U) : \ f(x) = 0 \text{ if } x \in U\},\ U \subset X \text{ open.}$$

The ideal sheaf $\mathscr{I}$ has the stalks

$$\mathscr{I}_y = \begin{cases} \mathfrak{m}_x & \text{if } y = x \\ \mathscr{O}_y & \text{if } y \neq x \end{cases}$$

Here

$$\mathfrak{m}_x \subset \mathscr{O}_x$$

is the maximal ideal of the local ring of convergent power series. The ideal sheaf $\mathscr{I}$ is coherent due to Proposition 4.29. The subsheaf

$$\mathscr{I} \cdot \mathscr{F} \subset \mathscr{F}$$

is finitely generated, hence a coherent submodule of the coherent $\mathscr{O}$-module $\mathscr{F}$. It fits into the exact sequence of sheaves

$$0 \to \mathscr{I} \cdot \mathscr{F} \to \mathscr{F} \to \mathscr{F} / \mathscr{I} \cdot \mathscr{F} \to 0$$

The quotient sheaf has the stalks

$$(\mathscr{F} / \mathscr{I} \cdot \mathscr{F})_y = \begin{cases} \mathscr{F}_x / \mathfrak{m}_x \mathscr{F}_x & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

which implies

$$H^0(X, \mathscr{F} / \mathscr{I} \cdot \mathscr{F}) \cong \mathscr{F}_x / \mathfrak{m}_x \cdot \mathscr{F}_x$$

Theorem *B*, see Theorem 6.22, implies

$$H^1(X, \mathscr{I} \cdot \mathscr{F}) = 0$$

Hence the long exact cohomology sequence has the segment

$$H^0(X, \mathscr{F}) \to \mathscr{F}_x / \mathfrak{m}_x \cdot \mathscr{F}_x \to 0$$

which shows the surjectivity of the evaluation

$$\mathscr{F}(X) = H^0(X, \mathscr{F}) \to \mathscr{F}_x / \mathfrak{m}_x \cdot \mathscr{F}_x$$

One applies the Nakayama lemma, Corollary 4.20, with

$$M := \mathscr{F}_x,\ N := span_{\mathscr{O}_x} < f_{1,x}, ..., f_{k,x} >$$

for a finite family $(f_j)_j$ of elements from $\mathscr{F}(X)$ such that the classes $(\overline{f}_{j,x})_j$ generate $\mathscr{F}_x / \mathfrak{m}_x \cdot \mathscr{F}_x$. The lemma implies that the family $(f_{j,x})_j$ generates the stalk $\mathscr{F}_x$ as $\mathscr{O}_x$-module. □

Note. In the end of the proof of Theorem 6.23 the quotient

$$\mathscr{F}_x / (\mathfrak{m}_x \cdot \mathscr{F}_x) = \mathscr{F}_x \otimes_{\mathscr{O}_x} (\mathscr{O}_x / \mathfrak{m}_x)$$

is a finite-dimensional vector space over the field $\mathbb{C} \simeq \mathscr{O}_x / \mathfrak{m}_x$. The Nakayama lemma lifts the surjectivity from the level of finite-dimensional complex vector spaces to generators at the level of finitely generated $\mathscr{O}_x$-modules.

Theorem 6.24 (Characterization of Stein manifolds). *For a complex manifold X are equivalent:*

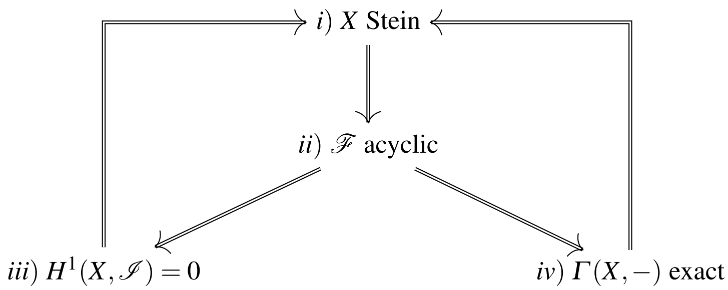
i) The manifold X is a Stein manifold according to Definition 6.10.

ii) Each coherent $\mathscr{O}_X$-module is acyclic.

iii) Each coherent ideal sheaf $\mathscr{J} \subset \mathscr{O}_X$ satisfies $H^1(X, \mathscr{J}) = 0$.

iv) The section-functor $\Gamma(X,-)$ is exact on short exact sequences of coherent $\mathscr{O}_X$-modules.

Proof. We prove the implications according to the following diagram



$i) \implies ii)$: Theorem B, see Theorem 6.22.

$ii) \implies iii)$: trivial

$ii) \implies iv)$: The long exact cohomology sequence from Theorem 3.12.

$iii) \implies i)$: We verify the three conditions from Definition 6.10:

- *Holomorphic separability*: For each pair of distinct points $x \neq y \in X$ consider the zero-dimensional analytic submanifold
$$A = \{x,y\} \subset X.$$
Its ideal sheaf $\mathscr{I}_A \subset \mathscr{O}_X$ is coherent due to Proposition 4.29. By assumption
$$H^1(X,\mathscr{I}_A) = 0.$$
Hence the short exact sequence
$$0 \to \mathscr{I}_A \to \mathscr{O}_X \to \mathscr{O}_A \to 0$$
induces the epimorphism
$$H^0(X,\mathscr{O}_X) \to H^0(X,\mathscr{O}_A)$$
Due to the isomorphy
$$H^0(X,\mathscr{O}_A) \xrightarrow{\simeq} \mathbb{C} \oplus \mathbb{C},\ f \mapsto (f(x), f(y)),$$
the surjectivity
$$H^0(X,\mathscr{O}_X) \to H^0(X,\mathscr{O}_A)$$

implies that X is holomorphically separable.

