Riemann Surfaces

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I prepared these notes for the students of my lecture and the subsequent seminar on Riemann surfaces. The two courses took place during the winter term 2019/20 and the summer term 2020 at the mathematical department of LMU (Ludwig-Maximilians-Universität) at Munich. The lecture in class presented the content of these notes up to and including Section 10.1. The following chapters served as the basis for the seminar. Compared to the oral lecture in class and to the seminar talks these written notes contain some additional material.

These lecture notes are permanently continued. Some undefined references in the actual release possibly refer to a forthcoming chapter of a later release.

These lecture notes are based on the book *Lectures on Riemann Surfaces* by O. Forster [8]. It serves as textbook for both courses. These notes make no claim of any originality in the presentation of the material. Instead they are just a commentary of selected chapters from [8] - and as it is mostly the case, the commentary is longer than the original! In addition, Chapter 12 relies on some notes of an unpublished lecture by O. Forster on Kähler manifolds.

I thank J. Bartenschlager and all participants who pointed out errors during the lecture in class and typos in the notes. Please report any further errors or typos to wehler@math.lmu.de

Release notes:

- Release 2.39: Chapters 1 - 10, minor revisions. Chapter 3, Update of Remark 3.32.
- Release 0.1: Chapter 1, Section 1.1 created.

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Introduction

A first course in complex analysis considers domains in the plane $\mathbb{C}$, and studies their holomorphic and meromorphic functions. But already the introduction of the point at infinity shows the necessity for a wider scope and calls for the introduction of the complex projective space $\mathbb{P}^1$.

The projective space is compact. Hence it does not embedd as a domain into $\mathbb{C}$. Instead, the projective space is a first non-trivial example of a compact complex manifold.

These lecture notes deal with Riemann surfaces, i.e. complex manifolds of complex dimension 1. A manifold is a topological space which is covered by open subsets homeomorphic to an open subset of the plane or of a higher-dimensional affine space. To obtain respectively a smooth or a complex structure on the manifold it is required that the local homeomorphisms transform respectively in a smooth or holomorphic way.

According to the definition topology covers the global structure of the manifold which can be quite different from the plane. While analysis determines the type of the manifold which is defined by the transformation type of the local homeomorphisms. These transormations are defined on open subets of the plane, see Figure 0.1.

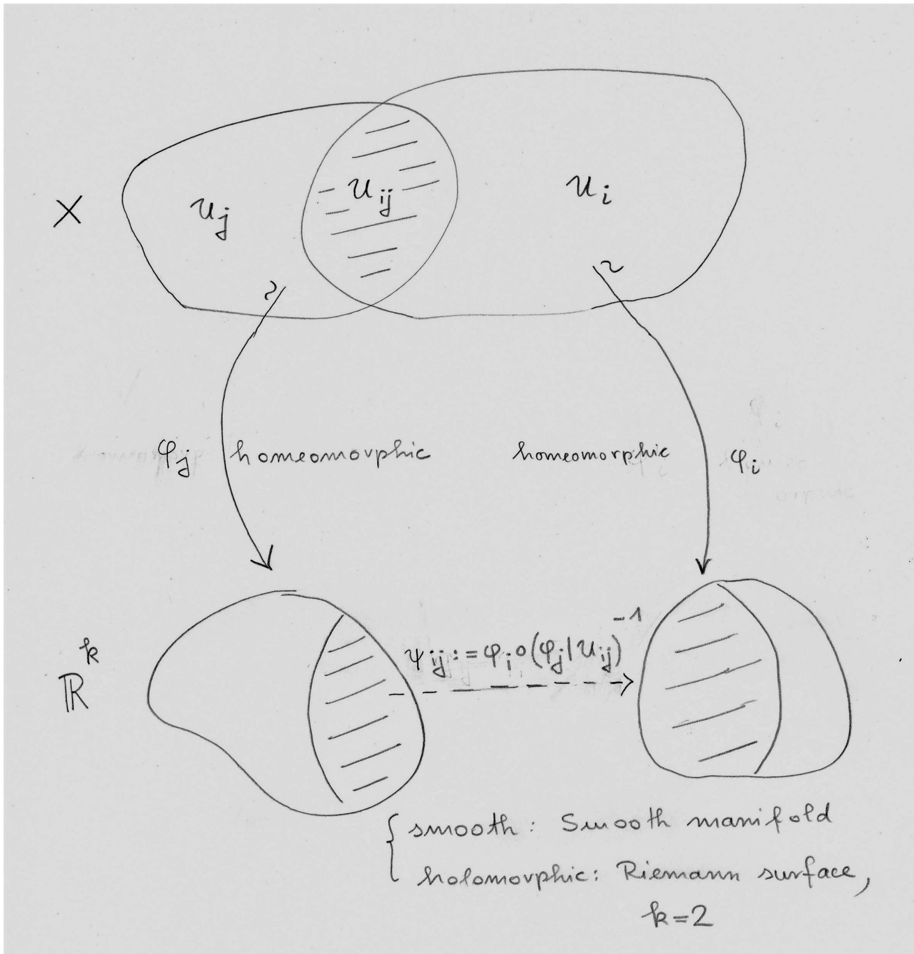


Fig. 0.1 Topological and analytical building blocks of a manifold X

Riemann surfaces split into two classes:

- *Compact Riemann surfaces*: The local representation by power series implies that on a compact Riemann surface all holomorphic functions are constant. Therefore the main emphasis lies on meromorphic functions and the location of their poles. The main result states: The set of meromorphic functions with poles of bounded order is a finite-dimensional vector space. The theorems of Riemann-Roch and Serre's duality theorem serve to compute its dimension.

- *Non-compact* alias *open Riemann surfaces*: This class comprises e.g., all domains in the plane $\mathbb{C}$. A deep result states that any open Riemann surface is a Stein manifold. A Stein manifold has many holomorphic and meromorphic functions. The Mittag-Leffler theorem and the Weierstrass product theorem generalize to Stein manifolds. Both problems are solvable on an open Riemann surface.

Part I
General Theory

Chapter 1
Riemann surfaces and holomorphic maps

This chapter gives the basic definitions of a Riemann surface and of holomorphic maps. The concept of a manifold allows to translate local properties from complex analysis in the plane to Riemann surfaces. One of the most fundamental properties is the local representation of a holomorphic function as a convergent power series. The chapter closes with some examples of compact Riemann surfaces. They show the importance of meromorphic maps on these manifolds.

1.1 The concept of the Riemann surface

Definition 1.1 (Topological manifold, chart, complex atlas and complex structure).

1. A *topological manifold* X of real dimension k is a topological Hausdorff space X such that each point $x \in X$ has an open neighbourhood U with a homeomorphism, named a *chart* around x,
$$\phi : U \xrightarrow{\simeq} V$$
onto an open set $V \subset \mathbb{R}^k$.

2. A *complex atlas* of a topological manifold X of real dimension 2, i.e. complex dimension 1, is a family $\mathscr{A}$ of *charts*
$$\mathscr{A} = (\phi_i : U_i \to V_i)_{i \in I}$$
with open subsets
$$V_i \subset \mathbb{C} \simeq \mathbb{R}^2,$$
such that
   -

$$X = \bigcup_{i \in I} U_i$$

- and for all pairs $i, j \in I$ and

  $$U_{ij} := U_i \cap U_j$$

  the *transition function* of the two charts

  $$\psi_{ij} := \phi_i \circ (\phi_j | U_{ij})^{-1} : \phi_j(U_{ij}) \to \phi_i(U_{ij})$$

  is holomorphic.

- Two complex atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ of X are *biholomorphically compatible* if their union

  $$\mathscr{A}_1 \cup \mathscr{A}_2$$

  is again a complex atlas. A maximal set of complex, biholomorphically compatible atlases of X is a *complex structure* Σ on X.

Definition 1.2 (Riemann surface, holomorphic map, meromorphic function).

1. A *Riemann surface* is a pair (X, Σ) with a 2-dimensional connected, topological manifold X with second-countable topology, i.e. having a countable base of the topology, and a complex structure Σ on X.

2. A continuous map

   $$f : (X, \Sigma_X) \to (Y, \Sigma_Y)$$

   between two Riemann surfaces is a *holomorphic map* if for each point $x \in X$ exists a chart around x from an atlas of Σ_X

   $$\phi : U \to V,$$

   and a chart around $f(x)$ from an atlas of Σ_Y

   $$\psi : S \to T,$$

   such that the composition

   $$\psi \circ f \circ (\phi | U \cap f^{-1}(S))^{-1} : \phi(U \cap f^{-1}(S)) \to \mathbb{C}$$

   is holomorphic. Note that the definition is independent from the choice of the charts.

   For an open set $U \subset X$: A map f on U is holomorphic iff the restriction of f to each component of U is holomorphic.

3. A *holomorphic function* on X is a holomorphic map

$$f : (X, \Sigma_X) \to \mathbb{C}.$$

The ring of all holomorphic functions on X with respect to addition and multiplication is denoted $\mathcal{O}(X)$.

4. A *meromorphic function* on X is given by an open set $U \subset X$ with $X \setminus U$ discrete and closed and a holomorphic function

$$f : U \to \mathbb{C}$$

such that for all $x_0 \in X \setminus U$

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} |f(x)| = \infty.$$

The points of $X \setminus U$ are named the *poles* and U is named the *domain* of the meromorphic function. The *order of a pole* $x \in X \setminus U$ is determined by the Laurent expansion of f with respect to a chart around x. The order is independent of the choice of the chart.

Two meromorphic functions f and g on X can be added and multiplied at each point which is neither a pole of f nor a pole of g. Extending the result to possibly removable singularities defines a meromorphic function on X. The ring of all meromorphic functions on X is denoted $\mathcal{M}(X)$. Because X is connected $\mathcal{M}(X)$ is even a field.

For an open set $U \subset X$: A function f on U is meromorphic iff the restriction of f to each component of U is meromorphic.

5. Consider an open set $Y \subset X$ and a meromorphic function on Y. One defines the *order* of f at a point $y \in Y$

$$ord(f;\, y) := \begin{cases} k & f \text{ has at } y \text{ a zero of order } k \in \mathbb{N} \\ -k & f \text{ has at } y \text{ a pole of order } k \in \mathbb{N} \\ \infty & f = 0 \text{ in a neighbourhood of } y \end{cases}$$

The requirement of second-countability of X in Definition 1.2 is made in order that X is paracompact. Paracompactness provides for each open covering a subordinate partition of unity, see Proposition 4.19.

Remark 1.3 (Holomorphic versus smooth).

1. Sometimes one uses for a chart of a Riemann surfaxe (X, Σ) the suggestive notation

$$z : U \to V \subset \mathbb{C}.$$

Then the decomposition into real part and imaginary part

$$z = x + i \cdot y$$

and identifying $\mathbb{C} \simeq \mathbb{R}^2$ defines a chart

$$(x,y) : U \to V \subset \mathbb{R}^2$$

of a smooth structure Σ_{smooth} on X: When considering a holomorphic transition function ψ as a function of two real variables then ψ has partial derivatives of arbitrary order. Hence the transition function is smooth, i.e. differentiable of class C^∞, and the complex structure Σ induces a smooth structure Σ_{smooth} on X and

$$(X, \Sigma_{smooth})$$

is a 2-dimensional paracompact smooth manifold. We will investigate a Riemann surface (X, Σ) by considering also its underlying smooth structure Σ_{smooth}.

2. If (X, Σ) is a Riemann surface then a map

$$f : X \to \mathbb{C}$$

is *smooth*, if f is smooth on (X, Σ_{smooth}). The ring of all smooth functions on X is denoted $\mathscr{E}(X)$.

In the following we will denote a Riemann surface (X, Σ) simply by X if the details of the complex structure Σ are not relevant.

Example 1.4 (Riemann surfaces).

1. *Connected open subsets of a Riemann surface*: If X is a Riemann surface, then also each open connected $Y \subset X$ is a Riemann surface.

2. *Domains in* $\mathbb{C}$: Apparently the plane $\mathbb{C}$ is a Riemann surface. According to Example 1) also each domain $X \subset \mathbb{C}$ is Riemann surface. Hence complex analysis of one variable is a specific part of the theory of Riemann surfaces.

3. *Projective space* $\mathbb{P}^1$: Consider the quotient

$$\mathbb{P}^1 := (\mathbb{C}^2 \setminus \{0\}) / \sim$$

with the equivalence relation

$$z = (z_0, z_1) \sim w = (w_0, w_1) :\Longleftrightarrow \exists \lambda \in \mathbb{C}^* : \ w = \lambda \cdot z \in \mathbb{C}^2 \setminus \{0\},$$

and the canonical projection onto equivalence classes

$$\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1,\ z \mapsto [z].$$

For $z = (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$ the expression

$$(z_0 : z_1) := \pi(z) \in \mathbb{P}^1$$

is named the *homogeneous coordinate* of $\pi(z)$.

We provide the set $\mathbb{P}^1$ with the quotient topology with respect to π, i.e. a subset $U \subset \mathbb{P}^1$ is open iff the inverse image

$$\pi^{-1}(U) \subset \mathbb{C}^2 \setminus \{0\}$$

is open. Then the topological space $\mathbb{P}^1$ is a connected Hausdorff space. The topology is second countable, i.e. it has a countable base of open sets. It is also compact because

$$\mathbb{P}^1 = \pi(S^3)$$

with the compact 3-sphere

$$S^3 := \{(z_0,\ z_1) \in \mathbb{C}^2 :\ |z_0|^2 + |z_1|^2 = 1\}.$$

On the topological space $\mathbb{P}^1$ we introduce the following complex atlas

$$\mathscr{A} = (\phi_i : U_i \to \mathbb{C})_{i=0,1} :$$

Set

$$U_i := \{(z_0 : z_1) \in \mathbb{P}^1 :\ z_i \neq 0\}$$

and define

$$\phi_i : U_i \to \mathbb{C}$$

by

$$\phi_i((z_0 : z_1)) := \begin{cases} z_1/z_0 & i = 0 \\ \\ z_0/z_1 & i = 1 \end{cases}$$

We have

$$U_0 \cup U_1 = \mathbb{P}^1$$

and

$$U := U_0 \cap U_1 = \{(z_0 : z_1) \in \mathbb{P}^1 :\ z_0 \neq 0 \text{ and } z_1 \neq 0\}$$

with

$$\phi_0(U) = \phi_1(U) = \mathbb{C}^*.$$

The transition functions are holomorphic:

$$\psi_{01} := \phi_0 \circ (\phi_1|U)^{-1} : \mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto \frac{1}{z}, \text{ and } \psi_{10} := \phi_1 \circ (\phi_0|U)^{-1} : \mathbb{C}^* \to \mathbb{C}^*,\ z \mapsto \frac{1}{z}$$

Therefore the *projective space* $\mathbb{P}^1$, provided with the complex structure induced by the complex atlas $\mathscr{A}$, is a compact Riemann surface. The atlas $\mathscr{A}$ is named the *standard atlas of* $\mathbb{P}^1$.

The *standard embedding* of $\mathbb{C}$ into $\mathbb{P}^1$ is the holomorphic map

$$j : \mathbb{C} \hookrightarrow \mathbb{P}^1,\ z \mapsto (1 : z).$$

Then

$$j(\mathbb{C}) = U_0$$

and

$$\mathbb{P}^1 = j(\mathbb{C}) \dot{\cup} \{(0:1)\}$$

with

$$\infty := (0:1) \in \mathbb{C}$$

named the point *infinity*.

4. *Torus*: Consider two complex constants $\omega_1, \omega_2 \in \mathbb{C}$ which are linearly independent over the field $\mathbb{R}$ and denote by

$$\Lambda := \Lambda(\omega_1, \omega_2) := \mathbb{Z} \cdot \omega_1 + \mathbb{Z} \cdot \omega_2 \subset \mathbb{C}$$

the lattice generated by (ω_1, ω_2). The lattice is a subgroup of the Abelian group $(\mathbb{C}, +)$. Hence the quotient

$$T := (\mathbb{C}/\Lambda, +)$$

is an Abelian group too. We denote by

$$\pi : \mathbb{C} \to T$$

the canonical quotient map and provide T with the induced quotient topology. Then π is an open map: For any open $U \subset \mathbb{C}$ the set

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in \Lambda} (\lambda + U) \subset \mathbb{C}$$

is open as the union of open sets. Hence $\pi(U) \subset T$ is open by definition of the quotient topology. As a topological space the torus T is Hausdorff, connected, and second countable. If

$$F := \{\lambda_1 \cdot \omega_1 + \lambda_2 \cdot \omega_2 \in \mathbb{C} :\ 0 \le \lambda_j < 1,\ j = 1,2\}$$

then the closure $\overline{F} \subset \mathbb{C}$ of the *fundamental parallelogram* F is compact. Hence the torus

$$T = \pi(\overline{F})$$

is compact.

To obtain a complex atlas $\mathscr{A}$ on the topological space T we choose for each point $y \in F$ an open neighbourhood $V_y \subset \mathbb{C}$ of y which does not contain two different points $y_1 \neq y_2$ with

$$y_1 - y_2 \in \Lambda.$$

Then for $x := \pi(y)$

$$\pi|V_y : V_y \to U_x := \pi(V_y)$$

is bijective, continuous and open, hence a homeomorphism. Set

$$\phi_x := (\pi|V_y)^{-1} : U_x \to V_y.$$

We define

$$\mathscr{A} := (\phi_{\pi(y)} : U_{\pi(y)} \to V_y)_{y \in F}$$

To compute the transition functions of two charts ϕ_1, ϕ_2

$$\psi_{12} = \phi_1 \circ (\phi_2|U)^{-1},\ U := U_1 \cap U_2,$$

consider $y \in \phi_2(U)$ and define

$$x := \pi(y) \in U.$$

Then

$$y = \phi_2(x) \text{ and } \psi_{12}(y) = \phi_1(x)$$

Hence

$$\psi_{12}(y) - y \in \Lambda.$$

The map

$$\psi_{12} - id$$

is locally constant on $\phi_2(U)$ because Λ is a discrete topological space and ψ_{12} is continuous. Hence ψ_{12} is holomorphic.

As a consequence: The torus T provided with the complex structure induced by $\mathscr{A}$ is a compact Riemann surface.

Remark 1.5 (Generalizations).