- *Holomorphic convexity*: For an indirect proof assume that X is not holomorphically convex. Proposition 6.3 provides the existence of an infinite closed discrete set $A \subset X$, such that each holomorphic function $f \in \mathscr{O}(X)$ stays bounded on A. W.l.o.g. A is countable. Due to its discreteness the closed set $A \subset X$ is an analytic submanifold of X. Hence its ideal sheaf $\mathscr{I}_A \subset \mathscr{O}_X$ is coherent due to Proposition 4.29. By assumption

$$H^1(X, \mathscr{I}_A) = 0$$

Hence the short exact sequence

$$0 \to \mathscr{I}_A \to \mathscr{O}_X \to \mathscr{O}_A \to 0$$

induces the epimophism

$$H^0(X, \mathscr{O}_X) \to H^0(X, \mathscr{O}_A) = \mathbb{C}^{\mathbb{N}}.$$

As a consequence, there exists a holomorphic function $f \in \mathscr{O}(X)$ which is unbounded on A, a contradiction.

- *Local uniformizability*: Consider a given point $a \in X$ and choose in a suitable neighbourhood U of a a chart of X

$$f = (f_1, ..., f_n) : U \to V \subset \mathbb{C}^n,\ n = dim\, X.$$

Then

$$A := \{a\} \subset X$$

is an analytic submanifold of X. Denote by $\mathscr{I}_A$ its ideal sheaf. It is coherent, and also the product $\mathscr{I}_A^2 \subset \mathscr{O}_X$ is a coherent ideal sheaf. By assumption

$$H^1(X, \mathscr{I}_A^2) = 0$$

Hence the short exact sequence

$$0 \to \mathscr{I}_A^2 \to \mathscr{O}_X \to \mathscr{O}_X / \mathscr{I}_A^2 \to 0$$

induces the epimorphism

$$H^0(X, \mathscr{O}_X) \to H^0(X, \mathscr{O}_X / \mathscr{I}_A^2).$$

As a consequence, for each $j = 1, ..., n$ exists a holomorphic function $F_j \in \mathscr{O}(X)$ with

$$F_{j,a} = \overline{f}_j \in H^0(X, \mathscr{O}_X / \mathscr{I}_A^2) \simeq \mathbb{C}\{z_1, ..., z_n\} / \mathfrak{m}^2$$

Due to Proposition 2.6 the map

$$F := (F_1, ..., F_n) : X \to \mathbb{C}^n$$

is a local isomorphism at a. Hence X is locally uniformizable.

iv) $\implies$ *i*): Alike to the previous implication *iii*) $\implies$ *i*), replacing the assumption

$$H^1(X, \mathscr{I}_A) = 0 \text{ resp. } H^1(X, \mathscr{I}_A^2) = 0$$

by the exactness of $\Gamma(X, -)$ provides the same epimorphisms on the level of sections, which prove the three requirements for X being a Stein manifold. □

A second application of Theorem B proves that each additive Cousin problem on a Stein manifold X has a solution. Possible obstructions against a solution of a multiplicative Cousin problem are of a topological - not complex analytic - nature.

Corollary 6.25 (Cousin problems on Stein manifolds). *Consider a Stein manifold X.*

1. *Each additive Cousin problem on X is solvable.*
2. *If $H^2(X, \mathbb{Z}) = 0$ then each multiplicative Cousin problem on X is solvable.*

Proof. Theorem 6.22 shows that the criteria from Corollary 3.24 are satisfied. □

A further application of Theorem B is a far reaching generalization of the Kugelsatz from by Corollary 1.15.

Corollary 6.26 (General Kugelsatz). *Consider a domain $G \subset \mathbb{C}^n$, $n \geq 2$, and a compact set $K \subset G$ with*

$$G \setminus K$$

connected. Then each holomorphic function $f \in \mathscr{O}(G \setminus K)$ extends uniquely to a holomorphic function $\tilde{f} \in \mathscr{O}(G)$.

Proof. i) *Translation to a problem in cohomology*: Consider the covering of $\mathbb{C}^n$ by two open sets

$$\mathscr{U} := (U_1 := G, U_2 := \mathbb{C}^n \setminus K)$$

Then

$$U_1 \cap U_2 = G \setminus K$$

each holomorphic function $f \in \mathscr{O}(G \setminus K)$ determines a 1-cocyle from $Z^1(\mathscr{U}, \mathscr{O})$. Because $\mathbb{C}^n$ is a Stein manifold we have

$$H^1(\mathbb{C}^n, \mathscr{O}) = 0$$

Due to Corollary 3.8

$$H^1(\mathbb{C}^n, \mathscr{O}) = 0 \implies H^1(\mathscr{U}, \mathscr{O}) = 0,$$

which provides a cochain $(f_1, f_2) \in C^0(\mathscr{U}, \mathscr{O})$ satisfying

$$f = f_2 - f_1$$

ii) *Applying the classical Kugelsatz*: The function f_1 is defined on all of G. Hence the aim is to extend

$$g := f_2 \in \mathscr{O}(\mathbb{C}^n \setminus K)$$

to a holomorphic function $\tilde{g} \in \mathscr{O}(\mathbb{C}^n)$ and then to restrict as $\tilde{g}|G$. Due to compactness of K we may choose an open polydisc $\Delta \subset \mathbb{C}^n$ with

$$K \subset \overline{\Delta}$$

and with a point

$$x_0 \in K \cap \partial\Delta$$

Hence there exists a non-empty open

$$U \subset (G \setminus \overline{\Delta}) \subset (G \setminus K),$$

see Figure 6.5.

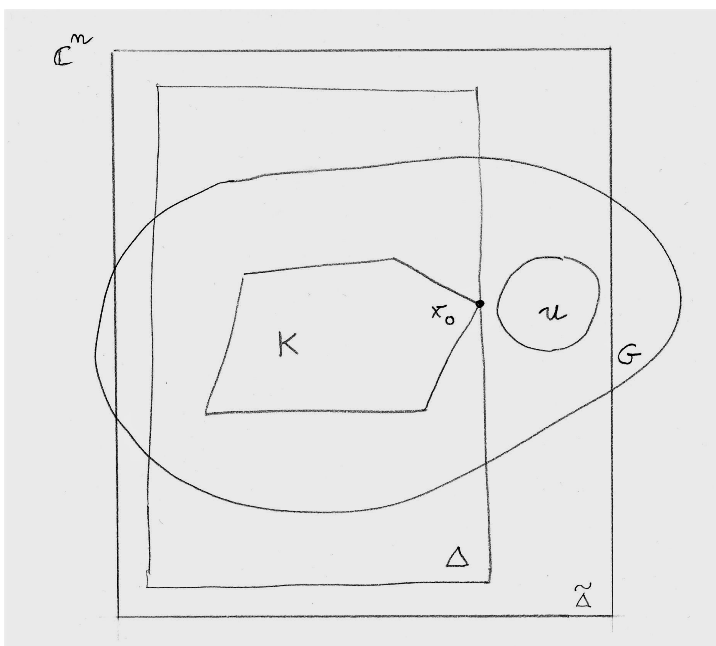


Fig. 6.5 $U \subset (G \setminus \overline{\Delta}) \subset (G \setminus K)$

The classical Kugelsatz, Corollary 1.15, applies for the pair of polydiscs $(\tilde{\Delta}, \Delta)$ with

$$\Delta \subset\subset \tilde{\Delta} \subset\subset \mathbb{C}^n \text{ and } U \subset \tilde{\Delta}$$

It extends $g|(\tilde{\Delta} \setminus \overline{\Delta})$ to a holomorphic function on $\tilde{\Delta}$, which then combines with $g \in \mathscr{O}(\mathbb{C}^n \setminus K)$ to an extension $\tilde{g} \in \mathscr{O}(\mathbb{C}^n)$ of g. Equality on U

$$\tilde{g}|U = g|U \text{ and connectedness of } G \setminus K$$

imply

$$\tilde{g}|G = g|G$$

Hence

$$f = (f_1 - f_2)|(G \setminus K) = (f_1 - \tilde{g})|(G \setminus K)$$

is the restriction of the holomorphic function

$$f_1 - (\tilde{g}|G) \in \mathscr{O}(G).$$

□

Proposition 6.27 (Analytic submanifolds of Stein manifolds). *Consider a Stein manifold X and an analytic submanifold of $Y \subset X$ with ideal sheaf $\mathscr{I}_Y \subset \mathscr{I}_Y$. If*

$$I(Y) \subset \mathscr{O}(X)$$

denotes the ideal of all global holomorphic functions on X which vanish on Y, then

$$Y = \{x \in X : \ f(x) = 0 \text{ for all } f \in I(Y)\},$$

i.e. Y is the "variety" of the ideal $I(Y)$.

Proof. We only show the non-trival inclusion. Claim: If there exists $x \notin Y$ then exists a function $f \in \mathscr{O}(X)$ satisfying

$$f(x) \neq 0 \text{ and } f|Y = 0$$

Then

$$I(Y) = \mathscr{I}_Y(X) = H^0(X, \mathscr{I}_Y)$$

If $x \in X \setminus Y$ then $\mathscr{I}_{Y,x} = \mathscr{O}_{X,x}$. Theorem *A*, see Theorem 6.23, provides a global function $f \in H^0(X, \mathscr{I}_A)$ such that the germ f_x generates $\mathscr{O}_x$ as $\mathscr{O}_x$-module. In particular $f(x) \neq 0$. □

Proposition 6.28 (Global extension of functions from analytic submanifolds of Stein manifolds). *Consider a Stein manifold X and an analytic submanifold $Y \subset X$. For each holomorphic function $f \in \mathscr{O}_Y(Y)$ exists a holomorphic function*

$$F \in \mathscr{O}_X(X) \text{ with } F|Y = f.$$

Proof. The short exact sequence

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \mathscr{O}_Y \to 0$$

induces the exact sequence

$$H^0(X, \mathscr{O}_X) \to H^0(Y, \mathscr{O}_Y) \to H^1(X, \mathscr{I}_Y) = 0,$$

see Theorem 6.24. □

Theorem 6.29 (Hilbert's Nullstellensatz). *Consider a Stein manifold X and finitely many global holomorphic functions $f_1, ..., f_k \in \mathscr{O}(X)$ without a common zero $x \in X$. Then exist holomorphic functions*

$$\alpha_1, ..., \alpha_k \in \mathscr{O}(X) \text{ with } \sum_{j=1}^{k} \alpha_j \cdot f_j = 1$$