1. The projective space $\mathbb{P}^1$ is the most simple example of the complex projective spaces $\mathbb{P}^n$, $n \geq 1$, which parametrize complex lines in $\mathbb{C}^{n+1}$. The projective spaces are generalized by the complex Grassmannians $Gr(k,n)$, the set of k-dimensional subspaces of an n-dimensional complex vector space.

2. For a torus T the map

$$T \times T \to T, (x,y) \mapsto x - y$$

is continuous and even holomorphic in the sense of complex analysis of several variables. Hence $(T,+)$ is a topological group and even a compact Abelian complex *Lie group*. The torus T generalizes to the complex Lie groups of higher-dimensional tori

$$T^n := \mathbb{C}^n/\Lambda,\ n \geq 1.$$

1.2 Holomorphic maps

Small open subsets of a Riemann surface cannot be distinguished from open sets in $\mathbb{C}$. Therefore those results from complex analysis, which refer to *local* properties, transfer at once to Riemann surfaces. Examples are given by Proposition 1.6 and Corollary 1.7.

But Riemann surfaces can be compact, see Example 1.4. It is a remarkable fact, which new properties this *global* property brings into play; properties which are not shared by domains in $\mathbb{C}$, see Proposition 1.8 and Theorem 1.9.

Proposition 1.6 (Local representation of a holomorphic map). *Consider a non-constant holomorphic map*

$$f : X \to Y$$

between two Riemann surfaces. For any $x \in X$ exist

- *a uniquely determined $k \in \mathbb{N}^*$, the* branching order *of f at x,*

- *a complex chart of X around x*

$$\phi : U \to V$$

- *and a complex chart of Y around $f(x)$*

$$\psi : S \to T,$$

such that

$$f(U) \subset S$$

and

$$g := \psi \circ f \circ \phi^{-1} : V \to T$$

has the form

$$g(z) = z^k,\ z \in V.$$

Proof. i) *Choosing charts*: We choose charts

$$\phi_1 : U_1 \to V_1$$

of X around x and

$$\psi : S \to T$$

of Y around $f(x)$ with

$$\phi_1(x) = \psi(f(x)) = 0 \in \mathbb{C}$$

and

$$f(U_1) \subset S.$$

Then

$$g_1 := \psi \circ f \circ \phi_1^{-1} : V_1 \to T$$

is a non-constant holomorphic function with

$$g_1(0) = 0.$$

If

$$k := ord(g_1;\ 0) \geq 1$$

then

$$g_1(w) = w^k \cdot h_1(w)$$

with h_1 holomorphic, having no zeros in a neighbourhood $D_r(0)$.

ii) *Existence of a k-th root*: The function h_1 has a k-th root

$$h := \sqrt[k]{h_1} : D_r(0) \to \mathbb{C}^*.$$

Because $h(0) \neq 0$ the function

$$D_r(0) \to \mathbb{C},\ w \mapsto w \cdot h(w),$$

is locally biholomorphic in a neighbourhood of zero by the inverse mapping theorem. Hence for suitable neighbourhoods of 0

$$V,\ V_2 \subset V_1$$

its restriction

$$\alpha : V_2 \xrightarrow{\simeq} V,\ w \mapsto w \cdot h(w),$$

is biholomorphic.

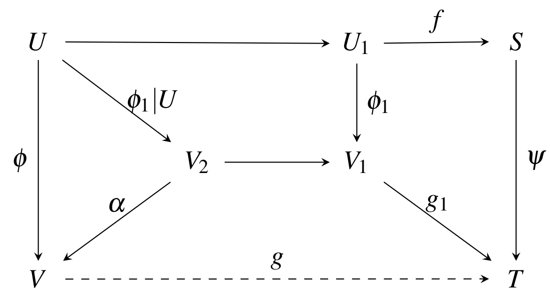
iii) *The definite chart*: Elements $z \in V$ satisfy

$$z = \alpha(w) = w \cdot h(w) \text{ or } \alpha^{-1}(z) = w$$

For $U := \phi_1^{-1}(V_2)$ the map

$$\phi := \alpha \circ (\phi_1 | U) : U \to V$$

is a complex chart of X around x.



Define

$$g := g_1 \circ \alpha^{-1} = \psi \circ f \circ \phi^{-1}.$$

For

$$z = \alpha(w) = w \cdot h(w) \in V$$

then

$$g(z) = g_1(\alpha^{-1}(z)) = g_1(w) = (w \cdot h(w))^k = z^k, \ q.e.d.$$

Corollary 1.7 (Open mapping theorem). *Each non-constant holomorphic map*

$$f : X \to Y$$

between two Riemann surfaces is an open map.

Proof. Proposition 1.6 implies that f maps neighbourhoods of a point $x \in X$ to neighbourhoods of $f(x)$ in Y, q.e.d.

As a consequence of Corollary 1.7 any injective holomorphic map between Riemann surfaces

$$f : X \to Y$$

is *biholomorphic* onto its image, i.e. the open set

$$Z := f(X) \subset Y$$

is a Riemann surface and the restriction

$$f : X \to Z$$

is bijective with holomorphic inverse

$$f^{-1} : Z \to X.$$

Proposition 1.8 (Maximum principle). *Consider a non-constant holomorphic function*

$$f : X \to \mathbb{C}$$

on a Riemann surface X. If f attains the maximum of its value at a point $x_0 \in X$ then f is constant. In particular: Any non-constant holomorphic function on a compact Riemann surface is constant.

Proof. Assume that f is not constant. Then Corollary 1.7 implies that f is open. Hence any neighbourhood U of x_0 contains a point $x \in U$ with

$$|f(x)| > |f(x_0)|,$$

a contradiction, q.e.d.

A corollary of Proposition 1.8 is Theorem 1.9.

Theorem 1.9 (Compact Riemann surfaces have no non-trivial holomorphic functions). *Each holomorphic function on a compact Riemann surface X is constant, i.e.*

$$\mathscr{O}(X) = \mathbb{C}.$$

Proof. For an indirect proof assume that f is not constant. Then Proposition 1.8 implies $f(X) = \mathbb{C}$ and $f(X)$ compact, a contradiction, q.e.d.

Theorem 1.10 (Meromorphic functions and holomorphic maps). *Meromorphic functions on a Riemann surface X are holomorphic maps*

$$X \to \mathbb{P}^1.$$

Conversely, any non-constant holomorphic map

$$f : X \to \mathbb{P}^1$$

is a meromorphic function $f \in \mathscr{M}(X)$ with domain $X \setminus f^{-1}(\infty)$.

Proof. i) Consider a meromorphic function $f \in \mathscr{M}(X)$. For each pole $x_0 \in X$ extend f by defining

$$f(x_0) := \infty \in \mathbb{P}^1.$$

Referring to the standard atlas of $\mathbb{P}^1$ from Example 1.4, 3

$$(\phi_i : U_i \to \mathbb{C})_{i=0,1},$$

for a suitable neighbourhood $U \subset X$ of x_0 the map

$$\phi_1 \circ (f|U) : U \to \mathbb{C}$$

is continuous and holomorphic on $U \setminus \{x_0\}$. By Riemann's theorem about removable singularities the map
$$\phi_1 \circ (f|U)$$
is holomorphic. Hence the extension
$$f : X \to \mathbb{P}^1$$
is a holomorphic map.

ii) Let
$$f : X \to \mathbb{P}^1$$
be a non-constant holomorphic map. Then
$$f^{-1}(\infty) \subset X$$
is discrete and closed with
$$f|X' : X' \to \mathbb{C},\ X' := X \setminus f^{-1}(\infty),$$
holomorphic and for any $x_0 \in f^{-1}(\infty)$
$$\lim_{x \to x_0} f(x) = \infty,\ q.e.d.$$

Proposition 1.11 (Meromorphic functions on $\mathbb{P}^1$). *The meromorphic functions on $\mathbb{P}^1$ are the rational functions, i.e.*
$$\mathscr{M}(\mathbb{P}^1) = \mathbb{C}(z)$$
as equality of fields.

Proof. i) The rational function
$$f(z) = a + z \in \mathbb{C}(z),\ a \in \mathbb{C},$$
is meromorphic on $\mathbb{P}^1$. Hence any rational function is meromorphic on $\mathbb{P}^1$ because $\mathscr{M}(\mathbb{P}^1)$ is a field. As a consequence
$$\mathbb{C}(z) \subset \mathscr{M}(\mathbb{P}^1).$$
ii) Consider a meromorphic function $f \in \mathscr{M}(\mathbb{P}^1)$. Compactness of $\mathbb{P}^1$ implies that f has a most finitely many poles
$$P := \{a_1, ..., a_n\}.$$
W.l.o.g. $\infty \notin P$, otherwise replace f by $1/f$. As a consequence, w.l.o.g. we may assume that f is meromorphic on $\mathbb{C}$. Let the rational functions

$$H_\nu(z) = \sum_{j=-k_\nu}^{-1} c_{\nu,j} \cdot (z - a_\nu)^j$$

be the principal parts of f at the poles $a_\nu \in P,\ \nu = 1, ..., n,$. Then

$$f - \sum_{\nu=1}^{n} H_\nu$$

is holomorphic on $\mathbb{P}^1$, hence constant due to Theorem 1.9. As a consequence f is rational. We proved

$$\mathscr{M}(\mathbb{P}^1) \subset \mathbb{C}(z),\ q.e.d.$$

Remark 1.12 (Meromorphic functions on the torus).

1. Consider a lattice $\Lambda \subset \mathbb{C}$. A meromorphic function f on $\mathbb{C}$ is *doubly periodic* or *elliptic* with respect to Λ if for each point $z \in \mathbb{C}$ from the domain of f: For all $\lambda \in \Lambda$
$$f(z+\lambda) = f(z).$$
Apparently the field of meromorphic functions on the torus $\mathbb{C}/\Lambda$ is isomorphic to the field of elliptic functions with respect to Λ.

2. A complex torus $T = \mathbb{C}/\Lambda$ has the field of meromorphic functions
$$\mathscr{M}(T) = \mathbb{C}(\wp, \wp').$$
Here $\wp$ denotes the Weierstrass $\wp$-function of the torus, which is transcendent over the field $\mathbb{C}$. Its derivative $\wp'$ satisfies the differential equation
$$\wp'^2 = 4 \cdot \wp^3 - g_2 \cdot \wp - g_3$$
with distinguished constants
$$g_2 = 60 \cdot G_{\Lambda,4} \text{ and } g_3 = 140 \cdot G_{\Lambda,6}$$
derived from the lattice Λ. In particular $\wp'$ is algebraic over the field $\mathbb{C}(\wp)$. For details cf. [40, Theor. 1.18].

Proposition 1.13 (Identity theorem). *Consider two holomorphic maps*

$$f_j : X \to Y,\ j = 1, 2,$$

between two Riemann surfaces. If for a set $A \subset X$ with accumulation point $a \in A$

$$f_1|A = f_2|A$$

then $f_1 = f_2$.

The proof of Proposition 1.13 reduces to the identity theorem of complex analysis in the plane by using charts around the accumulation point $a \in A$ and its image $f(a) \in Y$.

Chapter 2
The language of sheaves

A sheaf is when you do vertically algebra and horizontally topology.

2.1 Presheaf and sheaf

We define presheaves (deutsch: Prägarbe) and sheaves (deutsch: Garbe) of Abelian groups first. But the definition and results transfer to other objects of Abelian categories, i.e. to commutative rings R or R-modules and also to the category of sets.

Definition 2.1 (Presheaf of Abelian groups).

1. A *presheaf* $\mathscr{F}$ *of Abelian groups* on a topological space X is a family

$$\mathscr{F}(U),\ U \subset X \text{ open subset },$$

of Abelian groups, and for each pair $V \subset U$ of open subsets of X a homomorphism of Abelian groups

$$\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$$

satisfying:

$$\rho_U^U = id_{\mathscr{F}(U)}$$

and

$$\rho_W^V \circ \rho_V^U = \rho_W^U \text{ for } W \subset V \subset U.$$

The maps ρ_V^U are often named *restrictions* and denoted

$$f|V := \rho_V^U(f)$$

for $f \in \mathscr{F}(U)$, $V \subset U$ open.

The elements of the Abelian groups

$$\mathscr{F}(U),\ U \subset X \text{ open},$$

are named the *sections* of $\mathscr{F}$ on U.

2. A *morphism*

$$f : \mathscr{F} \to \mathscr{G}$$

between two presheaves of Abelian groups with restrictions respectively ρ and σ is a family of group homomorphisms

$$f_U : \mathscr{F}(U) \to \mathscr{G}(U),\ U \subset X \text{ open subset },$$

such that for any pair $V \subset U$ of open subsets of X the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ \rho^U_V \downarrow & & \downarrow \sigma^U_V \\ \mathscr{F}(V) & \xrightarrow{f_V} & \mathscr{G}(V) \end{array}$$

Remark 2.2 (Presheaf as a functor). Consider a fixed topological space X. Denote by $\underline{X}$ the category of open subsets of X:

- Objects of $\underline{X}$ are the open sets of $U \subset X$
- and for $V \subset U$ the only morphism from $Mor(V,U)$ is the injection $V \hookrightarrow U$, i.e.

$$Mor(V,U) := \begin{cases} \{V \hookrightarrow U\} & V \subset U \\ \emptyset & otherwise; \end{cases}$$

Then the presheaves $\mathscr{F}$ of Abelian groups on X are exactly the contravariant functors

$$\mathscr{F} : \underline{X} \to \underline{Ab}$$

to the category $\underline{Ab}$ of Abelian groups. A morphism

$$\mathscr{F} \to \mathscr{G}$$

between two presheaves is a functor morphism (natural transformation) from $\mathscr{F}$ to $\mathscr{G}$.

In general the concept of a presheaf is too weak to support any strong result on a Riemann surface. The stronger concept is a *sheaf*. It satisfies two additional sheaf-conditions. According to these conditions local sections which coincide on their common domain of definition glue to a unique global section.

Definition 2.3 (Sheaf). Consider a topological space X. A *sheaf* $\mathscr{F}$ of Abelian groups on X is a presheaf of Abelian groups on X, which satisfies the following two sheaf axioms:

For each open $U \subset X$ and for each open covering $\mathscr{U} = (U_i)_{i \in I}$ of U:

1. If two elements $f,\ g \in \mathscr{F}(U)$ satisfy for all $i \in I$

$$f|U_i = g|U_i$$

then

$$f = g,$$

i.e. local equality implies global equality.

2. If a family

$$f_i \in \mathscr{F}(U_i), i \in I,$$

satisfies for all $i, j \in I$

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j$$

then an element $f \in \mathscr{F}(U)$ exists satisfying for all $i \in I$

$$f|U_i = f_i,$$

i.e. local sections which agree on the intersections glue to a global element.

A *morphism of sheaves* is a morphism of the underlying presheaves.

If one paraphrases a presheaf as a family of *local* objects, then a sheaf is a family of local objects which fit together to make a unique *global* object. If it is not possible to make the parts fit, then cohomology theory is a means to measure the *obstructions*, see Chapter 6.

Definition 2.4 (Subsheaf of Abelian groups). Consider a presheaf $\mathscr{F}$ of Abelian groups on a topological space X.

1. A presheaf of Abelian groups $\mathscr{G}$ on X is a *subpresheaf* of $\mathscr{F}$
   - if for all open sets $U \subset X$

     $$\mathscr{G}(U) \subset \mathscr{F}(U)$$

     is a subgroup, and
   - if the restriction maps of $\mathscr{G}$ are induced by the restriction maps of $\mathscr{F}$.
2. If $\mathscr{F}$ is a sheaf, then a sheaf $\mathscr{G}$ is a *subsheaf* of $\mathscr{F}$ if $\mathscr{G}$ is a subpresheaf of $\mathscr{F}$.

Similar to presheaves and sheaves of Abelian groups one defines presheaves and sheaves with other algebraic structures like rings or modules.

Example 2.5 (Sheaves).

1. Let X be a topological space.
   - *Sheaf $\mathscr{C}$ of continuous functions*: For any open set $U \subset X$ define
   $$\mathscr{C}(U) := \{f : U \to \mathbb{C} \mid f \text{ continuous}\}$$
   as the complex vector space of continuous maps on U. The presheaf
   $$\mathscr{C}(U),\ U \subset X \text{ open},$$
   with the restriction of functions
   $$\rho_V^U : \mathscr{C}(U) \to \mathscr{C}(V),\ f \mapsto f|V,\ V \subset U,$$
   is a sheaf. It is named the sheaf $\mathscr{C}$ of continuous functions on X.
   - *Sheaf $\mathbb{Z}$ of locally constant functions*: Consider a topological space X. For each open set $U \subset X$ define
   $$\mathscr{F}(U) := \{f : U \to \mathbb{Z} \mid f \text{ constant}\}$$
   with the canonical restriction morphisms. The family
   $$\mathscr{F} := \mathscr{F}(U),\ U \subset X \text{ open},$$
   is a presheaf.

     In general, the presheaf $\mathscr{F}$ is not a sheaf: Assume
   $$X = X_1 \dot\cup X_2$$
   with two connected components. Then the family (f_1, f_2) with
   $$f_1(X_1) := \{1\} \text{ and } f_2(X_2) := \{2\}$$
   does not arise as
   $$f_1 = f|X_1 \text{ and } f_2 = f|X_2$$
   with a constant section $f \in \mathscr{F}(X)$.

     A slight change in the definition of $\mathscr{F}$ provides a sheaf on X: A function on an open set $U \subset X$ is *locally constant* if each point $x \in U$ has a neighbourhood V, such that the restriction $f|V$ is constant. One defines

$$\mathbb{Z}(U) := \{f : U \to \mathbb{Z} \mid f \text{ locally constant}\}.$$

Then

$$\mathbb{Z}(U),\ U \subset X \text{ open},$$

with the canonical restrictions is a sheaf. The sheaf is often denoted $\mathbb{Z}$ like the ring of integers. The context has to clarify whether the symbol denotes the ring of integers or the sheaf of locally constant integer-valued functions.

Similarly one defines the sheaf $\mathbb{C}$ of locally constant complex-valued functions. Note that both sheaves $\mathbb{Z}$ and $\mathbb{C}$ are named *constant* sheaves - not locally constant sheaves.

2. Let X be a Riemann surface.

- *Sheaf $\mathscr{O}$ of holomorphic functions*: Consider for each open $U \subset X$ the ring

  $$\mathscr{O}(U) := \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$$

  the ring of holomorphic functions on U. The presheaf

  $$\mathscr{O}(U),\ U \subset X \text{ open},$$

  with the canonical restriction of functions is a sheaf of rings. It is named the sheaf $\mathscr{O}$ of holomorphic functions on X or the *holomorphic structure sheaf*.

- *Sheaf $\mathscr{O}^*$ of holomorphic functions without zeros*: Consider for each open $U \subset X$ the multiplicative Abelian group

  $$\mathscr{O}^*(U) := \{f \in \mathscr{O}(U) : f(x) \neq 0 \text{ for all } x \in U\}.$$

  The presheaf

  $$\mathscr{O}^*(U),\ U \subset X \text{ open},$$

  with the canonical restriction of functions is a sheaf. It is named the sheaf $\mathscr{O}^*$ of holomorphic functions without zeros on X. Apparently $\mathscr{O}^*$ is the sheaf of units of $\mathscr{O}$.

- *Sheaf $\mathscr{M}$ of meromorphic functions*: Consider for each open $U \subset X$ the ring

  $$\mathscr{M}(U) := \{f \text{ meromorphic in } U\}$$

  the ring of meromorphic functions in U. The presheaf

  $$\mathscr{M}(U),\ U \subset X \text{ open},$$

  with the canonical restriction of functions is a sheaf of rings. It is named the sheaf $\mathscr{M}$ of meromorphic functions on X.

- *Sheaf $\mathscr{M}^*$ of meromorphic functions, non-vanishing on any component*: Consider for each open $U \subset X$ the multiplicative Abelian group

  $$\mathscr{M}^*(U) := \{f \in \mathscr{M}(U) : f \text{ does not vanish identically on any component of } U\}$$

  The presheaf
  $$\mathscr{M}^*(U),\ U \subset X \text{ open},$$
  with the canonical restriction of functions is a sheaf. It is named the sheaf $\mathscr{M}^*$ of non-zero meromorphic functions on X.

- *Sheaf $\mathscr{E}$ of smooth functions*: Consider for each open $U \subset X$ the ring

  $$\mathscr{E}(U) := \{f : U \to \mathbb{C} \mid f \text{ smooth}\}$$

  The presheaf
  $$\mathscr{E}(U),\ U \subset X \text{ open},$$
  with the canonical restriction of functions is a sheaf. It is named the sheaf $\mathscr{E}$ of smooth functions on X or the *smooth structure sheaf*.

The sheaves $\mathscr{O}$ and $\mathscr{M}$ are sheaves of rings. The sheaves $\mathscr{O}^*$ and $\mathscr{M}^*$ are sheaves of multiplicative groups. They are the sheaves of units of repectively $\mathscr{O}$ and $\mathscr{M}$.

Sheaves of locally constant functions like $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ are important for homology and cohomology in the context of algebraic topology. While sheaves of holomorphic and meromorphic functions are the basic objects on Riemann surfaces. A deep result on compact Riemann surfaces shows the relation between the cohomology of the sheaves from the topological context and those from the holomorphic context, see Theorem 12.41.

2.2 The stalk of a presheaf

Definition 2.6 (Stalk of a presheaf). Consider a presheaf $\mathscr{F}$ of Abelian groups on a topological space X, and a point $x \in X$. The *stalk* $\mathscr{F}_x$ of $\mathscr{F}$ at x is the set of equivalence classes with respect to the following equivalence relation on the union of all $\mathscr{F}(U)$, U open neighbourhood of x:

$$f_1 \in \mathscr{F}(U_1) \sim f_2 \in \mathscr{F}(U_2)$$

if for a suitable open neighbourhood V of x with $V \subset U_1 \cap U_2$

$$f_1|V = f_2|V.$$

Apparently, the stalk $\mathscr{F}_x$ is an Abelian group in a canonical way, and each canonical map

$$\pi_x^U : \mathscr{F}(U) \to \mathscr{F}_x$$

is a group homomorphism. The elements from $\mathscr{F}_x$ are named the *germs* of $\mathscr{F}$ at x.

Remark 2.7 (Stalks).

1. The stalk of a presheaf $\mathscr{F}$ at a point $x \in X$ is the *inductive limit* of the sections from $\mathscr{F}(U)$ for all neighbourhoods U of x.

2. Let X be a Riemann surface. For a point $x \in X$ consider the stalks

$$R := \mathscr{O}_x \text{ and } K := \mathscr{M}_x.$$

Using a chart around x shows

$$R = \mathbb{C}\{z\}, \text{ the ring of convergent power series with center } = 0,$$

and

$$K = Q(R) = \mathbb{C}(z), \text{ the quotient field of } R,$$

a statement about germs. The quotient field $Q(R)$ is the field of convergent Laurent series with center $= 0$, having only finitely many terms with negative exponents.

In general, this local statement does not necessarily generalize to a global statement: On one hand, for $X = \mathbb{C}$ one has

$$\mathscr{M}(X) = Q(\mathscr{O}(X))$$

due to Weierstrass product theorem, a statement about global sections. The same statement holds even for any domain $X \subset \mathbb{C}$. On the other hand, on a compact Riemann surface X one has

$$\mathscr{O}(X) = \mathbb{C}$$

but

$$\mathscr{M}(X) \neq \mathbb{C}, \text{ e.g. } \mathscr{M}(\mathbb{P}^1) = \mathbb{C}(z).$$

For more advanced results see [30, Kapitel 4* §1.5 Satz, §2.4].

3. Any morphism of presheaves on X

$$f : \mathscr{F} \to \mathscr{G}$$

induces for any $x \in X$ a morphism

$$f_x : \mathscr{F}_x \to \mathscr{G}_x$$

of the corresponding stalks such that the following diagram diagram commutes

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{f_U} & \mathscr{G}(U) \\ \pi_x^U \downarrow & & \downarrow \tau_x^U \\ \mathscr{F}_x & \xrightarrow{f_x} & \mathscr{G}_x \end{array}$$

Here the vertical maps are the canonical group homomorphisms from Definition 2.6. In general, these maps are not surjective. But each germ $f_x \in \mathscr{F}_x$ has an open neighbourhood $U \subset X$ and a representative $f \in \mathscr{F}(U)$. The neighbourhood U may depend on f_x.

4. On a Riemann surface X sections of a sheaf like $\mathscr{O}$ can be considered at least from the following different topological viewpoints:
   - At a point $x \in X$ one considers the value $f(x) \in \mathbb{C}$ of a function f holomorphic in an open neighbourhood of x.
   - At a point $x \in X$ one considers the germ $f_x \in \mathbb{C}(z)$ of a function f holomorphic in an open neighbourhood of x.
   - In a given open neighbourhood $U \subset X$ of a point $x \in X$ one considers a holomorphic function $f \in \mathscr{O}(U)$.
   - One considers a globally defined holomorphic function $f \in \mathscr{O}(X)$.

Definition 2.8 (Exact sheaf sequence). Consider a topological space X.

1. A *sequence of sheaves* on X is a family

$$(f_i : \mathscr{F}_i \to \mathscr{F}_{i+1})_{i \in \mathbb{Z}}$$

of morphisms of sheaves. The family is a *complex* if for all $x \in X$ on the level of stalks the induced family of morphisms of Abelian groups

$$(f_{i,x} : \mathscr{F}_{i,x} \to \mathscr{F}_{i+1,x})_{i \in \mathbb{Z}}$$

satisfies for all $i \in \mathbb{Z}$

$$f_{i,x} \circ f_{i-1,x} = 0.$$

The family is *exact* if for all $x \in X$ on the level of stalks the induced family of morphisms of Abelian groups

$$(f_{i,x} : \mathscr{F}_{i,x} \to \mathscr{F}_{i+1,x})_{i \in \mathbb{Z}}$$

is exact, i.e. if for all $i \in \mathbb{Z}$

$$ker[f_{i,x} : \mathscr{F}_{i,x} \to \mathscr{F}_{i+1,x}] = im[f_{i-1,x} : \mathscr{F}_{i-1,x} \to \mathscr{F}_{i,x}].$$

2. A *short exact sequence of sheaves* is an exact sheaf sequence of the form

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0.$$

3. A morphism of sheaves

$$f : \mathscr{F} \to \mathscr{G}$$

is respectively, *injective* or *surjective* or *bijective* if the corresponding property holds on the level of stalks

$$f_x : \mathscr{F}_x \to \mathscr{G}_x$$

for all $x \in X$.

Note: An exact sequence of sheaves has to satisfy in particular for all $i \in \mathbb{Z}$

$$f_{i+1} \circ f_i = 0.$$

Remark 2.9 (Exactness of a sheaf sequence).

1. Exactness of a sheaf sequence is a statement about the induced morphisms of the *stalks*. It is not required that the corresponding sequence of morphisms of the groups of *sections*

$$f_{i,U} : \mathscr{F}_i(U) \to \mathscr{F}_{i+1}(U),\ U \text{ open neighbourhood of } x \in X,\ i \in \mathbb{Z},$$

is exact.

One has to distinguish between a statement on the level of germs and a local statement on the level of neighbourhoods. It is exactly the task of cohomology theory, see Chapter 6, to measure the difference between exactness on the level of germs and exactness on the level of sections, in particular on the level of global sections.

2. A sequence of Abelian groups

$$0 \to F_1 \xrightarrow{f} F \xrightarrow{g} F_2 \to 0$$

is exact iff

$$f \text{ injective},\ g \text{ surjective, and } im\ f = ker\ g.$$

Proposition 2.10 (Exponential sequence). *The* exponential sequence *on a Riemann surface X is the following exact sequence of sheaves of Abelian groups*

$$0 \to \mathbb{Z} \xrightarrow{j} \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$$

Here the morphism j is the canonical inclusion. And the exponential

$$\mathscr{O} \xrightarrow{ex} \mathscr{O}^*$$

is defined for open sets $U \subset X$ as

$$ex_U : \mathscr{O}(U) \to \mathscr{O}^*(U),\ f \mapsto exp(2\pi i \cdot f).$$

Proof. To prove that the sheaf sequence is exact we consider for arbitrary but fixed $x \in X$ the sequence of stalks

$$0 \to \mathbb{Z}_x = \mathbb{Z} \xrightarrow{j_x} \mathscr{O}_x \xrightarrow{ex_x} \mathscr{O}_x^* \to 0$$

Exactness at $\mathscr{O}_x$: Each holomorphic function f defined on a domain and satisfying

$$e^{2\pi i f} = 1$$

is an integer constant and vice versa.

Exactness at $\mathscr{O}_x^*$: The surjectivity of the morphism ex_x follows from the fact, that any holomorphic function without zeros defined in a disk has a holomorphic logarithm, q.e.d.

Note that the exponential sequence from Proposition 2.10 is not exact on the level of global sections: For $X = \mathbb{C}^*$ the morphism

$$\mathscr{O}(X) \xrightarrow{ex_X} \mathscr{O}^*(X),\ f \mapsto exp(2\pi i \cdot f)$$

is not surjective, because the holomorphic function

$$z|X \in \mathscr{O}^*(X)$$

has no holomorphic logarithm. The counter example will be continued in Chapter 6.

Example 2.11 (The twisted sheaves $\mathscr{O}(k)$ on $\mathbb{P}^1$). We consider the Riemann surface $\mathbb{P}^1$ with its standard atlas $\mathscr{A}$ from Example 1.4. Set

$$U_{01} := U_0 \cap U_1$$

and

$$g_{01} : U_{01} \to \mathbb{C}^*,\ g_{01}(z_0 : z_1) := \frac{z_1}{z_0}$$

1. *Twisted sheaf*: For arbitray but fixed $k \in \mathbb{Z}$ the presheaf

$$\mathscr{O}(k)(U),\ U \subset \mathbb{P}^1 \text{ open},$$

with

$$\mathscr{O}(k)(U) := \{(s_0, s_1) \in \mathscr{O}(U \cap U_0) \times \mathscr{O}(U \cap U_1) |\ s_0 = g_{01}^k \cdot s_1 \text{ on } U \cap U_{01}\}$$

and the canonical restrictions is a sheaf $\mathscr{O}(k)$. It is named *a twist* of the structure sheaf, because for $k = 0$

$$\mathscr{O}(0) = \mathscr{O}.$$

2. *Local representation of global sections*: The holomorphic functions

$$s_0,\ s_1 \text{ and } g_{01}$$

are defined on open sets of the Riemann surface $\mathbb{P}^1$. Using the standard coordinates we derive holomorphic functions on open subsets of the plane: Set

$$u := \frac{z_1}{z_0} = \phi_0(z_0 : z_1) \text{ and } \mathrm{v} := \frac{z_0}{z_1} = \phi_1(z_0 : z_1)$$

and define the holomorphic functions

$$f_j : \mathbb{C} \to \mathbb{C},\ j = 0, 1,$$

with

$$f_0(u) := s_0(1 : u) \text{ and } f_1(\mathrm{v}) := s_1(\mathrm{v} : 1).$$

These holomorphic functions have the Taylor expansions

$$f_0(u) = \sum_{n=0}^{\infty} c_{0,n} \cdot u^n \text{ and } f_1(\mathrm{v}) = \sum_{n=0}^{\infty} c_{1,n} \cdot \mathrm{v}^n.$$

The transformation

$$f_0(u) = s_0(1:u) = g_{01}^k(1:u) \cdot s_1(1:u) = u^k \cdot s_1(1:u) = u^k \cdot s_1(1/u:1) = u^k \cdot f_1(1/u)$$

implies for all $u \in \mathbb{C}^*$

$$\sum_{n=0}^{\infty} c_{0,n} \cdot u^n = u^k \cdot \sum_{n=0}^{\infty} c_{1,n} \cdot (1/\mathrm{u})^n.$$

Comparing coefficients implies for the global sections of the twisted sheaves $\mathscr{O}(k)$:

- If $k \geq 0$ then

$$\mathscr{O}(k)(\mathbb{P}^1) \simeq \left\{ \sum_{n=0}^{k} c_n \cdot u^n :\ c_n \in \mathbb{C},\ n = 0, ..., k \right\} \simeq \mathbb{C}^{k+1}$$

- If $k < 0$ then
$$\mathscr{O}(k)(\mathbb{P}^1) = 0.$$

3. *Global sections and homogeneous polynomials*: An appropriate representation of the complex vector spaces
$$\mathscr{O}(k),\ k \geq 0,$$
are the vector spaces of homogeneous polynomials: For $k \geq 0$ denote by
$$HPol(k) \subset \mathbb{C}[z_0, z_1]$$
the $k+1$-dimensional vector space of complex *homogeneous polynomials* of degree k in two variables. The vector space $HPol(k)$ is generated by the monomials
$$z_0^\alpha \cdot z_1^\beta \text{ with } \alpha + \beta = k.$$
The $\mathbb{C}$-linear map
$$g : HPol(k) \to \mathscr{O}(k)(\mathbb{P}^1),\ P(z_0, z_1) \mapsto (s_0, s_1),$$
with
$$s_0(z_0 : z_1) := P(1, z_1/z_0) \text{ and } s_1(z_0 : z_1) := P(z_0/z_1, 1)$$
is well-defined: Homogeneity implies for any $\lambda \in \mathbb{C}^*$
$$P(\lambda \cdot z_0, \lambda \cdot z_1) = \lambda^k \cdot P(z_0, z_1).$$
Therefore
$$s_0(z_0 : z_1) = P(1, z_1/z_0) = (1/z_0)^k \cdot P(z_0, z_1)$$
and
$$s_1(z_0 : z_1) = P(z_0/z_1, 1) = (1/z_1)^k \cdot P(z_0, z_1)$$
which implies
$$s_0(z_0 : z_1) = (z_1/z_0)^k \cdot s_1(z_0 : z_1) = g_{01}(z_0 : z_1)^k \cdot s_1(z_0 : z_1)$$
The map g is injective, and therefore also surjective because domain and range have the same dimension $k+1$. As a consequence for $k \geq 0$
$$\mathscr{O}(k)(\mathbb{P}^1) \simeq HPol(k) \subset \mathbb{C}[z_0, z_1].$$
The global sections of the first twists are
$$\mathscr{O}(0)(\mathbb{P}^1) \simeq span_{\mathbb{C}} < 1 >$$
$$\mathscr{O}(1)(\mathbb{P}^1) \simeq span_{\mathbb{C}} < z_0,\ z_1 >$$
$$\mathscr{O}(2)(\mathbb{P}^1) \simeq span_{\mathbb{C}} < z_0^2,\ z_0 \cdot z_1,\ z_1^2 > .$$

2.3 General sheaf constructions

The section investigates some methods to build new sheaves.

Definition 2.12 (Image sheaf). Consider a continuous map

$$f : X \to Y$$

between topological spaces. For any sheaf $\mathscr{F}$ on X the family

$$f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}V),\ V \subset Y \text{ open},$$

with the induced restriction maps is a sheaf on Y, namd the *image sheaf* or *direct image* $f_*\mathscr{F}$ of $\mathscr{F}$.

Remark 2.13 (Image sheaf).

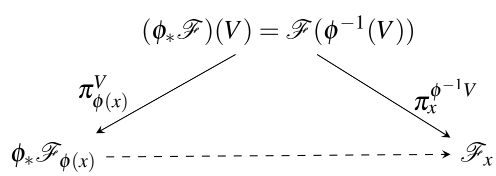
1. Let

   $$\phi : X \to Y$$

   be a continuous map and $\mathscr{F}$ a sheaf on X. For each point $x \in X$ one has a canonical morphism of stalks

   $$(\phi_*\mathscr{F})_{\phi(x)} \to \mathscr{F}_x$$

   induced from the canonical maps



   with $V \subset Y$ open neighbourhood of $\phi(x)$.