Proof. Consider the free $\mathscr{O}$-module $\mathscr{O}^k$ and its canoncial basis $(e_j)_{j=1,.,,,k}$ of global sections. The induced morphism of $\mathscr{O}$-modules

$$\alpha : \mathscr{O}^k \to \mathscr{O},\ e_j \mapsto f_j,\ j = 1, ..., k,$$

is surjective due to the assumption that the functions $f_1, ..., f_k$ have no common zero. The kernel

$$\mathscr{K} := ker\,[\alpha : \mathscr{O}^k \to \mathscr{O}]$$

is coherent. It fits into the exact sequence

$$0 \to \mathscr{K} \to \mathscr{O}^k \xrightarrow{\alpha} \mathscr{O} \to 0$$

Theorem 6.22 implies $H^1(X, \mathscr{K}) = 0$. Hence the sequence

$$H^0(X, \mathscr{O}^k) \xrightarrow{\alpha_X} H^0(X, \mathscr{O}) \to 0$$

is exact, which proves the claim. □

Remark 6.30 (Hilbert's Nullstellensatz).

1. In Algebraic Geometry Hilbert's Nullstellensatz (weak version) states: Consider an algebraically closed field k. Each proper ideal $I \subsetneq k[X_1, ..., k_n]$ of polynomials has a non-empty variety

$$Var(I) := \{x \in \mathbb{A}^n :\ f(x) = 0 \text{ for all } f \in I\} \neq \emptyset$$

The Nullstellensatz compares an algebraic property with a geometric property.

2. The weak form is equivalent to the statement: For any family

$$f_1,...,f_k \in k[X_1,...,X_n]$$

of polynomials without a common zero exist polynomials $\alpha_1,...,\alpha_k \in k[X_1,...,X_n]$ satisfying

$$\sum_{j=1}^{k} \alpha_j \cdot f_j = 1$$

Proof: Set $I := < f_1,...,f_k >$.

1) $\implies$ *2)*: If $(f_1,...,f_k)$ have no common zero, the ideal

$$I := < f_1,...,f_k > \subset k[X_1,...,X_n]$$

satisfies $V(I) = \emptyset$. Part 1) implies:

$$I = k[X_1,...,X_n], \text{ in particular } 1 \in I.$$

2) $\implies$ *1)*: Proof by contraposition. Assume $V(I) = \emptyset$. Then $(f_1,...,f_k)$ have no common zero. Part 2) implies a representation

$$1 = \sum_{j=1}^{k} \alpha_j \cdot f_j$$

Hence $1 \in I$ which implies $I = k[x_1,...,X_n]$.

3. Hence Theorem 6.29 is the analogue of the weak Nullstellensatz for ideals of global holomorphic functions on Stein manifolds.

Theorem 6.31 (Holomorphic de Rham theorem). *Consider a Stein manifold X and denote by*

$$Rh^j_\omega(X) := \frac{ker\ [\Omega^j(X) \xrightarrow{d} \Omega^{j+1}(X)]}{im\ [\Omega^{j-1}(X) \xrightarrow{d} \Omega^j(X)]},\ j \in \mathbb{N},$$

the holomorphic de Rahm groups of X. Then for all $j \in \mathbb{N}$

$$Rh^j_\omega(X) \simeq H^j(X,\mathbb{C})$$

Proof. Poincarè's Lemma for the d-operator for differential forms on an n-dimensional complex manifold X shows the exactness of the following sequence of sheaf morphisms

$$0 \to \mathbb{C} \to \mathscr{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} ... \xrightarrow{d} \Omega^n \to 0$$

Each $\mathscr{O}$-module is locally free of finite rank due to the local representation of differential forms in coordinates. In particular, each $\Omega^j,\ j \in \mathbb{N}$, is coherent. Theorem 6.22 shows $H^1(X,\Omega^j) = 0,\ j \in \mathbb{N}$, and the abstract de Rahm theorem, Theorem 3.16, proves the claim. □

Corollary 6.32 (Cohomology of Stein manifolds). *For an n-dimensional Stein manifold X holds for all* $j > n$

$$H^j(X,\mathbb{C}) = 0$$

Proof. For $j > n$ holds $\Omega^j = 0$. Hence Theorem 6.31 proves the claim. □

Chapter 7
Outlook

7.1 Stein manifolds

Remark 7.1 (Embedding of Riemann surfaces). Consider a Riemann surface X.

1. For compact X there exists a projective embedding

$$X \hookrightarrow \mathbb{P}^3$$

   The proof uses the existence of a very ample line bundle $\mathscr{L} \in Pic(X)$ which defines an embedding

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$

   The existence of $\mathscr{L}$ is implied by the Riemann-Roch theorem and makes use of Serre duality, cf. [32, Chap. 11]. In case $n > 3$ it is possible to project the image

$$\phi_{\mathscr{L}}(X) \subset \mathbb{P}^n$$

   injectively into $\mathbb{P}^3 \subset \mathbb{P}^n$, see [18, Chap. IV, Cor. 3.6].

2. An open Riemann surface is a Stein manifold, see [32, Chap. 15]. Due to Theorem 7.2 it embeds as analytic submanifold into an affine spcae $\mathbb{C}^3$.

Theorem 7.2 (Affine embedding of Stein manifolds). *Each n-dimensional Stein manifold X embeds as analytic submanifold into an affine space*

$$X \hookrightarrow \mathbb{C}^m.$$

For $n \geq 2$ *one may choose* $m = 2n$*, while for* $n = 1$ *the choice* $m = 3$ *is possible.*

The theorem is due to Remmert, see [20, Theor. 57.4].

Complex analysis on a Stein manifold X with structure sheaf $\mathscr{O}$ often deals with the global holomorphic functions on X, i.e. with the elements of the $\mathbb{C}$-algebra

$$\mathscr{O}(X) = \Gamma(X, \mathscr{O})$$

Provided with the Fréchet topology of compact convergence, $\mathscr{O}(X)$ becomes a topological $\mathbb{C}$-algebra. For a Stein manifold X the relation between the manifold and the topological algebra is very close: It is formalized by the *spectrum* of $\mathscr{O}(X)$.

Definition 7.3 (Spectrum of a topological $\mathbb{C}$-algebra). Consider a topological $\mathbb{C}$-algebra A with unity.

1. A *character* of A is a continuous morphism of $\mathbb{C}$-algebras with unit

$$\chi : A \to \mathbb{C}$$

2. The *spectrum* of A is the set of all characters of A

$$S(A) := \{\chi : \text{character of } A\}$$

3. Each element $f \in A$ defines on $S(A)$ the evaluation map

$$\hat{f} : S(A) \to \mathbb{C},\ \chi \mapsto \hat{f}(\chi) := \chi(f).$$

   One provides the set $S(A)$ with the corresponding *initial topology*, i.e. with the coarsest topology such that

$$(\hat{f} : S(A) \to \mathbb{C})_{f \in A}$$

   becomes a family of continuous maps.

Note that $S(A) \subset \mathbb{C}^A$ is a subspace of a product of the Hausdorff spaces $\mathbb{C}$, hence the spectrum $S(A)$ is a Hausdorff space itself.

Theorem 7.4 (Character theorem for Stein manifolds). *Consider a complex manifold X and denote by*

$$A := \Gamma(X, \mathscr{O})$$

the topological $\mathbb{C}$-algebra of its global holomorphic functions provided with the canonical Fréchet topology. Then are equivalent:

1. The manifold X is a Stein manifold.

2. The canonical map

$$\varepsilon : X \to S(A),\ x \mapsto \varepsilon_x \in S(A),$$

   defined as the point character

$$\varepsilon_x : A \to \mathbb{C},\ \varepsilon_x(f) := f(x),$$

   is a homeomorphism.

For a proof of Theorem 7.4 see [2, Anhang Kap. VI, Satz 7] and [20, Theor. 57.3]. The injectivity of ε is equivalent to X being holomorphically separable. The difficult part is to show the surjectivity of ε, i.e. to prove for a Stein manifold X that each character of $\Gamma(X,\mathscr{O})$ is a point character.

A further topic deals with *duality theorems* on Stein manifolds. The prototype of a duality theorem is Serre's theorem on a compact Riemann surface:

Proposition 7.5 (Duality on compact Riemann surfaces). *On a compact Riemann surface X denote by*

$$\omega := \Omega$$

the sheaf of holomorphic differential forms. For each line bundle on X with invertible sheaf $\mathscr{L}$ of holomorphic sections the bilinear composition of finite-dimensional complex vector spaces

$$H^j(X,\mathscr{L}) \times H^{1-j}(X,\mathscr{L}^\vee \otimes_{\mathscr{O}} \omega) \to H^1(X,\omega) \simeq \mathbb{C},\ j=0,1,$$

is a pairing. In particular,

$$H^j(X,\mathscr{L})^\vee \simeq H^{1-j}(X,\mathscr{L}^\vee \otimes_{\mathscr{O}} \omega)$$

Here

$$\mathscr{L}^\vee := \mathscr{H}om_{\mathscr{O}}(\mathscr{L},\mathscr{O})$$

denotes the dual sheaf of $\mathscr{L}$, corresponding to the holomorphic sections of the dual line bundle of L.

For a proof see [32, Chap. 9].

Proposition 7.5 generalizes to compact manifolds of dimension $n \geq 1$. The dualizing sheaf ω becomes the sheaf of holomorphic differential forms of highest degree

$$\omega := \Omega^n.$$

If one replaces the locally free sheaf $\mathscr{L}$ by an arbitrary coherent sheaf $\mathscr{F}$ then one has to take into account also the *Ext*-groups $Ext^j(\mathscr{F},\omega)$.