2. A holomorphic map

   $$\phi : X \to Y$$

   between two Riemann surfaces induces via pullback of holomorphic functions from Y to X a sheaf morphism

   $$\tilde{\phi} : \mathscr{O}_Y \to \phi_*\mathscr{O}_X$$

   defined as follows: For an open set $V \subset Y$

   $$\tilde{\phi}_V : \mathscr{O}_Y(V) \to (\phi_*\mathscr{O}_X)(V) = \mathscr{O}_X(\phi^{-1}(V)),\ f \mapsto f \circ \phi.$$

In particular, one obtains for each $y \in Y$ and each

$$x \in X_y := \phi^{-1}(y)$$

a morphism of stalks

$$\mathscr{O}_{Y,y} \to (\phi_* \mathscr{O}_X)_{\phi(x)} \to \mathscr{O}_{X,x}.$$

Conversely, a continuous map

$$\phi : X \to Y$$

between two Riemann surfaces is holomorphic if ϕ induces via pullback a sheaf morphism

$$\tilde{\phi} : \mathscr{O}_Y \to \phi_* \mathscr{O}_X.$$

According to the saying "A sheaf is when you do vertically algebra and horizontally topology" one can translate the basic constructions from commutative algebra to sheaves. We will distinguish a basic sheaf of rings $\mathscr{R}$ and introduce the concept of a sheaf $\mathscr{F}$ of $\mathscr{R}$-modules.

Definition 2.14 ($\mathscr{O}$-module sheaf). Consider a Riemann surfaces X. Recall from Example 2.5 the holomorphic structure sheaf $\mathscr{O}$.

1. A sheaf $\mathscr{F}$ of $\mathscr{O}$-modules - for short an *$\mathscr{O}$-module sheaf* or even an *$\mathscr{O}$-module $\mathscr{F}$* - is a sheaf $\mathscr{F}$ such that $\mathscr{F}(U)$ is an $\mathscr{O}(U)$-module for each open $U \subset X$, and the corresponding ring multiplication is compatible with restrictions, i.e. for each open $V \subset U$ the following diagram commutes

$$\begin{array}{ccc} \mathscr{O}(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow \\ \mathscr{O}(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \end{array}$$

   Here the horizontal morphisms define the module structure on the sections, and the vertical morphisms are the restrictions.

2. A morphism

$$f : \mathscr{F} \to \mathscr{G}$$

   between two $\mathscr{O}$-module sheaves is an *$\mathscr{O}$-module morphism* if for all $U \subset X$

$$f_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

   is a morphism of $\mathscr{O}(U)$-modules.

Definition 2.15 (Algebraic constructions with sheaves).

1. Let X be a topological space and consider two sheaves $\mathscr{F}$, $\mathscr{G}$ of Abelian groups on X.

   - *Direct sum*: The presheaf

     $$\mathscr{F}(U) \oplus \mathscr{G}(U),\ U \subset X \text{ open},$$

     with the induced restriction maps is a sheaf on Y. It is named the *direct sum*

     $$\mathscr{F} \oplus \mathscr{G}$$

     of $\mathscr{F}$ and $\mathscr{G}$.

   - *Restriction*: For any open $Y \subset X$ the presheaf

     $$(\mathscr{F}|Y)(U) := \mathscr{F}(U),\ U \subset Y \text{ open},$$

     with the induced restriction maps is a sheaf on Y. It is named the *restriction* $\mathscr{F}|Y$ of $\mathscr{F}$ to Y.

   - *Extension*: Consider a closed set $Y \subset X$ and a sheaf $\mathscr{F}$ on $X \setminus Y$. The presheaf

     $$\mathscr{F}^X(U),\ U \subset X \text{ open},$$

     with

     $$\mathscr{F}^X(U) := \begin{cases} \mathscr{F}(U) & Y \cap U = \emptyset \\ 0 & Y \cap U \neq \emptyset \end{cases}$$

     with the restrictions induced from $\mathscr{F}$ is a sheaf. It is named the *extension* $\mathscr{F}^X$ of $\mathscr{F}$ to X.

   - *Sheaf of sheaf morphisms*: The presheaf

     $$Hom(\mathscr{F},\mathscr{G})(U) := Hom(\mathscr{F}|U,\ \mathscr{G}|U),\ U \subset X \text{ open},$$

     with induced restrictions is a sheaf on X. It is named

     $$\mathscr{H}om(\mathscr{F},\mathscr{G})$$

     the sheaf of sheaf morphisms from $\mathscr{F}$ to $\mathscr{G}$. Note the difference between the two Abelian groups

     $$Hom(\mathscr{F}|U,\ \mathscr{G}|U) \text{ and } Hom(\mathscr{F}(U),\ \mathscr{G}(U)).$$

2. Let X be a Riemann surface and $\mathscr{F}$, $\mathscr{G}$ two $\mathscr{O}$-module sheaves on X.

   - The sheaf $\mathscr{F}$ is

– a *free sheaf* of rank $= k$ if
$$\mathscr{F} \simeq \mathscr{O}^{\oplus k},$$
– a *locally free sheaf* of rank $= k$ if any point $x \in X$ has an open neighbourhood $U \subset X$ such that the restriction
$$\mathscr{F}|U$$
is a free sheaf on U of rank $= k$. A locally free sheaf of rank $= 1$ is named an *invertible sheaf.*

Note. All stalks of a locally free sheaf $\mathscr{F}$ of rank $= k$ at X are isomorphic, i.e. for all $x \in X$
$$\mathscr{F}_x \simeq \mathscr{O}_x^{\oplus k}.$$

- *Sheaf of $\mathscr{O}$-module morphisms*: For open $U \subset X$ denote by
$$Hom_{\mathscr{O}|U}(\mathscr{F}|U, \mathscr{G}|U)$$
the $\mathscr{O}(U)$-module of $\mathscr{O}|U$-module morphisms between $\mathscr{F}|U$ and $\mathscr{G}|U$. The presheaf
$$Hom_{\mathscr{O}|U}(\mathscr{F}|U, \mathscr{G}|U),\ U \subset X \text{ open},$$
is an $\mathscr{O}$-module sheaf, named
$$\mathscr{H}om_{\mathscr{O}}(\mathscr{F}, \mathscr{G}).$$
- *Dual sheaf*: For an $\mathscr{O}$-module sheaf $\mathscr{F}$ the sheaf
$$\mathscr{F}^{\vee} := \mathscr{H}om_{\mathscr{O}}(\mathscr{F}, \mathscr{O})$$
is named the *dual sheaf* of $\mathscr{F}$.

The twisted sheaves $\mathscr{O}(k)$ from Example 2.11 are invertible sheaves on $\mathbb{P}^1$. Chapter 10 will introduce line bundles and investigate the relation between line bundles and invertible sheaves.

Analogously one may consider the smooth structure (X, Σ_{smooth}) and take the sheaf $\mathscr{E}$ as its structure sheaf. One defines in an analogous way $\mathscr{E}$-module sheaves.

Definition 2.16 carries over the definition of a sheaf from all open sets of a topological space to a base of the open sets. Proposition 2.17 allows to extend a sheaf with respect to a base to the whole topological space. This results facilitates the construction of sheaves because one has to define sections only on small open sets.

Definition 2.16 (Sheaf with respect to a base). Let X be a topological space and consider a base $\mathscr{B}$ of the topology of X. A *$\mathscr{B}$-sheaf $\mathscr{F}$* of Abelian groups is a family

$$\mathscr{F}(U),\ U \in \mathscr{B},$$

with restriction maps for basic open sets $U,\ V$

$$\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$$

satisfying

- for each basic open set U
$$\rho_U^U = id_{\mathscr{F}(U)}$$
- and for basic open sets $W \subset V \subset U$
$$\rho_W^V \circ \rho_V^U = \rho_W^U$$

such that for each basic open set $Y \in \mathscr{B}$ and each covering $\mathscr{U}$ of Y by basic open sets the following two axioms hold:

1. If two elements $f, g \in \mathscr{F}(Y)$ satisfy for all $U \in \mathscr{U}$
$$f|U = g|U$$
then
$$f = g.$$
2. If a family
$$f_U \in \mathscr{F}(U),\ U \in \mathscr{U},$$
satisfies for each pair $U_1, U_2 \in \mathscr{U}$ and for each $V \in \mathscr{B}$ with $V \subset U_1 \cap U_2$
$$f_{U_1}|V = f_{U_2}|V,$$
then an element $f \in \mathscr{F}(Y)$ exists such that for all $U \in \mathscr{U}$
$$f|U = f_U.$$

Proposition 2.17 (Constructing a sheaf bottom up from a $\mathscr{B}$-sheaf). *Let X be a topological space and consider a base $\mathscr{B}$ of the topology of X. Then each $\mathscr{B}$-sheaf $\mathscr{F}_{\mathscr{B}}$ induces a sheaf $\mathscr{F}$ on X by the following construction:*

For any open $U \subset X$ define

$$\mathscr{F}(U) := \{(f_V)_V \in \prod_{\substack{V \subset U \\ V \in \mathscr{B}}} \mathscr{F}_{\mathscr{B}}(V) :\ f_V|W = f_W \text{ for all basic sets } W \subset V\}$$

For $\tilde{U}, U \in \mathscr{B}$ with $U \subset \tilde{U}$ define the restriction

$$\rho_U^{\tilde{U}} : \mathscr{F}(\tilde{U}) \to \mathscr{F}(U)$$

by the following commutative diagram

$$\begin{array}{ccc}
\mathscr{F}(\tilde{U}) & \xrightarrow{\rho_U^{\tilde{U}}} & \mathscr{F}(U) \\
\downarrow & & \downarrow \\
\prod_{\substack{V \subset \tilde{U} \\ V \in \mathscr{B}}} \mathscr{F}(V) & \dashrightarrow & \prod_{\substack{V \subset U \\ V \in \mathscr{B}}} \mathscr{F}(V)
\end{array}$$

with vertical injections and the map

$$\prod_{\substack{V \subset \tilde{U} \\ V \in \mathscr{B}}} \mathscr{F}(V) \to \prod_{\substack{V \subset U \\ V \in \mathscr{B}}} \mathscr{F}(V)$$

induced by the universal property of the product by the projections for $W \subset U,\ W \in \mathscr{B}$*:*

$$\prod_{\substack{V \subset \tilde{U} \\ V \in \mathscr{B}}} \mathscr{F}(V) \to \mathscr{F}(W)$$

Proof. One has to verify that the restriction of a compatible family within the product is again compatible, q.e.d.

The construction of $\mathscr{F}(U)$ in Proposition 2.17 is the *projective limit* while the stalk $\mathscr{F}_x$ in Definition 2.6 is the *inductive limit* or *direct limit* of certain families of Abelian groups of sections.

Chapter 3
Covering projections

The chapter first recalls some fundamental results from covering theory. These results will then be applied in the context of Riemann surfaces. Classifying the concepts and issues, which have been introduced, we distinguish the following steps of increasing abstraction:

- Complex analysis in domains of $\mathbb{C}$
- Riemann surfaces and holomorphic maps
- Sheaf theory on Riemann surfaces and topological spaces
- Étale space of the structure sheaf as a holomorphic unbranched covering projection.

The basic object of investigation in this chapter is the étale space of a presheaf, see Definition 3.9. We then give the following applications of this concept:

- Sheafification of a presheaf, Theorem 3.11.
- The tensor product sheaf of two $\mathscr{O}$-module sheaves, Definition 3.17.
- The maximal global analytic continuation of a holomorphic germ, Theorem 3.31.
- The Riemann surface of an algebraic function over a compact Riemann surface, for a scetch see Remark 3.33.

3.1 Branched and unbranched covering projections

Definition 3.1 (Covering projection). Consider a map

$$p : X \to Y$$

between two topological spaces X and Y.

1. The map p is a *local homeomorphism* or an *étale* map if any $x \in X$ has an open neighbourhood $U \subset X$ such that $p(U) \subset Y$ is open and the restriction

   $$p|U : U \xrightarrow{\simeq} p(U)$$

   is a homeomorphism.

2. The map p is a *covering projection*, if p is continuous, open and discrete, i.e. each fibre

   $$X_y := p^{-1}(y),\ y \in Y,$$

   is a discrete topological space when equipped with the subspace topology of $X_y \subset X$. The spaces X and Y are named respectively the *total space* and the *base* of the covering projection.

3. If p is a covering projection then a point $x \in X$ is a *branch point* of p if for any neighbourhood U of x the restriction

   $$p|U : U \to Y$$

   is not injective. If $A \subset X$ denotes the set of branch points of p, then

   $$B := p(A) \subset Y$$

   is the set of *critical values* of p. A covering projection without branch points is named *unbranched*.

4. The map p is an *unbounded, unbranched covering projection* if each point $y \in Y$ has an open neighbourhood V such that

   $$p^{-1}(V) = \dot{\bigcup}_{i \in I} U_i \text{ (disjoint union)}$$

   and for each $i \in I$ the restriction

   $$p|U_i : U_i \to V$$

   is a homeomorphism.

Apparently, a local homeomorphism is a continuous and open map. Each unbranched and unbound covering projection in the sense of Definition 3.1, part 4 is a covering projection in the sense of part 2. It is unbranched in the sense of part 3.

Lemma 3.2 (Local homeomorphism). *A map*

$$p : X \to Y$$

between topological spaces is an unbranched covering projection if and only if p is a local homeomorphism.

Proof. i) Assume that p is an unbranched covering projection. Any point $x \in X$ has an open neighbourhood U with $p|U$ injective. Because p is open, the set

$$V := p(U) \subset Y$$

is open. The map

$$p|U : U \to V$$

is bijective, continuous and open, hence a homeomorphism.

ii) Assume that p is a local homeomorphism. Then any point $x \in X$ has an open neighbourhood U such that

$$p|U : U \to V$$

is a homeomorphism onto an open set

$$V := p(U) \subset Y.$$

In particular $p|U$ is injective, which implies that p is unbranched and

$$\{x\} = U \cap p^{-1}(p(x)) = U \cap X_{p(x)}$$

Hence each fibre

$$X_y,\ y \in Y,$$

is discrete. Moreover p is open and continuous, q.e.d.

Unbounded, unbranched covering projections play an important role in the category of topological spaces and homotopic maps:

- They facilitate the computation of the *fundamental group* of a topological space, Definition 3.3.

- They satisfy a *lifting criterion* depending on the fundamental group, Proposition 3.4: Whether a map
$$f : X \to B$$
into the base B of an unbounded, unbranched covering projection
$$p : E \to B$$
lifts to a map into its total space E only depends on the induced maps of the fundamental groups.

- They have the *homotopy lifting property*: Whether a map
$$f : X \to B$$
into the base of an unbounded, unbranched covering projection
$$p : E \to B$$
lifts to a map into the total space E only depends on the homotopy class of f, Proposition 3.5.

To recall these results from algebraic topology we recommend the textbooks [20, Chap. 1.1, 1.3] and [36, Chap. 2, Sect. 2, 4]. Note that books from algebraic topology name "covering projection" a map which is an unbounded, unbranched covering projection in the sense of Definition 3.1. If one distinguishes in a topological space X a point $x_0 \in X$, then the pair (X, x_0) is named a *pointed topological space* with *base point* x_0.

Let $I = [0,1] \subset \mathbb{R}$ denote the real unit interval.

Definition 3.3 (Fundamental group and simply connectedness). Consider a path-connected topological space X.

1. After choosing an arbitrary but fixed distinguished point $x_0 \in X$ the *fundamental group* $\pi_1(X, x_0)$ of X with respect to the basepoint x_0 is the set of homotopy classes of closed paths, i.e. of continuous maps
$$\alpha : I \to X \ with\ \alpha(0) = \alpha(1) = x_0$$
with the catenation
$$(\alpha_1 * \alpha_2)(t) := \begin{cases} \alpha_1(2t) & \text{if } 0 \le t \le 1/2 \\ \alpha_2(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$
as group multiplication.

2. The topological space X is *simply-connected* if $\pi_1(X, x_0) = 0$.

Apparently closed paths can also be considered continuous maps

$$S^1 \to X$$

from the 1-sphere, i.e. from the unit circle. One checks that the catenation defines a group structure on the set of homotopy classes. In addition, the fundamental group - as an abstract group - does not depend on the choice of the basepoint. Therefore one often writes writes $\pi_1(X,*)$ or even $\pi_1(X)$.

A morphism

$$f : (X, x_0) \to (Y, y_0)$$

of path-connected pointed topological spaces, i.e. satisfying $f(x_0) = y_0$, induces a group homorphism of the fundamental groups

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0), [\alpha] \mapsto [f \circ \alpha].$$

In case of an unbounded, unbranched covering projection f the induced map $\pi_1(f)$ is injective. The fundamental group is a covariant functor from the homotopy category of path-connected pointed topological spaces, i.e. the category of path-connected pointed topological spaces with morphism the homotopy classes of continuous maps respecting the base point, to the category of groups.

Proposition 3.4 (Lifting criterion). *Consider an unbounded, unbranched covering projection*

$$p : (E, e_0) \to (B, b_0)$$

of path-connected, pointed topological spaces and a continuous map

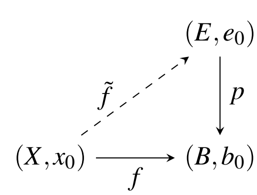
$$f : (X, x_0) \to (B, b_0)$$

with X path-connected and locally path-connected. Then the following properties are equivalent:

1. *The map f has a unique lift to (E, e_0), i.e. a continuous map*

$$\tilde{f} : (X, x_0) \to (E, e_0)$$

exists such that the following diagram commutes



2. The induced map of the fundamental groups

$$\pi_1(f) : \pi_1(X, x_0) \to \pi_1(B, b_0)$$

satisfies

$$[im : \pi_1(X, x_0) \xrightarrow{\pi_1(f)} \pi_1(B, b_0)] \subset [im : \pi_1(E, e_0) \xrightarrow{\pi_1(p)} \pi_1(B, b_0)].$$

In particular, any continous map $f : (X, x_0) \to (B, b_0)$ from a simply-connected topological space X lifts to a continous map $\tilde{f}$ into the covering space (E, e_0). Choosing the simply-connected topological space

$$X = I$$

shows: Any path

$$f \text{ in } B \text{ with } f(0) = b_0$$

lifts to a unique path

$$\tilde{f} \text{ in } B \text{ with } \tilde{f}(0) = e_0$$

But note: If f is a closed path, the lift $\tilde{f}$ is not necessarily closed.

Proposition 3.5 (Homotopy lifting property). *Consider an unbounded, unbranched covering projection*

$$p : (E, e_0) \to (B, b_0).$$

If a continous map with connected X

$$f : (X, x_0) \to (B, b_0)$$

into the base lifts to a map

$$\tilde{f} : (X, x_0) \to (E, e_0)$$

into the covering space then also any homotopy F of f, which fixes the base point, lifts uniquely to a homotopy of $\tilde{f}$, i.e. expressing the homotopy lifting property *in a formal way: Assume the existence of*

- *a homotopy of f relative $\{x_0\}$, i.e. a continuous map*

  $$F : (X, x_0) \times I \to (B, b_0)$$

  with

  $$F(-, 0) = f \text{ and } F(x_0, -) = b_0,$$

- *and of a continuous map $\tilde{f} : (X, x_0) \to (E, e_0)$ with $p \circ \tilde{f} = f$.*

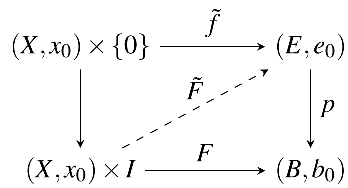
Then a unique homotopy of $\tilde{f}$ relative $\{x_0\}$ exists

$$\tilde{F} : (X, x_0) \times I \to (E, e_0)$$

with

$$F = p \circ \tilde{F},$$

i.e. such that the following diagram commutes:



Lemma 3.6 (Poincaré-Volterra). *Consider a connected topological manifold X, a second-countable topological space Y and a continuous map*

$$p : X \to Y$$

with discrete fibres. Then also X is second-countable.

For the proof see [8, Lemma 23.2].

We now apply covering theory from algebraic topology to the theory of sheaves and holomorphic maps between Riemann surfaces. This method has been prepared by Proposition 1.6.

Proposition 3.7 (Holomorphic maps and covering projections). *Each non-constant holomorphic map*

$$f : X \to Y$$

between two Riemann surfaces is a covering projection.

Proof. The map is continuous. It is also open according to Corollary 1.7. If f were not discrete then for at least one $y \in f(X)$ the fibre $X_y \subset X$ were not discrete. Then X_y has an accumulation point, which contradicts the identity theorem from Proposition 1.13, q.e.d.

Proposition 3.8 (Pullback of the complex structure along unbranched covering projections). *Consider a Riemann surface Y, a connected Hausdorff space X and an unbranched covering projection*

$$p : X \to Y.$$

Then on X exists a unique structure of a Riemann surface such that p becomes a holomorphic map.

Proof. Because p is a local homeomorphism, also X is a topological manifold. Lemma 3.6 implies second countability of X. We define a complex atlas $\mathscr{A}$ on X as the family of the following charts: If

$$\phi_1 : U_1 \to V$$

is a complex chart of the complex structure of Y such that an open set $U \subset X$ exists with

$$p|U : U \to U_1$$

a homeomorphism, then the homeomorphism

$$\phi := \phi_1 \circ (p|U) : U \to V$$

belongs to $\mathscr{A}$. To check that the charts of $\mathscr{A}$ are biholomorphic compatible we assume two charts

$$\phi = \phi_1 \circ (p|U) : U \to V \text{ and } \phi' = \phi_1' \circ (p|U') : U' \to V'$$

with $U \cap U' \neq \emptyset$. Then the transition function - we do not indicate all restrictions -

$$\psi_{U'U} := \phi' \circ \phi^{-1} = \phi_1' \circ (p|U \cap U') \circ (p|U \cap U')^{-1} \circ (\phi_1)^{-1} = \phi_1' \circ (\phi_1)^{-1}$$

is holomorphic because the charts of Y are biholomorphically compatible. After providing the topological space X with the complex structure Σ induced by the atlas $\mathscr{A}$ we obtain a Riemann surface (X,Σ), and the map

$$p : (X,\Sigma) \to Y$$

is holomorphic and even locally biholomorphic. If (X,Σ') is a second complex structure such that

$$p : (X,\Sigma') \to Y$$

is holomorphic, then

$$id_X : (X,\Sigma) \to (X,\Sigma')$$

is locally biholomorphic, hence biholomorphic, q.e.d.

We recall: For a continous map $p : X \to Y$ between topological spaces a *section* over an open set $U \subset Y$ is a continuous map

$$s : U \to X$$

with $p \circ s = id_U$.

Definition 3.9 (Étale space of a presheaf). Consider a presheaf $\mathscr{F}$ on a topological space X. Define the disjoint union of all stalks

$$|\mathscr{F}| := \dot{\bigcup}_{x\in X} \mathscr{F}_x$$

and the canonical projection

$$p : |\mathscr{F}| \to X,\ f_x \in \mathscr{F}_x \mapsto x \in X.$$

For each open set $U \subset X$ and $f \in \mathscr{F}(U)$ consider the set of corresponding germs of f

$$[U, f] := \{f_x :\ x \in U\}.$$

The set $\mathscr{B}$ of all sets $[U, f]$ is the base of a topology on $|\mathscr{F}|$, and the topological space $|\mathscr{F}|$ is named the *étale space* of the presheaf $\mathscr{F}$.

Proposition 3.10 (Étale space of a presheaf). *Let $\mathscr{F}$ be a presheaf on a topological space X.*

1. *The étale space $|\mathscr{F}|$ from Definition 3.9 is a topological space.*

2. *The canonical projection*

$$p : |\mathscr{F}| \to X,\ f_x \mapsto x \text{ for } f_x \in \mathscr{F}_x,$$

is a local homeomorphism.

Proof. 1. We have to show that $\mathscr{B}$ is the base of a topology: Assume $x \in X$, $f_x \in \mathscr{F}_x$, and

$$f_x \in [U, f] \cap [V, g].$$

Then $x \in U \cap V$ and

$$f \in \mathscr{F}(U) \text{ and } g \in \mathscr{G}(V)$$

determine the same germ $f_x \in \mathscr{F}_x$. Hence an open subset $W \subset U \cap V$ exists such that

$$f|W = g|W.$$

As a consequence $[W, f|W] \in \mathscr{B}$ and

$$f_x \in [W, f|W] \subset [U, f] \cap [V, g].$$

As a consequence, the set $\mathscr{B}$ is the base of the topology $\mathscr{T}$ on $|\mathscr{F}|$ with elements the arbitrary unions of elements from $\mathscr{B}$. Here we follow the convention that the union of an empty family is the empty set and the intersection of an empty family is the whole set.

2. For each open set $U \subset X$ the inverse image

$$p^{-1}(U) = \bigcup \{[V, f] :\ V \subset U \text{ open and } f \in \mathscr{F}(V)\}$$

is open. Apparently
$$p([U,f]) = U \subset X \text{ open}.$$
Hence p is continuous and open. The restriction
$$p|[U,f] : [U,f] \to U$$
is bijective, because for all $y \in U$
$$(p|[U,f])^{-1}(y) = \{f_y\}.$$
Hence p is a local homeomorphism, q.e.d.

As a first application of the étale space construction we attach to each presheaf a sheaf, named its sheafification.

Theorem 3.11 (Sheafification of a presheaf). *Consider a presheaf $\mathscr{F}$ on a topological space X. If one defines for each open set $U \subset X$*
$$\mathscr{F}^{sh}(U) := \{s : U \to |\mathscr{F}| : \ s \text{ section of } p\},$$
then the family
$$\mathscr{F}^{sh}(U),\ U \subset X \text{ open},$$
with the canonical restriction of sections is a sheaf, the sheafification $\mathscr{F}^{sh}$ *of the presheaf $\mathscr{F}$.*

Proof. Apparently the presheaf
$$\mathscr{F}^{sh}(U),\ U \subset X \text{ open},$$
with the canonical restriction of sections is a sheaf $\mathscr{F}^{sh}$: In the present context both sheaf axioms deal with continuous maps to $|\mathscr{F}|$, q.e.d.

If the presheaf $\mathscr{F}$ is already a sheaf, then $\mathscr{F}^{sh} \simeq \mathscr{F}$, i.e. the sheafification of a sheaf is the sheaf itself.

Definition 3.12 (Presheaf satisfying the identity theorem). Let X be a topological space. A presheaf $\mathscr{F}$ on X satisfies the *identity theorem* if for any connected open subset $Y \subset X$ holds: Two sections
$$f,\ g \in \mathscr{F}(Y),$$
which define the same germ for at least one point $y \in Y$, are equal, i.e.
$$f_y = g_y \implies f = g.$$

Apparently the structure sheaf $\mathscr{O}$ of a Riemann surface satisfies the identity theorem.

Proposition 3.13 (Hausdorff property of the étale space). *Let X be a locally-connected Hausdoff space and $\mathscr{F}$ a presheaf on X which satisfies the identity theorem from Definition 3.12. Then the étale space $|\mathscr{F}|$ is a Hausdorff space.*

Proof. We denote by
$$p : |\mathscr{F}| \to X$$
the canonical projection. Consider two distinct elements $f_{x_1} \neq f_{x_2} \in |\mathscr{F}|$.

i) If $x_1 \neq x_2$ then for $j = 1,\ 2$ we may choose disjoint neighbourhoods $U_j \subset X$ of x_j. Apparently
$$p^{-1}(U_1) \text{ and } p^{-1}(U_2)$$
are disjoint neighbourhoods of respectively f_{x_1} and f_{x_2}.

ii) If $x_1 = x_2 =: x$ then for $j = 1,\ 2$ we represent each germ $f_{x_j} \in \mathscr{F}_x$ by a section
$$f_j \in \mathscr{F}(U_j)$$
with open neighbourhoods $U_j \subset X$ of x_j. We choose a connected neighbourhood U of x with
$$U \subset U_1 \cap U_2$$
and obtain open neighbourhoods
$$[U, f_j|U] \subset |\mathscr{F}|$$
of $f_{x_j},\ j = 1, 2$.

Assume: Their intersection
$$[U, f_1|U] \cap [U, f_2|U]$$
is not empty. Then we obtain a point $y \in U$ with
$$f_y = f_{1,y} = f_{2,y}$$
Because $\mathscr{F}$ satisfies the identity theorem we conclude
$$f_1|U = f_2|U,$$
which implies $f_{x_1} = f_{x_2}$, a contradiction, q.e.d.

Remark 3.14 (Complex structure on the étale space of the structure sheaf). Consider a Riemann surface X. Then the étale space $|\mathscr{O}_X|$ has a complex structure such each connected component

$$Y \subset |\mathscr{O}_X|$$

becomes a Riemann surface and the restriction

$$p|Y : Y \to X$$

is a locally biholomorphic map between Riemann surfaces.

Proof. The result follows from Proposition 3.8, Proposition 3.10 and Proposition 3.13, q.e.d.

Proposition 3.15 (Sections of a sheafification). *Let X be a topological space and $\mathscr{F}$ a presheaf on X. Consider the étale space*

$$p : |\mathscr{F}| \to X$$

and a section of p

$$s : U \to |\mathscr{F}|,\ U \subset X \text{ open}.$$

Then: Each point $x \in U$ has an open neighbourhood $V \subset U$ and an element $f \in \mathscr{F}(V)$ satisfying for all $y \in V$

$$s(y) = f_y \in \mathscr{F}_y.$$

As a consequence, an element

$$f \in \mathscr{F}^{sh}(U),\ U \subset X \text{ open},$$

is a family of compatible *sections of $\mathscr{F}$, i.e. a family $f = (f_i)_{i \in I}$*

- *with an open covering $\mathscr{U} = (U_i)_{i \in I}$ of U depending on f*
- *and elements $f_i \in \mathscr{F}(U_i),\ i \in I$, such that*

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j,\ i, j \in I.$$

In particular, for each $x \in X$ holds the isomorphy of stalks

$$\mathscr{F}_x^{sh} \simeq \mathscr{F}_x.$$

Remark 3.16 (Sections of the sheafification).

1. Proposition 3.15 characterizes sections of the sheafification $\mathscr{F}^{sh}$ as compatible families of sections of the presheaf $\mathscr{F}$. Sections in a sheaf are equivalent to compatible local sections due to the two sheaf axioms. Hence the sheafification of a sheaf $\mathscr{F}$ reproduces the sheaf, i.e. $\mathscr{F}^{sh} \simeq \mathscr{F}$.

2. For a sheaf $\mathscr{F}$ on X the étale space

$$p : |\mathscr{F}| \to X$$

allows to define sections of $\mathscr{F}$ over arbitrary subsets $A \subset X$:

$$\mathscr{F}(A) := \{s : A \to |\mathscr{F}| : \ p \circ s = id_A\}$$

In particular, for any point $x \in X$ one obtains the stalk at x as

$$\mathscr{F}(\{x\}) = \mathscr{F}_x$$

An application of Theorem 3.11 is the definition of the tensor product sheaf.

Definition 3.17 (Tensor product). Let X be a Riemann surface. For two $\mathscr{O}$-module sheaves $\mathscr{F}$, $\mathscr{G}$ consider the presheaf

$$\mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U),\ U \subset X \text{ open},$$

with restrictions induced from the restrictions of the two factors. Its sheafification is defined as the *tensor product*

$$\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G}$$

of $\mathscr{F}$ and $\mathscr{G}$.

Remark 3.18 (Tensor product).

1. *Stalk of a tensor product*: For two $\mathscr{O}$-module sheaves $\mathscr{F}$ and $\mathscr{G}$ on a Riemann surface X for all $x \in X$

$$(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G})_x \simeq \mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{G}_x.$$

The proof relies on the fact that tensoring commutes with taking the inductive limits, see [37].

Dennote by $\mathscr{H}$ the presheaf

$$\mathscr{H}(U) := \mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U),\ U \subset X \text{ open},$$

with restrictions induced from the restrictions of the factors. Proposition 3.15 implies: For $x \in X$ the germs, i.e. the elements of the stalk

$$(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G})_x$$

are the equivalence classes of elements from

$$\mathscr{H}(U) = \mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U),\ U \subset X \text{ open neighbourhood of } x.$$

With

$$\varinjlim_{x \in U} \mathscr{O}(U) = \mathscr{O}_x$$

we obtain

$$(\mathscr{F} \otimes_{\mathscr{O}} \mathscr{G})_x = \mathscr{H}_x = \varinjlim_{x \in U} (\mathscr{F}(U) \otimes_{\mathscr{O}(U)} \mathscr{G}(U)) =$$

$$= \varinjlim_{x \in U} \mathscr{F}(U) \otimes_{\mathscr{O}_x} \varinjlim_{x \in U} \mathscr{G}(U) = \mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{G}_x$$

with the second last equality due to interchanging tensor product and direct limit.

2. *Tensor product of twisted sheaves*: For two integers $k_1,\ k_2 \in \mathbb{Z}$ the tensor product of the corresponding twisted sheaves on $\mathbb{P}^1$ from Example 2.11 satisfies

$$\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2) \simeq \mathscr{O}(k_1 + k_2).$$

Proof. i) *Sheaf morphism*: We define a sheaf morphism

$$f : \mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2) \to \mathscr{O}(k_1 + k_2)$$

as follows: The domain is a sheafification. Proposition 3.15 implies: Each element

$$s \in (\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2))(U),\ U \subset X \text{ open},$$

is a family of compatible sections from

$$\mathscr{O}(k_1)(V) \otimes_{\mathscr{O}(V)} \mathscr{O}(k_2)(V),\ V \subset U \text{ suitable open sets depending on } s.$$

The tensor product of two sections

$$s_1 \otimes s_2 \in \mathscr{O}(k_1)(V) \otimes_{\mathscr{O}(V)} \mathscr{O}(k_2)(V)$$

defines a section in

$$\mathscr{O}(k_1 + k_2)(V)$$

by multiplying the representing holomorphic functions, because the product satisfies the correct transformation law with $g_{01}^{k_1+k_2}$. The family of resulting sections from

$$\mathscr{O}(k_1 + k_2)(V)$$

is compatible and defines the section

$$f(s) \in \mathscr{O}(k_1 + k_2)(U).$$

ii) Alternative formulation of part i) by using the étale space: Consider the presheaf

$$\mathscr{H}(U) := \mathscr{O}(k_1)(U) \otimes_{\mathscr{O}(U)} \mathscr{O}(k_2)(U),\ U \subset \mathbb{P}^1 \text{ open},$$

with the canonical restrictions induced from the factors. If

$$p_{|\mathscr{H}|} : |\mathscr{H}| \to \mathbb{P}^1$$

denotes the covering projection of its étale space then by definition of $\mathscr{H}^{sh}$ for each open $U \subset \mathbb{P}^1$

$$(\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2))(U) := \mathscr{H}^{sh}(U) = \{s : U \to |\mathscr{H}| : \text{ s section of } p_{|\mathscr{H}|}\}$$

Here we have due to part 1 for each $x \in U$

$$s(x) = s_1(x) \otimes s_2(x) \in \mathscr{O}(k_1)_x \otimes_{\mathscr{O}_x} \mathscr{O}(k_2)_x$$

Analogously

$$\mathscr{O}(k_1 + k_2)(U) = \{\sigma : U \to |\mathscr{O}(k_1 + k_2)| : \ \sigma \text{ section of } p_{|\mathscr{O}(k_1+k_2)|}\}$$

Therefore

$$f_U : (\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2))(U) \to \mathscr{O}(k_1 + k_2)(U), s = s_1 \otimes s_2 \mapsto \sigma := s_1 \cdot s_2$$

is well-defined.

iii) *Isomorphism*: Due to part 1 the morphism

$$f : \mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2) \to \mathscr{O}(k_1 + k_2)$$

induces for each $x \in \mathbb{P}^1$ a morphism of stalks

$$f_x : (\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2))_x = \mathscr{O}(k_1)_x \otimes_{\mathscr{O}_x} \mathscr{O}(k_2)_x \to \mathscr{O}(k_1 + k_2)_x$$

Using a fixed complex chart around x we identify

$$\mathscr{O}(k_1)_x \simeq \mathscr{O}(k_2)_x \simeq \mathscr{O}(k_1 + k_2)_x \simeq \mathscr{O}_x$$

then

$$\mathscr{O}_x \simeq \mathscr{O}_x \otimes_{\mathscr{O}_x} \mathscr{O}_x \xrightarrow{f_x} \mathscr{O}_x, \ s_1 \otimes s_2 \mapsto s_1 \cdot s_2,$$

is an isomorphism of stalks. Hence f is an isomorphism of sheaves.