Moreover, for a non-compact manifold X the cohomology groups with values in a coherent $\mathscr{O}$-module $\mathscr{F}$

$$H^q(X,\mathscr{F})$$

are no longer finite-dimensional in general. As a consequence, to obtain duality results one has to introduce topologies and to consider topological duals and to consider cohomology with compact support

$$H^q_c(X,\mathscr{F}).$$

Proposition 7.6 (Cohomology with compact support). *Consider an n-dimensional Stein manifold X and a coherent $\mathscr{O}$-module $\mathscr{F}$. Then*

$$H_c^q(X,\mathscr{F}) = 0 \textit{ for } q > n$$

For the proof see [3, Chap. I, Lem. 2.5]. One uses

$$H_c^q(X,\mathscr{F}) = \varinjlim_{k} H_c^q(P_k,\mathscr{F}|P_k)$$

for an exhaustion of X by a sequence of analytic polyhedra $P_k \subset\subset X$. Over each analytic polyhedron $P \subset X$ one has a finite resolution of $\mathscr{F}$ by free $\mathscr{O}$-modules of finite rank. The proof of the claim follows from the Dolbeault resolution of $\mathscr{O}$

$$0 \to \mathscr{O} \to \mathscr{E}^{0,0} \xrightarrow{d''} \mathscr{E}^{0,1} \xrightarrow{d''} \dots \xrightarrow{d''} \mathscr{E}^{0,n} \to 0$$

by tensoring over $\mathscr{O}$ with $\mathscr{F}$.

For a proof of Proposition 7.6 see [3, Chap. I, Lem. 2.5].

Theorem 7.7 is a first example from duality theory of coherent sheaves on non-compact complex manifolds.

Theorem 7.7 (Duality on Stein manifolds). *Consider an n-dimensional Stein manifold X with structure sheaf $\mathscr{O}$ and denote by*

$$\omega := \Omega^n$$

the sheaf of holomorphic differential forms on X of highest order. For each coherent $\mathscr{O}$-module $\mathscr{F}$ exists a Fréchet topology on the Ext-groups

$$Ext^j_{\mathscr{O}}(\mathscr{F},\omega),\ j \in \mathbb{N},$$

such that their dual space of linear continuous functionals is algebraically isomorphic to the cohomology groups of $\mathscr{F}$ with compact support:

$$Ext^{n-j}_{\mathscr{O}}(\mathscr{F},\omega)' \simeq H_c^j(X,\mathscr{F})$$

For a proof of Theorem 7.7 see [3, Chap. I, Theor. 2.1].

Theorem 7.8 shows: Being a Stein manifold X has remarkable consequences for the homology of X.

Theorem 7.8 (Stein manifold as CW-complex). *Each n-dimensional Stein manifold X is homotopy equivalent to a CW-complex of real dimension $\leq n$. A consequence is the vanishing of higher homology groups:*

$$H_q(X, \mathbb{Z}) = 0 \text{ if } q > n.$$

Moreover, the homology group $H_n(X, \mathbb{Z})$ of highest dimension is free.

The theorem is due to Andreotti and Frankel, see also [21] and [10, Cor. 3.12.2].

7.2 Stein theory with singularities

The concept of complex manifold generalizes to the concept of a *complex space*. A complex space is more general than a complex manifold in two aspects:

- The local models of a complex space build not necessarily on open subsets in $\mathbb{C}^n$, but on analytic subsets of open subsets in $\mathbb{C}^n$. Analytic sets may have singularities, the local rings $\mathscr{O}_{A,x}, x \in A$, are quotients of rings of convergent power series.

- The holomorphic functions of a local model are not necessarily determined by their values on the analytic set, but may be nilpotent. Nilpotent holomorphic functions may be zero on open sets without defining the zero germs.

For a short introduction to complex spaces, written in the spirit of local models, see the first pages of [5].

Definition 7.9 (Complex space).

1. The *local model* of a complex space is a pair $(A, \mathscr{O}_A)$ with an analytic set

$$A \subset V,\ V \subset \mathbb{C}^n \text{ open},$$

and on A a sheaf of rings

$$\mathscr{O}_A := \mathscr{O}_V / \mathscr{J}_A$$

with a coherent sheaf of ideals

$$\mathscr{I}_A \subset \mathscr{O}_V \text{ with } supp\ \mathscr{O}_A \subset A,$$

i.e. the stalks satisfy

$$\mathscr{O}_{A,x} = 0 \text{ for all } x \in V \setminus A.$$

The sheaf $\mathscr{I}_A$ is named the *ideal sheaf* of the local model, while $\mathscr{O}_A$ is named its *structure sheaf*.

2. A *complex space* is a pair $(X, \mathscr{O}_X)$ with a Hausdorff space X and a sheaf of rings $\mathscr{O}_X$ on X satisfying the following property: For each point $x \in X$ exists an open neighbourhood $U \subset X$ and a local model $(A, \mathscr{O}_A)$ with an isomorphism of $\mathbb{C}$-ringed spaces
$$(U, \mathscr{O}_X|U) \simeq (A, \mathscr{O}_A),$$
see [5, Chap. 0.14].

For the local model $(A, \mathscr{O}_A)$ each stalk $\mathscr{O}_{A,x},\ x \in A$, is a local $\mathbb{C}$-algebra. A morphism
$$f : A \to B$$
between two $\mathbb{C}$-algebras has to map the maximal ideals, i.e.
$$f(\mathfrak{m}_A) \subset \mathfrak{m}_B.$$

Definition 7.10 (Nilradical and reduction of a complex space).