3. *Tensor product of presheaves*: For the twisted sheaves $\mathscr{O}(k)$ on $\mathbb{P}^1$:

- On one hand, due to part 2 we have the isomorphy of sheaves

$$\mathscr{O}(k_1) \otimes_{\mathscr{O}} \mathscr{O}(k_2) \simeq \mathscr{O}(k_1 + k_2).$$

- On the other hand, tensoring global sections gives for $k_1 = 1, \ k_2 = -1$

$$\mathscr{O}(1)(\mathbb{P}^1) \otimes_{\mathscr{O}(\mathbb{P}^1)} \mathscr{O}(-1)(\mathbb{P}^1) = \mathbb{C}^2 \otimes_{\mathbb{C}} 0 = 0 \text{ (first taking sections, then tensoring)}$$

while

$$(\mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{O}(-1))(\mathbb{P}^1) = \mathscr{O}(\mathbb{P}^1) = \mathbb{C} \text{ (first tensoring then taking sections)}.$$

Therefore

$$0 = \mathscr{O}(1)(\mathbb{P}^1) \otimes_{\mathscr{O}(\mathbb{P}^1)} \mathscr{O}(-1)(\mathbb{P}^1) \subsetneq (\mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{O}(1))(\mathbb{P}^1) = \mathbb{C}.$$

As a consequence, tensoring does not commute with taking sections, and the presheaf

$$\mathscr{O}(1)(U) \otimes_{\mathscr{O}(U)} \mathscr{O}(-1)(U),\ U \subset \mathbb{P}^1 \text{ open},$$

is not a sheaf.

3.2 Proper holomorphic maps

The present section combines the result of the local representation of a holomorphic map from Proposition 1.6 and the topological properties of proper maps to obtain a global result about the fibres of holomorphic maps between compact Riemann surfaces, see Theorem 3.22.

Definition 3.19 (Proper map). A continous map

$$f : X \to Y$$

between two locally compact Hausdorff spaces is *proper* if the inverse image of any compact subset of Y is compact.

Remark 3.20 (Proper map).

1. *Compact domain implies properness*: For compact X any continuous map to a locally compact Hausdorff space is proper.

2. *Proper maps are closed*: Each proper map

$$f : X \to Y$$

is closed, i.e. each closed set $A \subset X$ has a closed image $f(A) \subset Y$.

Proof. We recall that in a locally compact space a set is closed iff its intersection with any compact set is compact.

Now consider a closed set $A \subset X$. We have to show: For any compact $K \subset Y$ the set

$$f(A) \cap K \subset Y$$

is closed. We have the equality

$$f(A) \cap K = f(A \cap f^{-1}(K)).$$

By assumption

$$f^{-1}(K) \subset X$$

is compact, hence also

$$A \cap f^{-1}(K) \subset X$$

is compact. As a consequence the image

$$f(A \cap f^{-1}(K))$$

is compact, q.e.d.

3. *Neighbourhood of a fibre*: Consider a proper map

$$f : X \to Y$$

and a point $y \in Y$. Then for any open neighbourhood $U \subset X$ of the fibre X_y exists an open neighbourhoood $V \subset Y$ of y with

$$f^{-1}(V) \subset U.$$

Proof. The complement

$$X \setminus U \subset X$$

is closed. Part 2 implies that

$$A := f(X \setminus U) \subset Y$$

is closed. Because $y \notin A$ the complement

$$V := Y \setminus A \subset Y$$

is an open neighbourhood of y. It satisfies

$$f^{-1}(V) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A) =$$

$$= X \setminus f^{-1}(f(X \setminus U)) \subset X \setminus (X \setminus U) = U, \ q.e.d.$$

4. *Proper unbranched coverings are unbounded*: A proper, unbranched covering projection

$$p : X \to Y$$

between locally compact Hausdorff spaces is an unbounded, unbranched covering projection with finite fibres. For connected Y the cardinality of the fibres

$$X_y, y \in Y,$$

does not depend on $y \in Y$.

Proof. Consider an arbitrary but fixed point $y \in f(X)$. Because f is proper and discrete, the fibre $X_y,\ y \in Y$, is discrete and compact, hence finite

$$X_y = \{x_1, ..., x_n\}.$$

Because p is unbranched, for each $j = 1, ..., n$ the point $x_j \in X_y$ has an open neighbourhood W_j such that

$$p|W_j : W_j \to V_j := p'(W_j)$$

is a local homeomorphism onto an open set $V_j \subset Y$. W.l.o.g. the sets

$$W_j,\ j = 1, ..., n,$$

are pairwise disjoint because X_y is discrete. Part 3 implies the existence of an open neighbourhood of y

$$V \subset \bigcap_{j=1}^{n} V_j$$

with

$$p^{-1}(V) \subset \bigcup_{j=1}^{n} W_j.$$

For $j = 1, ..., n$ we set

$$U_j := W_j \cap p^{-1}(V).$$

Then

$$p^{-1}(V) = \dot{\bigcup}_{j=1,...,n} U_j,$$

and for each $j = 1, .., n$ the restriction

$$p|U_j : U_j \to V$$

is a homeomorphisms. The map f is proper and open, in particular closed and open. Hence

$$f(X) = Y$$

if Y is connected. The cardinality of the fibres X_y depends continuously on y. Hence for connected Y this number is independent from $y \in Y$, q.e.d.

If a holomorphic map has finite fibres, Definition 3.21 gives the cardinality of the fibres. The definition includes the fibres at critical values.

Definition 3.21 (Cardinality of a finite fibre). Consider a holomorphic map

$$f : X \to Y$$

with finite fibres. For a point $x \in X$ one defines the *multiplicity* of f at x as

$$\text{v}(f;\, x) := k$$

with $k \in \mathbb{N}$ the number from the local representation of f at x according to Proposition 1.6. For each $y \in Y$ one defines the *cardinality* of the fibre

$$X_y := f^{-1}(y)$$

as

$$card\ X_y := \sum_{x \in X_y} \text{v}(f;\, x).$$

Apparently, Definition 3.21 counts the cardinality according to multiplicity.

Theorem 3.22 (Value attainment of proper holomorphic maps). *Consider a non-constant proper holomorphic map*

$$f : X \to Y$$

between two Riemann surfaces X and Y. Then f assumes every value $y \in Y$ with the same multiplicity, i.e. all fibres

$$X_y,\ y \in Y,$$

have the same cardinality, counted according to multiplicity.

Proof. Corollary 1.7 and Remark 3.20 imply that the holomorphic map f is open and closed. Connectedness of Y implies $f(X) = Y$, i.e. f is surjective.

i) *Unbounded, unbranched covering projection outside the critical fibres*: According to Proposition 1.6 the function f has locally the form

$$f(z) = z^k,\ k \geq 1.$$

Hence its set A of branch points is discrete and closed, which implies that also the set

$$B := f(A) \subset Y$$

of critical values is closed. Set

$$Y' := Y \setminus B \text{ and } X' := X \setminus f^{-1}(B).$$

The restriction

$$f' := f|X' : X' \to Y'$$

is an unbranched covering projection. Properness of f implies that also f' is proper. Remark 3.20, part 4 implies that

$$f' : X' \to Y'$$

is an unbounded unbranched covering projection with finite fibres. The set X' is connected. Hence also Y' is connected and the cardinality of the fibres of f' has a constant value $n \in \mathbb{N}^*$.

ii) *Cardinality of the fibres at critical values*: Let $y_0 \in B$ be a critical value of f with fibre

$$X_{y_0} = \{x_1, ..., x_r\},\ k_j := \text{v}(f;\ x_j) \in \mathbb{N},\ j = 1, ..., r.$$

Proposition 1.6 implies: For each $j = 1, .., r$ exist an open neighbourhood

$$U_j \subset X$$

of x_j and an open neigbhourhood

$$V_j \subset Y$$

of y_0 such that for each point $y \in V_j \setminus \{y_0\}$ the set

$$f^{-1}(y) \cap U_j$$

has

$$k_j = \text{v}(f, x_j)$$

distinct points. We may assume the open sets U_j pairwise disjoint. Remark 3.20, part 3 implies the existence of an open neighbourhood of y_0

$$V \subset \bigcap_{j=1}^{r} V_j$$

with

$$f^{-1}(V) \subset \bigcup_{j=1}^{r} U_j.$$

Hence for each point $y \in V \cap Y'$ the fibre X_y has

$$n = k_1 + ... + k_r$$

points while

$$card\ X_{y_0} := \sum_{j=1}^{r} \text{v}(f; x_j) = k_1 + ... + k_r,\ q.e.d.$$

Corollary 3.23 (Surjectivity of holomorphic maps). *Any non-constant holomorphic map $f : X \to Y$ between two Riemann surfaces with X compact is surjective.*

Proof. The map f is proper, q.e.d.

Corollary 3.24 (Poles and zeros of a meromorphic function). *A non-constant meromorphic function on a compact Riemann surface X has the same number of poles and zeros, counted according to multiplicity. In particular: A polynomial of degree $n \in \mathbb{N}^*$, when considered as a meromorphic function on $\mathbb{P}^1$, has exactly n zeros, counted according multiplicity.*

Proof. According to Theorem 1.10 a meromorphic function f on a Riemann surface X can be considered a holomorphic map

$$f : X \to \mathbb{P}^1.$$

The map is proper because X is compact. Hence Theorem 3.22 proves the claim. A polynomial of degree $n \in \mathbb{N}^*$ has a single pole. The pole is at $\infty \in \mathbb{P}^1$ and has the order $= n$. Hence the polynomial has exactly n zeros, q.e.d.

3.3 Analytic continuation

Definition 3.25 (Analytic continuation of a germ along a path). Consider a Riemann surface X, a path

$$\gamma : I \to X$$

from a point $a \in X$ to a point $b \in X$. A germ

$$f_b \in \mathscr{O}_b$$

originates from a germ $f_a \in \mathscr{O}_a$ by *analytic continuation along* γ if the following properties are satisfied:

- For each $t \in I$ exists a germ

  $$f_{\gamma(t)} \in \mathscr{O}_{\gamma(t)}$$

  such that

  $$f_{\gamma(0)} = f_a \text{ and } f_{\gamma(1)} = f_b$$

- and for each $t \in I$ exists an open neighbourhood $T \subset I$ of t, an open set $U \subset X$ with $\gamma(T) \subset U$, and a holomorphic function $f \in \mathscr{O}(U)$ such that for all $\tau \in T$

  $$f_{\gamma(\tau)} = \pi^U_{\gamma(\tau)}(f).$$

See Figure 3.1, upper part.

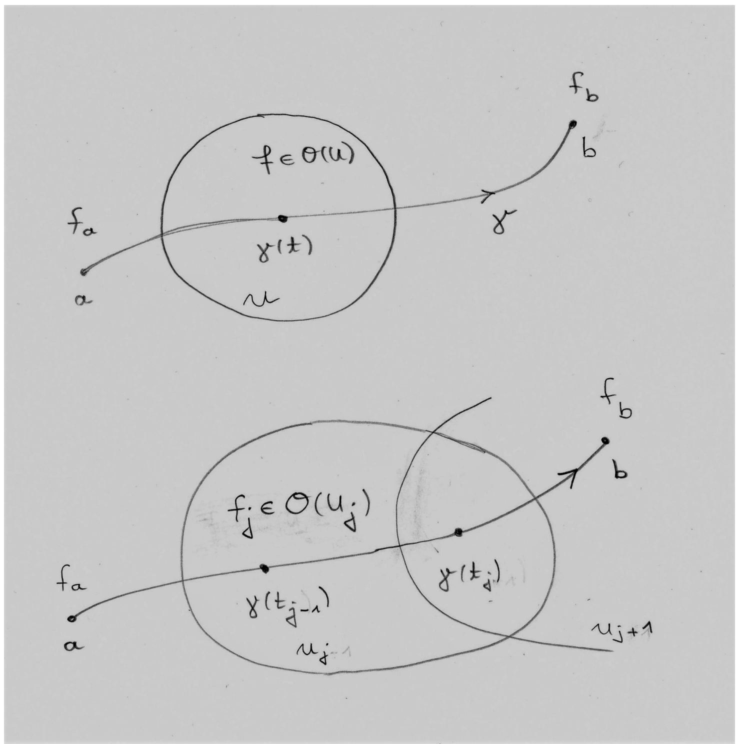


Fig. 3.1 Analytic continuation

Because I is compact, Definition 3.25 is equivalent to the following "Kreiskettenverfahren", see Figure 3.1, lower part: There exist

- a finite subdivision of I

$$0 < t_0 < t_1 < ... < t_{n-1} < t_n = 1$$

- a family

$$\mathscr{U} = \{U_1, ..., U_n\}$$

  of domains in $U_j \subset X$ with

$$\gamma([t_{j-1}, t_j]) \subset U_j, \ j = 1, ..., n,$$

- and for each $j = 1, ..., n$ a holomorphic function $f_j \in \mathscr{O}(U_j)$ with

$$f_a = \pi_a^{U_1}(f_1) \text{ and } f_b = \pi_b^{U_n}(f_n)$$

such that on the component of $U_j \cap U_{j+1}$ which contains the point $\gamma(t_j)$

$$f_j = f_{j+1},\ j = 1, ..., n-1.$$

Note that Definition 3.25 defines the analytic continuation of germs as a relation between germs, not between functions.

The concept of analytic continuation of a holomorphic germ along a path translates to lifting the path to the étale space of the structure sheaf.

Theorem 3.26 (Analytic continuation of holomorphic germs and the étale space of the structure sheaf). *Consider a Riemann surface X, two points $a,\ b \in X$ with germs*

$$f_a \in \mathscr{O}_a \text{ and } f_b \in \mathscr{O}_b$$

and a path

$$\gamma : I \to X$$

with

$$\gamma(0) = a \text{ and } \gamma(1) = b.$$

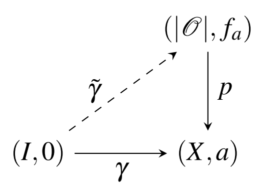
Then are equivalent:

- *The germ $f_b \in \mathscr{O}_b$ is the analytic continuation of f_a along γ*

- *The path γ lifts to a path*

  $$\tilde{\gamma} : I \to |\mathscr{O}|$$

  according to



  and the lifting $\tilde{\gamma}$ satisfies

  $$\tilde{\gamma}(1) = f_b$$

Proof. i) Assume that $f_b \in \mathscr{O}_b$ is the analytic continuation of $f_a \in \mathscr{O}_b$ along γ. The family $(f_{\gamma(t)})_{t \in I}$ defines the map

$$\tilde{\gamma} : I \to |\mathscr{O}|,\ t \mapsto f_{\gamma(t)}$$

By definition of the topology of $|\mathscr{O}|$ the map $\tilde{\gamma}$ is continuous and the diagram from Theorem 3.26 commutes.

ii) Conversely, assume the existence of a lift $\tilde{\gamma}$ according to the diagram. We define a family of germs $(f_{\gamma(t)})_{t\in I}$ by

$$f_t := \tilde{\gamma}(t) \in \mathscr{O}_{\gamma(t)}.$$

For given $\tau \in I$ choose an open neighbourhood $[U,f]$ of $\tilde{\gamma}(\tau)$. By continuity exists an open neighbourhood T of τ in I such that

$$\tilde{\gamma}(T) \subset [U,f], \text{ i.e.}$$

$$\gamma(T) \subset U \text{ and } f_{\gamma(t)} = \tilde{\gamma}(t) = \pi_t^U(f) \text{ for all } t \in T.$$

By definition

$$f_b := \tilde{\gamma}(1)$$

is an analytic continuation of f_a along γ, q.e.d.

Theorem 3.26 states concerning a germ $f_a \in \mathscr{O}_a$: The analytic continuations of f_a along a path γ equals the endpoint $\tilde{\gamma}(1)$ of the lift $\tilde{\gamma}$ of γ. According to Theorem 3.27 the analytic continuation along a path γ depends only on the homotopy class of γ.

Proposition 3.27 (Monodromy). *Let X be a Riemann surface. Consider two paths*

$$\gamma_0,\ \gamma_1 : I \to X$$

with

$$a := \gamma_0(0) = \gamma_1(0) \text{ and } b := \gamma_0(1) = \gamma_1(1)$$

and a homotopy $(\gamma_s)_{s\in I}$ relative $\{0,1\}$ from γ_0 to γ_1. Moreover consider a germ $f_a \in \mathscr{O}_a$ and assume that f_a has an analytic continuation along every path $\gamma_s,\ s \in I$. Then f_a extends along γ_0 and along γ_1 to the same germ $f_b \in \mathscr{O}_b$, see Figure 3.2.

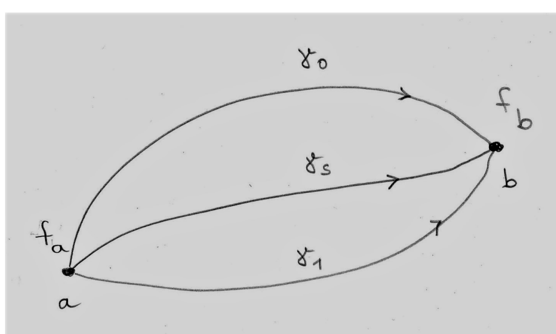


Fig. 3.2 Monodromy

Proof. Theorem 3.26 implies that each curve $\gamma_s,\ s \in I$, lifts to a curve

$$\tilde{\gamma}_s,\ s \in I,$$

which starts at $f_a \in |\mathscr{O}|$. One checks that the map

$$\tilde{\Phi} : I \times I \to X,\ (t,s) \mapsto \tilde{\gamma}_s(t)$$

is continuous because p is a local homeomorphism, see [8, Theor. 4.10]. In particular, the endpoints of $\tilde{\gamma}_s,\ s \in I$, depend continuously on $s \in I$. They vary in the fibre $p^{-1}(b)$, hence they are constant, i.e. each lift $\tilde{\gamma}_s,\ s \in I$, terminates at $f_b \in |\mathscr{O}|$, q.e.d.

Remark 3.28 (Pushdown of holomorphic germs along holomorphic, unbranched covering projections). A holomorphic, unbranched covering projection

$$p : Y \to X$$

between two Riemann surfaces is locally biholomorphic. Hence for each $x \in X$ and each $y \in Y_x$ the composition with the canonical maps from Remark 2.13, 2

$$p^* := [\mathscr{O}_{X,x} \to (p_*\mathscr{O}_Y)_x \to \mathscr{O}_{Y,y}]$$

is an isomorphism of stalks. We denote by

$$p_* : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$$

its inverse, the *pushdown* of germs.

We now investigate the analytic continuation of a holomorphic germ to a global holomorphic function. In particular, we have to provide a domain of definition for the analytic extension.

Definition 3.29 (Global analytic continuation of a germ). Consider a Riemann surface X, a point $a \in X$ and a germ $f_a \in \mathscr{O}_a$.

1. A *global analytic continuation* of f_a is a triple

$$(p, f, b)$$

with

- a holomorphic, unbranched covering of pointed Riemann surfaces.

$$p : (Y, b) \to (X, a)$$

- and a holomorphic function
$$f : Y \to \mathbb{C}$$
such that the germ
$$f_b \in \mathscr{O}_{Y,b}$$
of f in $b \in Y$ satisfies
$$p_*(f_b) = f_a \text{ with } p_* : \mathscr{O}_{Y,b} \to \mathscr{O}_{X,a}$$
the pushdown from Remark 3.28.

2. A global analytic continuation (p, f, b) of f_a is *maximal* or *universal* iff each analytic contiuation of f_a factorizes via (p, f, b), i.e. iff for any analytic continuation
$$(p', f', b')$$
of f_a exists a holomorphic map
$$F : Y' \to Y$$
such that the following diagram commutes
$$\begin{array}{ccc} (Y', b') & \overset{F}{\dashrightarrow} & (Y, b) \\ & {\scriptstyle p'} \searrow \quad \swarrow {\scriptstyle p} & \\ & (X, a) & \end{array}$$
and the pullback of f satisfies
$$F^*(f) := f \circ F = f'.$$

Lemma 3.30 proves for a global analytic continuation (Y, p, b) of $f_a \in \mathscr{O}_{X,a}$: Any path starting at $b \in Y$ induces an analytic continuation of f_a along the induced path in X.

Lemma 3.30 (Global analytic continuation and analytic continuations along paths). *Let X be a Riemann surface and consider a global analytic continuation*
$$(p, f, b)$$
of a germ $f_a \in \mathscr{O}_{X,a}$. Then for any point $y \in Y$ the germ at $x := p(y)$
$$f_x := p_*(f_y) \in \mathscr{O}_{X,x}$$
originates from the germ

$$f_a \in \mathscr{O}_{X,a}$$

by analytic continuation along a suitable path in X from a to x.

Proof. Because the Riemann surface Y is connected and therefore also path-connected, we may choose a path in Y from b to y

$$\alpha : (I, 0) \to (Y, b).$$

The path α projects to the path $p \circ \alpha$ in X from a to x according to the commutative diagram

$$\begin{array}{ccc} & & (Y,b) \\ & \overset{\alpha}{\nearrow} & \downarrow p \\ (I,0) & \underset{p\circ\alpha}{\dashrightarrow} & (X,a) \end{array}$$

The locally biholomorphic map pushes down the holomorphic germs along α to a compatible family of holomorphic germs along $p \circ \alpha$, see Figure 3.3, q.e.d.

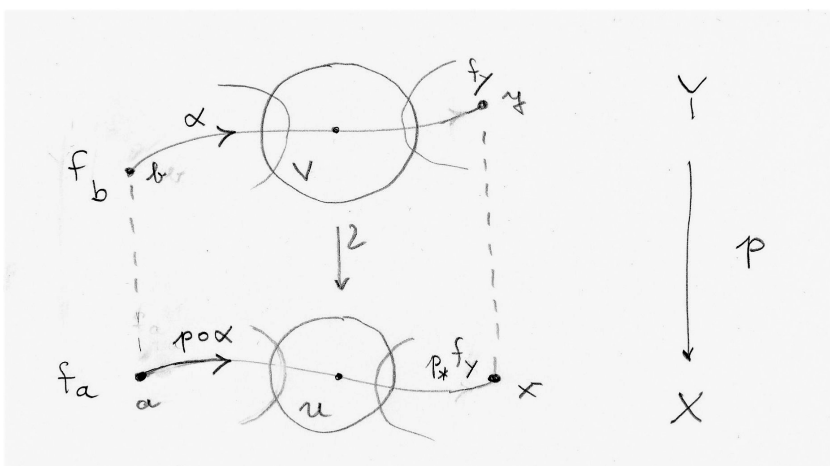


Fig. 3.3 Paths in the étale space and analytic continuation

Summing up the statements of Lemma 3.30 and Theorem 3.26: Consider a Riemann surface X, a point $a \in X$ and a germ $f_a \in \mathscr{O}_{X,a}$. Assume a global analytic continuation (p, f, b) of f_a.

- Each point $y \in Y$ defines a germ

$$p_*(f_y) \in \mathscr{O}_{X,x},\ x := p(y),$$

which originates from f_a by analytic continuation along a path in X.

- If $Y \subset |\mathscr{O}_X|$ is the component which contains f_a and

$$p : Y \to X$$

the canonical unbranched covering, then the germs, which originate from f_a by analytic continuation along a path in X, correspond bijectively to the points of Y.

Theorem 3.31 shows: The maximal global analytic extensions of the germs of holomorphic functions on a Riemann surface X are the restrictions of the étale space of the structure sheaf $\mathscr{O}$

$$p : |\mathscr{O}| \to X$$

to its connected components $Y \subset |\mathscr{O}|$. Hence we now make the step from the level of stalks to the level of global holomorphic functions which are implicitely defined by the germs of the stalk.

Theorem 3.31 (Existence of the maximal global analytic continuation). *Consider a Riemann surface X and a point $a \in X$. Then any germ $f_a \in \mathscr{O}_a$ has a maximal global analytic continuation.*

Proof. We have to define a triple (p, f, b) with the properties from Definition 3.29.

i) *Definition of p*: We denote by $Y \subset |\mathscr{O}|$ the component which contains f_a and restrict the canonical projection

$$|\mathscr{O}| \to X$$

to obtain a holomorphic unbranched covering projection

$$p : Y \to X,$$

see Remark 3.14. We set

$$b := f_a \in Y.$$

ii) *Definition of f*: We define

$$f : Y \to \mathbb{C}$$

in the following tautological way: By definition each point $y \in Y$ is a germ $f_{p(y)} \in \mathscr{O}_{p(y)}$. We define

$$f(y) := f_{p(y)}(p(y))$$

attaching to $y \in Y$ the value of the germ

$$f_{p(y)} \in \mathscr{O}_{p(y)}$$

at the point $p(y) \in X$.

To show that f is holomorphic we note that the component $Y \subset |\mathscr{O}|$ is open because the topological space $|\mathscr{O}|$ is locally connected. A given point $y \in Y$ with

$$x := p(y) \in X$$

has an open neighbourhood $V \subset Y$ such

$$U := p(V) \subset X$$

is open, the restriction

$$p|V : V \to U$$

is biholomorphic, and the germ f_x has a holomorphic representative

$$f_U : U \to \mathbb{C}.$$

The composition

$$f|V = f_U \circ (p|V)$$

shows the holomorphy of $f|V$. As a consequence

$$(p, f, b)$$

is a global analytic extension of $f_a \in \mathscr{O}_a$.

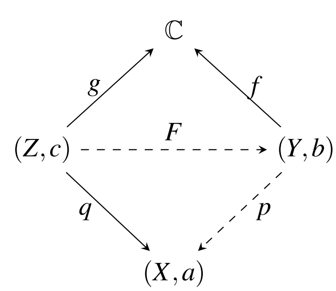
iii) *Maximality*: Now consider a further global analytic continuation (q, g, c) of $f_a \in \mathscr{O}_a$ with

$$q : (Z, c) \to (X, a).$$

We have to construct a holomorphic function

$$F : (Z, c) \to (Y, b)$$

such that the following diagram commutes:



We define

$$F : (Z, c) \to (Y, b)$$

as follows: For a given point $\zeta \in Z$ choose a path in Z from c to ζ

$$\tilde{\alpha} : (I, 0) \to (Z, c).$$

- *Pushing down* $\tilde{\alpha}$ *via* q: Lemma 3.30 implies for the path

$$\gamma := q \circ \tilde{\alpha} : (I, 0) \to (X, a) :$$

The germ

$$f_a = q_*(g_c) \in \mathscr{O}_{X,x}$$

extends along γ to the analytic germ

$$f_x := q_*(g_\zeta) \in \mathscr{O}_{X,x},\ x := q(\zeta),$$

see Figure 3.5.

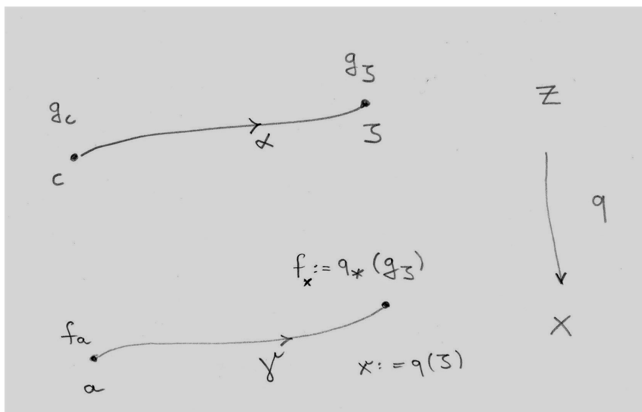


Fig. 3.4 Definition of F

- *Lifting* γ *via* p: According to Theorem 3.26 the germs, which originate from$f_a \in \mathscr{O}_{X,a}$ by analytic continuations along the path γ in X, correspond bijectively to the points of Y. Hence the germ

$$f_x \in \mathscr{O}_{X,x}$$

determines a unique point $y \in Y$ with

$$p_*(f_y) = f_x = q_*(g_\zeta) \in \mathscr{O}_{X,x}$$

We define

$$F(\zeta) := y \in Y.$$

One checks that F is holomorphic and satisfies

$$F^*(f) = g,\ q.e.d.$$

[25, Chap. 3] presents some interesting examples of maximal global analytic continuations.

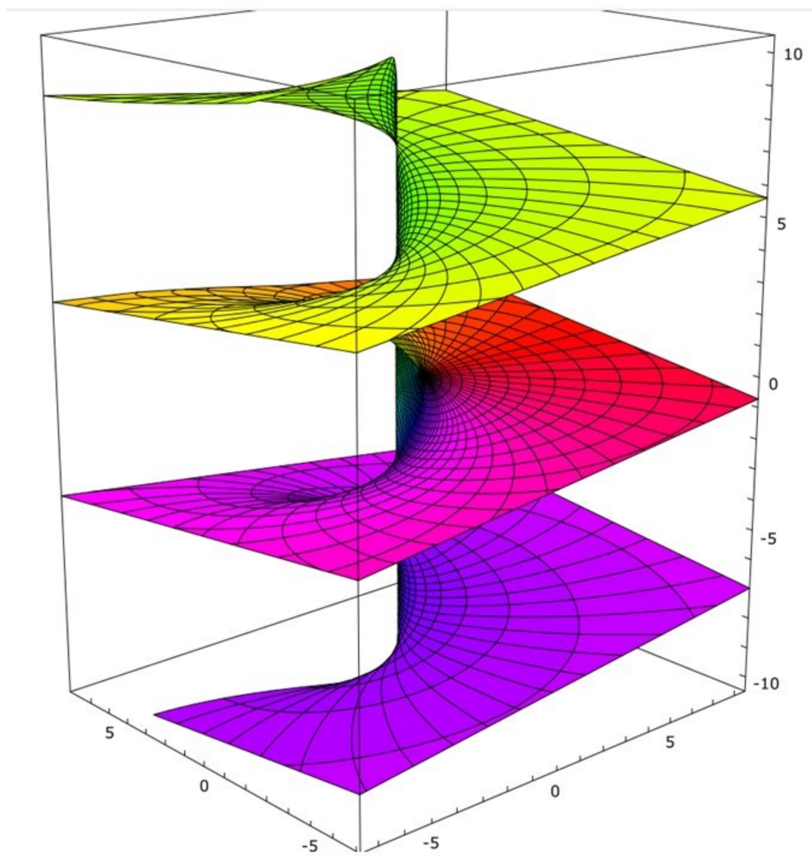


Fig. 3.5 The "mysterious" spiral staircase: Maximal global analytic continuation of the logarithm (due to Leonid 2)

Remark 3.32 (The "mysterious" spiral staircase). Figure 3.5 visualizes the exponential map as the maximal global analytic continuation (p, f, b) of the germ of the principal value *Log* of the logarithm at the point

$$a = 1 \in \mathbb{C}^*.$$

The figure shows the total space Y of the unbranched covering projection

$$p : (Y, b) \to (X, a) = (\mathbb{C}^*, a)$$

The projection maps in vertical direction onto (X, a).

i) *Construction of* (p, f, b): The Riemann surface Y is obtained by gluing the family

$$Y_k,\ k \in \mathbb{Z},$$

of copies $Y_k \simeq \mathbb{C}^*$ along the negative real axis $\mathbb{R}^-$: After passing $\mathbb{R}^-$ from the second quadrant in Y_k one enters the third quadrant in Y_{k+1}. After gluing Y_k with Y_{k+1}, the limit points when approaching the negative real axis from the upper Y_{k+1} are considered elements of the lower Y_k. Define

$$Y := \dot{\bigcup}_{k \in \mathbb{Z}} Y_k$$

Then elements of Y are pairs

$$(y, k) \in \mathbb{C}^* \times \mathbb{Z}.$$

Set

$$a := 1 \in \mathbb{C}^* \text{ and } b := (1, 0) \in Y.$$

The map p projects Y in vertical direction onto $\mathbb{C}^*$ as

$$p : (Y, b) \to (\mathbb{C}^*, a),\ (y, k) \mapsto y.$$

On Y exists a global logarithm, the holomorphic function

$$log : Y \to \mathbb{C},\ log(y, k) := Log(y) + k \cdot 2\pi i$$

with *Log* the principal value of the logarithm. The triple

$$(p, log, b)$$

is a global analytic continuation of the germ of *Log* at $a = 1$.

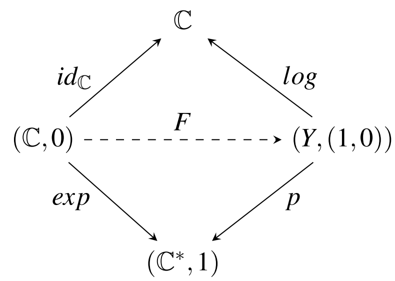
ii) *The exponential map and global analytic continuation of Log*: The Riemann surface Y is simply connected because there is no central fibre Y_0. As a consequence Y is homeomorphic - and a posteriori biholomorphic - to the total space $\mathbb{C}$ of the universal covering of $\mathbb{C}^*$

$$exp : \mathbb{C} \to \mathbb{C}^*$$

A biholomorphic map

$$F : \mathbb{C} \xrightarrow{\simeq} Y$$

making commutative the diagram



can be obtained as follows: Let

$$D := \{z \in \mathbb{C} : -i\pi < Im\, z \leq i\pi\}$$

denote a fundamental domain of the exponential map and set

$$D_k := D + k \cdot 2\pi i$$

Each restriction

$$exp|D_k : D_k \to \mathbb{C}^*$$

is a bijective continuous map. The maps

$$F_k : D_k \to Y_k, (z,k) \mapsto (exp(z),k),\ k \in \mathbb{Z},$$

combine to a biholomorphic map

$$F : \mathbb{C} = \dot{\bigcup}_{k\in\mathbb{Z}} D_k \to Y = \dot{\bigcup}_{k\in\mathbb{Z}} Y_k$$

with

$$F(0) = (1,0) \in Y$$

One computes for each $k \in \mathbb{Z}$

$$F^*(f)|D_k : D_k \to \mathbb{C}, z \mapsto log(exp(z),k) = Log(exp(z)) + k \cdot 2\pi i = z,$$

hence

$$log \circ F = id_{\mathbb{C}}.$$

As a consequence the two global analytic continuations of the germ of *Log*

$$(p, f, b) \text{ and } (exp, id_{\mathbb{C}}, 0)$$

with

$$exp : (\mathbb{C}, 0) \to (\mathbb{C}^*, 1)$$

are isomorphic.

iii) *Maximality*: If (q,g,c) with

$$q:(Z,c)\to(X,a),\ Z\subset|\mathscr{O}_X|,$$

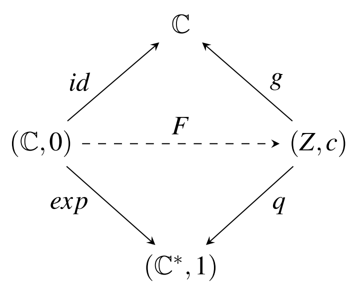
denotes the maximal global continuation of the germ

$$f_a = Log_1$$

then the maximality induces a holomorphic map

$$F:\mathbb{C}\to Z$$

such that the following diagram commutes



and

$$F^*(g) := g\circ F = id_{\mathbb{C}}.$$

Hence F is injective. Because *exp* and q are surjective unbranched covering projections, the general theory of covering projections implies that also F is a surjective covering projection. Hence the map F is a holomorphic homeomorphism, and therefore biholomorphic, q.e.d.

We conclude this section with an outlook. It shows how the étale space of the structure sheaf can be used to consider a multiple-valued meromorphic function on a compact Riemann surface X as a well-defined meromorphic function on a covering of X.