1. For a complex space $(X, \mathscr{O})$ the *nilradical*
$$\mathscr{N} \subset \mathscr{O}_X$$
is the sheafification of the presheaf on X
$$\{f \in \mathscr{O}_X(U) :\ f^k = 0 \text{ for a suitable } k \in \mathbb{N}\},\ U \subset X \text{ open}$$
2. The complex space $(X, \mathscr{O}_X)$ is *reduced* if $\mathscr{N} = 0$.
3. The *reduction* of a complex space $(X, \mathscr{O}_X)$ is the reduced complex space $(X, \mathscr{O}_X/\mathscr{N})$.

The nilradical $\mathscr{N} \subset \mathscr{O}_X$ of a complex space is a coherent $\mathscr{O}_X$-module.

The definition of a Stein space is literally the same as the classical Definition 6.10 for Stein manifolds.

Definition 7.11 (Stein space). A complex space $(X, \mathscr{O}_X)$ with second countable topology is a *Stein space* if it satisfies all of the following properties:

1. *Holomorphically convex*: For each compact $K \subset X$ the holomorphically convex hull
$$\hat{K} := \bigcap_{f \in \mathscr{O}(X)} \{x \in X :\ |f(x)| \le \|f\|_K\}$$
is compact.

2. *Holomorphically separable*: For each pair of points $x, y \in X$ exists a global holomorphic function $f \in \mathscr{O}(X)$ with $f(x) \neq f(y)$.

3. *Locally holomorphically uniformizable*: Each point $x \in X$ has an open neighbourhood $U \subset X$ and finally many global holomorphic holomorphic functions

$$f_1, ..., f_k \in \mathscr{O}(X)$$

such that the restriction

$$(f_1, ..., f_k)|U : U \to A \subset \Delta$$

induces an isomorphism of $(U, \mathscr{O}_X|U)$ onto a local model $(A, \mathscr{O}_A)$ with an analytic subset $A \subset \Delta$ in a polydisc $\Delta \subset \mathbb{C}^k$.

Due to Grauert's simplification of the definition: A complex space X is already a Stein space if X is holomorphically convex and holomorphically spreadable, compare Remark 6.11.

Proposition 7.12 (Reduction and normalization of a Stein space).

- *A complex space X is a Stein space iff its reduction X^{red} is a Stein space.*
- *A reduced complex space X is a Stein space iff its normalization X^{norm} is a Stein space.*

For the proof see [20, Prop. 52.19] and [5, Chap. 2.32, Cor.]

7.3 Affine schemes

Grothendieck showed how to geometrize algebra: Each commutative ring with unity can be represented as a topological space equipped with a canonical structure sheaf. The resulting pair is named the *affine scheme* of the ring. All commutative rings in the following will be assumed to have a unit.

Definition 7.13 (Affine scheme). Consider a commutative ring A.

- *Spectrum Spec A*: Denote by

$$Spec\, A := \{\mathfrak{p} \subset A : \ \mathfrak{p} \text{ prime ideal}\}$$

the set of prime ideals of A.

- *Zariski topology on Spec A*: For each ideal $\mathfrak{a} \subset A$ denote by

$$V(\mathfrak{a}) := \{\mathfrak{p} \in Spec\, A : \mathfrak{a} \subset \mathfrak{p}\}$$

the set of prime ideals lying over $\mathfrak{a}$, i.e. containing $\mathfrak{a}$. For an element $f \in A$ denote by

$$D_f := \{\mathfrak{p} :\ f \notin \mathfrak{p}\}$$

the set of prime ideals which do not contain f. The *Zariski topology* of set *Spec A* has the closed sets

$$V(\mathfrak{a}),\ \mathfrak{a} \subset A,$$

respectively the base of open sets

$$D_f,\ f \in A.$$

- *Sheaves on Spec A*: For each basic open set $D(f) \subset Spec\, A$ denote by A_f the localization of A by the multiplicatively closed set $\{f^n :\ n \in \mathbb{N}\}$. The attachment

$$D(f) \mapsto A_f, f \in A,$$

defines the *structure sheaf* $\mathscr{O}_{Spec\, A}$ on *Spec A*. More general, for each A-module M the attachement

$$D(f) \mapsto M_f := M_{\{f^n:\ n \in \mathbb{N}\}} = A_f \otimes_A M,\ f \in A,$$

defines the $\mathscr{O}$-module sheaf $\tilde{M}$ on *Spec A*.

- *Stalks*: The structure sheaf $\mathscr{O} = \mathscr{O}_{Spec\, A}$ of *Spec A* has at a point $\mathfrak{p} \in Spec\, A$ the stalk

$$\mathscr{O}_{\mathfrak{p}} = A_{\mathfrak{p}},$$

the localization of A by the multiplicatively closed set $A \setminus \mathfrak{p}$. The stalk $\mathscr{O}_{\mathfrak{p}}$ is a local ring. The quotient field

$$k(\mathfrak{p}) := \mathscr{O}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}},$$

with $\mathfrak{m}_{\mathfrak{p}} \subset \mathscr{O}_{\mathfrak{p}}$ the maximal ideal, is named the *residue field* at the point $\mathfrak{p} \in Spec\, A$.

Moreover for each A-module M and each point $\mathfrak{p} \in X$ the sheaf $\tilde{M}$ on *Spec A* has the stalk

$$\tilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}},$$

the localisation with respect to $A \setminus \mathfrak{p}$.

- *Coherent $\mathscr{O}$-modules*: Each A-module M can be recovered from the $\mathscr{O}$-module $\tilde{M}$ according to

$$\Gamma(Spec\, A, \tilde{M}) = M,$$

in particular

$$\Gamma(X, \mathscr{O}_X) = A.$$

The $\mathscr{O}$-modules of the form $\tilde{M}$ are named *quasi-coherent*. If M is even finitely generated then $\tilde{M}$ is named *coherent*.

- *Affine scheme*: An *affine scheme* is a pair of the form $X = (Spec\,A, \mathscr{O}_{Spec\,A})$.

Remark 7.14 (Affine scheme).

1. *Modules and quasicoherent sheaves*: Consider a fixed affine scheme $(Spec\,A, \mathscr{O}_{Spec\,A})$. The canonical covariant functor

$$A\text{-}\underline{mod} \to \mathscr{O}_{Spec\,A}\text{-}\underline{mod}$$

$$M \mapsto \tilde{M} \text{ and } (g : M_1 \to M_2) \mapsto (\tilde{g} : \tilde{M}_1 \to \tilde{M}_2)$$

from the category of A-modules to the category of $\mathscr{O}_{Spec\,A}$-modules is fully faithful, i.e. bijective on morphisms. The functor maps exact sequences to exact sequences. See [18, Chap. II, Prop. 5.2].

2. *Change of rings and change of affine schemes*: The pullback of a prime ideal along a morphism of commutative rings is again a prime ideal: Consider a morphism

$$\phi : A \to B$$

of commutative rings. Then for each prime ideal $\mathfrak{p} \subset B$ the inverse image

$$\phi^{-1}(\mathfrak{p}) \subset A$$

is also a prime ideal. As a consequence, each morphism between two commutative rings induces a morphism in the opposite direction between their affine schemes.

There exists a contravariant functor from the category of commutative rings to the category of locally ringed spaces:

- *Mapping objects*:

$$A \mapsto (Spec\,A, \mathscr{O}_{Spec\,A})$$

- *Mapping morphisms*:

$$(\phi : A \to B) \mapsto ((f, f^\sharp) : (Spec\,B, \mathscr{O}_{Spec\,B}) \to (Spec\,A, \mathscr{O}_{Spec\,A}))$$

Here $f : Spec\,B \to Spec\,A$ is the pullback of prime ideals. And

$$f^\sharp : \mathscr{O}_{Spec\,A} \to f_* \mathscr{O}_{Spec\,B}$$

is defined on sections over basis open sets as

$$\mathscr{O}_{Spec\ A}(D_g) = A_g \mapsto f_*\mathscr{O}_{Spec\ B}(D_g) := \mathscr{O}_{Spec\ B}(f^{-1}(D_g)) = \mathscr{O}_{Spec\ B}(D_{\phi(g)}) = B_{\phi(g)}$$

The functor is a full functor, i.e. it is surjective on morphisms, see [18, Chap II, Prop. 2.3].

Remark 7.15 (Cohomology of affine schemes). Consider an affine scheme

$$X = (Spec\ A, \mathscr{O}_{Spec\ A})$$

with a Noetherian ring A and an A-module M.

1. *Injective resolution*: There exists an injective resolution of A-modules

$$0 \to M \to I^0 \to I^1 \to ...$$

   see [18, Chap. III, Prop. 2.1].

2. *Flasque resolution*: The exactness of the functor from Remark 7.14 induces the resolution of $\tilde{M}$

$$0 \to \tilde{M} \to \tilde{I}^0 \to \tilde{I}^1 \to ...$$

   All $\mathscr{O}_X$-modules $\tilde{I}^j,\ j \in \mathbb{N}$, are flasque, see [18, Chap. III, Prop. 3.4], and therefore acyclic. Hence

$$H^j(X,\tilde{M}) = \frac{ker\ [\Gamma(X,\tilde{I}^j) \to \Gamma(X,\tilde{I}^{j+1})]}{im\ [\Gamma(X,\tilde{I}^{j-1}) \to \Gamma(X,\tilde{I}^j)]}$$

Proposition 7.16 (Theorem B for affine schemes). *On an affine scheme*

$$X = (Spec\ A, \mathscr{O}_{Spec\ A})$$

each quasi-coherent $\mathscr{O}_X$-module $\tilde{M}$ is acyclic.

For a proof under the additional assumption that A is Noetherian see [18, Chap. III, Theor. 3.5].

Theorem 7.17 (Characterization of affine schemes). *For a Noetherian scheme X the following properties are equivalent:*

1. *The scheme X is affine, i.e. there exists a commutative ring A with unit such that $X = (Spec\ A, \mathscr{O}_{Spec\ A})$.*

2. *All quasi-coherent $\mathscr{O}_X$-modules are acyclic.*

3. *Each quasi-coherent ideal sheaf $\mathscr{J} \subset \mathscr{O}_X$ satisfies*

$$H^1(X, \mathscr{J}) = 0.$$

The proof is due to Serre, see [18, Chap. III, Theor. 3.7].

Due to Theorem 6.24 the Stein manifolds from *complex analysis* correspond to complex affine schemes without singularities from *algebraic geometry*.

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FINIS

References

The main reference for the present text are the notes [6].

An excellent textbook is [20] by L. Kaup and B. Kaup. Their book covers also the theory of Stein *spaces*, i.e. it considers also singularities. In addition, it proves the fundamental theorems from local analytic geometry.

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