Remark 3.33 (Algebraic extensions of the field of meromorphic functions). Consider a compact Riemann surface X and the field

$$k := \mathscr{M}(X)$$

of global meromorphic functions on X. Each finite field extension K/k of k has a primitive element: There exist an element F with

$$K = k(F).$$

The field extension poses the following problem:

- How to realize F as a meromorphic funtion $F \in \mathscr{M}(Y)$ defined on a suitable Riemann surface Y and

- how to relate Y to X?

i) *The minimal polynomial*: We consider the *minimal polynomial* $P \in k[T]$ of K/k, an irreducible polynomial of degree

$$n := [K : k].$$

Let

$$P(T) = T^n + c_1 \cdot T^{n-1} + ... + c_{n-1} \cdot T + c_n \in k[T]$$

For $j = 1, ..., n$ the coefficient $(-1)^j \cdot c_j \in k$ is the j-th elementary symmetric polynomial

$$s_j(F_1, ..., F_n) \in k$$

in the n-roots of P.

ii) *Construction within the étale space* $|\mathscr{O}_X|$: Denote by $\Delta \in k[T]$ the discriminant of P. There exists a discrete set $A \subset X$ such that each coefficient $c_j,\ j = 1, ..., n$, is holomorphic in a neighbourhood of all points of

$$X' := X \setminus A$$

and such that Δ has no zeros in X'. Set

$$Y' := \{g_x \in |\mathscr{O}_X| : x \in X' \text{ and } P(g_x) = 0\}$$

with the canonical projection

$$\pi' : Y' \to X'.$$

One defines the holomorphic function

$$F_1 : Y' \to X',\ F_1(g_x) := g_x(x).$$

It satisfies $P(F_1) = 0$, i.e. for all $y \in Y'$ and for all $x := \pi'(y) \in X'$

$$0 = P(F_1)(y) = F_1(y)^n + c_1(x) \cdot F_1(y)^{n-1} + ... + c_{n-1}(x) \cdot F_1(y) + c_n(x).$$

iii) *Extending the unbranched covering to include branch points over* X: One checks, that π' is an unbounded, unbranched covering projection, which extends to a proper, holomorphic, but possibly branched covering projection

$$\pi : Y \to X$$

of Riemann surfaces according to the following commutative diagram

$$\begin{array}{ccc} Y' & \dashrightarrow & Y \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \longrightarrow & X \end{array}$$

The holomorphic function F_1 on Y' extends to a meromorphic function F on Y annihilated by the pullback $\pi^*(P)$ of the minimal polynomial, i.e. on Y

$$(\pi^*(P))(F) = 0.$$

For the details of this construction as well as for an illustrative example see [8, Theorem 8.9 and Example 8.10].

Chapter 4
Differential forms

Exterior derivation is the means to define derivatives on smooth manifolds. The exterior derivation generalizes the partial derivations in affine space. In order that the result on a manifold is independent from the used charts one has to define the exterior derivation of a functions as a first order differential form. Analogously, the exterior derivation of a first order differential form has to be defined as a second order differential form.

The main result on exterior derivation on a Riemann surface, i.e. on a manifold with an additional complex structure, is the exactness of the Dolbeault sequence and the de Rham sequence, see Section 5.1 and 5.2 later on.

4.1 Cotangent space

This section starts with considering the smooth structure of a Riemann manifold. First, we study some algebraic properties of the stalks of the sheaf $\mathscr{E}$ of smooth functions. We recall that a *local* ring is a ring with exactly one maximal ideal.

Lemma 4.1 (Ring of germs of smooth functions). *Denote by R the ring of smooth functions defined in a neighbourhood of* $0 \in \mathbb{R}^2$*, i.e. R is the quotient of the set*

$$\{f : U \to \mathbb{C} | \ U \subset \mathbb{R}^2 \text{ open neighbourhood of } 0, \ f \text{ smooth}\}$$

when identifying two functions with the same restriction to a common neighbourhood of zero. The ring R is isomorphic to the stalks $\mathscr{E}_{X,x}$ *of the structure sheaf* $\mathscr{E}_X$ *of the smooth - not the complex - structure of a Riemann surface X at an arbitrary point* $x \in X$*.*

1. The set

$$\mathfrak{m} := \{f \in R : f(0) = 0\}$$

is the unique maximal ideal of R, hence R is a local ring.

2. The product satisfies

$$\mathfrak{m}^2 = \left\{ f \in R : f(0) = \frac{\partial f}{\partial x}(0) = \frac{\partial f}{\partial y}(0) = 0 \right\}.$$

Proof. 1) If $f \in R$ but $f(0) \neq 0$ then the reciproque

$$f^{-1} := \frac{1}{f} \in R,$$

hence f is a unit in R. Conversely: For every unit f the reciproque $f^{-1} \in R$ exists which implies $f(0) \neq 0$.

Hence $R \setminus \mathfrak{m}$ is the set of units of R. Any ideal $\mathfrak{a} \subset R$ which contains an element $f \in R \setminus \mathfrak{m}$ equals R. As a consequence $\mathfrak{m} \subset R$ is a maximal ideal, and it is the only maximal ideal.

2) If

$$h := f \cdot g \in \mathfrak{m}^2,\ f,\ g \in \mathfrak{m},$$

then by the product rule

$$\frac{\partial h}{\partial x}(0) = \frac{\partial f}{\partial x}(0) \cdot g(0) + f(0) \cdot \frac{\partial g}{\partial x}(0) = 0$$

and similarly for $\dfrac{\partial h}{\partial y}(0)$. Conversely assume $h \in \mathfrak{m}$ satisfying

$$\frac{\partial h}{\partial x}(0) = \frac{\partial h}{\partial y}(0) = 0.$$

For (x,y) in a ball neighbourhood of $0 \in \mathbb{R}^2$ we have

$$h(x,y) := \int_0^1 \frac{d}{dt}\, h(tx,ty)\ dt = \int_0^1 \left(\frac{\partial h}{\partial x}(tx,ty) \cdot x + \frac{\partial h}{\partial y}(tx,ty) \cdot y \right)\ dt =$$

$$= x \cdot \int_0^1 \frac{\partial h}{\partial x}(tx,ty)\ dt + y \cdot \int_0^1 \frac{\partial h}{\partial y}(tx,ty)\ dt = x \cdot h_1(x,y) + y \cdot h_2(x,y)$$

with

$$h_1,\ h_2 \in \mathfrak{m} \text{ and thus } h \in \mathfrak{m}^2$$

because by assumption

$$h_1(0,0) = \int_0^1 \frac{\partial h}{\partial x}(0,0)\, dt = \frac{\partial h}{\partial x}(0,0) \cdot 1 = 0$$

and similarly

$$h_2(0,0) = 0, \; q.e.d.$$

A smooth 2-dimensional manifold is locally isomorphic to an open neighbourhood of $0 \in \mathbb{R}^2$. Hence the result of Lemma 4.1 carries over to the stalks of the structure sheaf $\mathscr{E}$ of the smooth structure underlying a Riemann surface.

Corollary 4.2 (The stalk of the structure sheaves $\mathscr{E}$ and $\mathscr{O}$). *Let X be a Riemann surface and fix an arbitrary point $p \in X$.*

1. *The stalk $\mathscr{E}_p$ of the smooth structure sheaf $\mathscr{E}$ on X is a commutative ring with* 1.

2. *An element $f \in \mathscr{E}_p$ is a unit if and only if*

$$f(p) \neq 0.$$

The non-units of $\mathscr{E}_p$ form the unique maximal ideal

$$\mathfrak{m}_{\mathscr{E},p} = \{f \in \mathscr{E}_p : \; f(p) = 0\}.$$

The ring $\mathscr{E}_p$ is a local ring with residue field

$$k(p) := \mathscr{E}_p/\mathfrak{m}_{\mathscr{E},p} = \mathbb{C}$$

and the injection

$$k(p) \hookrightarrow \mathscr{E}_p$$

embeds the germs of locally constant functions.

3. *The subring*

$$\mathscr{O}_p \subset \mathscr{E}_p,$$

the stalk of the holomorphic structure sheaf on X, is a local ring with maximal ideal

$$\mathfrak{m}_{\mathscr{O},p} = \{f \in \mathscr{O}_p : \; f(p) = 0\}$$

and residue field

$$k_{\mathscr{O}}(p) := \mathscr{O}_p/\mathfrak{m}_{\mathscr{O},p} = \mathbb{C}.$$

Note. In the holomorphic context the powers of the maximal ideal

$$\mathfrak{m}^k_{\mathscr{O},p}, \; k \in \mathbb{N}^*,$$

are principal ideals. The ideal $\mathfrak{m}_{\mathscr{O},p}$ is generated by the germ of the coordinate function z. The product $\mathfrak{m}^k_{\mathscr{O},p}$ is generated by the germ of power of z^k.

For a Riemann surface X we now use charts around a given point to carry over the concept of the cotangent space from the affine spaces $\mathbb{R}^2 \simeq \mathbb{C}$ to the smooth and to the complex structure of X. We start with the smooth cotangent space and identify the holomorphic cotangent space as a subspace. If not stated otherwise we will identify $\mathbb{C}$ and its open subsets with $\mathbb{R}^2$ and its open subsets.

Definition 4.3 (Partial derivations and Wirtinger operators). Let X be a Riemann surface. A chart of X

$$z = x + i \cdot y : U \to V \subset \mathbb{C}$$

defines for any function $f \in \mathscr{E}(U)$ a smooth function

$$f \circ z^{-1} : V \to \mathbb{C}$$

according to the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbb{C} \\ {\scriptstyle z}\downarrow & \nearrow_{f \circ z^{-1}} & \\ V & & \end{array}$$

One defines the *partial derivations*

$$\frac{\partial}{\partial x} : \mathscr{E}(U) \to \mathscr{E}(U),\ \frac{\partial f}{\partial x} := \frac{\partial (f \circ z^{-1})}{\partial x} \circ z$$

$$\frac{\partial}{\partial y} : \mathscr{E}(U) \to \mathscr{E}(U),\ \frac{\partial f}{\partial y} := \frac{\partial (f \circ z^{-1})}{\partial y} \circ z$$

and the *Wirtinger operators*

$$\partial := \frac{\partial}{\partial z} := \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y} \right) \text{ and } \overline{\partial} := \frac{\partial}{\partial \overline{z}} := \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y} \right)$$

Note. The partial derivatives from Definition 4.3 depend on the choice of the chart which is used for the definition.

Remark 4.4 (Wirtinger operators). Consider a Riemann surface X, a chart of X

$$z : U \to V \subset \mathbb{R}^2$$

and a smooth function

$$f : U \to \mathbb{C}.$$

i) The Wirtinger operators on the Riemann surface X relate to the Wirtinger operators on $\mathbb{C}$:

$$\partial f := \frac{\partial(f \circ z^{-1})}{\partial z} \circ z \text{ and } \overline{\partial} f := \frac{\partial(f \circ z^{-1})}{\partial \overline{z}} \circ z$$

In both equations the Wirtinger operator on the left-hand side relates to the open set $U \subset X$ of the Riemann surface, while the Wirtinger operator on the right-hand side relates to the open set $V \subset \mathbb{C}$ of the plane. As a consequence, the smooth function f is holomorphic iff

$$\overline{\partial} f = 0.$$

ii) With respect to complex conjugation the Wirtinger operators satisfy

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \overline{f}}{\partial \overline{z}} \text{ and } \frac{\partial f}{\partial \overline{z}} = \overline{\frac{\partial \overline{f}}{\partial z}}.$$

iii) We have

$$\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}} = \frac{1}{i} \cdot \frac{\partial}{\partial y}$$

Definition 4.5 (Smooth cotangent space and differential). Consider a Riemann surface X and fix a point $p \in X$. Denote by

$$\mathfrak{m}_{\mathscr{E},p} \subset \mathscr{E}_p$$

the maximal ideal in the ring of germs of smooth functions in a neighbourhood of p.

1. The quotient

$$T_p^1 := \mathfrak{m}_{\mathscr{E},p} / \mathfrak{m}_{\mathscr{E},p}^2$$

is in a canonical way a vector space over the residue field

$$\mathbb{C} \simeq k(p) = \mathscr{E}_p / \mathfrak{m}_{\mathscr{E},p},$$

named the *smooth cotangent space* of X at $p \in X$. Its elements are called *smooth cotangent vectors* of X at p.

2. The canonical map

$$d_p : \mathscr{E}_p \to T_p^1,\ f \mapsto d_p f := f - f(p) \mod \mathfrak{m}_p^2$$

is named the *differential*. Because

$$f - f(p) \in \mathfrak{m}_p$$

its residue class in $\mathfrak{m}_{\mathscr{E},p} / \mathfrak{m}_{\mathscr{E},p}^2$ is well-defined.

The smooth cotangential space T_p^1 is an invariant of the stalk $\mathscr{E}_p$ of the smooth structure sheaf $\mathscr{E}$ of the Riemann surface X. Because all rings $\mathscr{E}_p, p \in X$, are isomorphic to the same ring R, the cotangent space T_p^1 is also denoted T^1R and equated with the cotangent space of the ring R.

Proposition 4.6 (Basis of the smooth cotangent space). *Consider a Riemann surface X and a point $p \in X$. If*

$$z : U \to V \subset \mathbb{C}$$

is chart around p with decomposition

$$z = x + i \cdot y$$

then the two differentials

$$(d_p x, d_p y)$$

form a basis of the smooth cotangent space T_p^1. The cotangent vector derived from a germ $f \in \mathscr{E}_p$

$$d_p f = f - f(p) \mod \mathfrak{m}_p^2$$

has the basis representation

$$d_p f = \frac{\partial f}{\partial x}(p) \cdot d_p x + \frac{\partial f}{\partial y}(p) \cdot d_p y$$

Proof. The claim is local. Hence we may assume $U \subset \mathbb{C} \simeq \mathbb{R}^2$ a disk with center $p = 0$.

i) *Generators*: The idea of the proof is to represent a smooth function $f \in \mathscr{E}(U)$ by its Taylor expansion

$$f(x,y) = f(0) + \frac{\partial f}{\partial x}(0) \cdot x + \frac{\partial f}{\partial y}(0) \cdot y + r(x,y)$$

with the rest term $r \in \mathscr{E}(U)$ satisfying

$$r(0,0) = \frac{\partial r}{\partial x}(0,0) = \frac{\partial r}{\partial y}(0,0) = 0.$$

Lemma 4.1 implies $r \in \mathfrak{m}_p^2$. We obtain

$$f - f(0) \equiv \frac{\partial f}{\partial x}(0) \cdot x + \frac{\partial f}{\partial y}(0) \cdot y \mod \mathfrak{m}_p^2$$

Hence

$$d_p f = \frac{\partial f}{\partial x}(0) \cdot d_p x + \frac{\partial f}{\partial y}(0) \cdot d_p y$$

ii) *Linear independence*: If the cotangent vector satisfies

$$c_1 \cdot d_p x + c_2 \cdot d_p y = 0 \in \mathfrak{m}_p/\mathfrak{m}_p^2$$

then its representing function

$$f := c_1 \cdot x + c_2 \cdot y \in \mathfrak{m}_p$$

already belongs to $\mathfrak{m}_p^2$. Lemma 4.1 implies that the partial derivatives at $p = 0$ vanish, i.e.

$$0 = \frac{\partial f}{\partial x}(p) = c_1 \text{ and } 0 = \frac{\partial f}{\partial y}(p) = c_2, \ q.e.d.$$

The smoooth cotangent space T_p^1 is a 2-dimensional complex vector space:

- It is a complex vector space because it is defined by using complex-valued functions.

- It is 2-dimensional because the differentials of the two real coordinate functions x and y form a basis.

The cotangent space T_p^1 is attached to the point $p \in X$. The index p must not suggest that its elements are germs - they are not. We will see that the cotangent vectors from T_p^1 are the *values* at the point p of the germs of certain differential forms defined in an open neighbourhood of p.

Proposition 4.7 (Splitting the smooth cotangent space). *Let X be a Riemann surface. Consider a point $p \in X$. A chart around p*

$$z : U \to V$$

splits the smooth cotangent space as the direct sum of two 1*-dimensional complex subspaces*

$$T_p^1 = T_p^{1,0} \oplus T_p^{0,1}$$

with

$$T_p^{1,0} := \mathbb{C} \cdot d_p z \text{ and } T_p^{0,1} := \mathbb{C} \cdot d_p \bar{z}.$$

The splitting is independent from the choice of the chart. As a consequence, the differential d_p splits as

$$d'_p : \mathscr{E}_p \to T_p^{1,0}, \ d'_p(f) := \frac{\partial f}{\partial z} \cdot d_p z,$$

and

$$d''_p : \mathscr{E}_p \to T_p^{0,1}, \ d''_p(f) := \frac{\partial f}{\partial \bar{z}} \cdot d_p \bar{z},$$

satisfying

$$d_p f = d'_p f + d''_p f.$$

The complex subspace

$$T_p^{1,0} \subset T_p^1$$

is named the holomorphic cotangent space *of X at p.*

Proof. We assume the existence of a second chart of X around $p \in X$

$$w : S \to T$$

Then

$$U \cap V \neq \emptyset.$$

With respect to the chart z we have the Taylor expansions of w

$$w - w(p) = \frac{\partial w}{\partial z}(p) \cdot (z - z(p)) + \frac{\partial w}{\partial \overline{z}}(p) \cdot (\overline{z} - \overline{z}(p)) \quad \text{mod } \mathfrak{m}^2_{\mathscr{E},p}$$

and of $\overline{w}$

$$\overline{w} - \overline{w}(p) = \frac{\partial \overline{w}}{\partial z}(p) \cdot (z - z(p)) + \frac{\partial \overline{w}}{\partial \overline{z}}(p) \cdot (\overline{z} - \overline{z}(p)) \quad \text{mod } \mathfrak{m}^2_{\mathscr{E},p}$$

They imply

$$d_p w = \frac{\partial w}{\partial z}(p) \cdot d_p z + \frac{\partial w}{\partial \overline{z}}(p) \cdot d_p \overline{z}$$

$$d_p \overline{w} = \frac{\partial \overline{w}}{\partial z}(p) \cdot d_p z + \frac{\partial \overline{w}}{\partial \overline{z}}(p) \cdot d_p \overline{z}.$$

The holomorphy of the transition function

$$w \circ z^{-1}$$

implies

$$\frac{\partial w}{\partial \overline{z}} = 0.$$

and together with Remark 4.4

$$\frac{\partial \overline{w}}{\partial z} = \overline{\frac{\partial w}{\partial \overline{z}}} = 0.$$

Hence

$$d_p w = \frac{\partial w}{\partial z}(p) \cdot d_p z \text{ and } d_p \overline{w} = \frac{\partial \overline{w}}{\partial \overline{z}}(p) \cdot d_p \overline{z}$$

with non-zero coefficients because $w \circ z^{-1}$ is locally biholomorphic. As a consequence the splitting

$$T_p^1 = T^{1,0} \oplus T^{0,1}$$

is independent from the chosen chart, q.e.d.

4.2 Exterior derivation

From real analysis it is well-known that differential forms of higher order are generated by the exterior product of first oder differentials. We recall the underlying algebraic construction, the *exterior algebra* of a vector space.

Remark 4.8 (Exterior product). For a complex vector space V the *exterior product*

$$\bigwedge^2 V$$

is the complex vector space with elements

$$\mathrm{v}_1 \wedge \mathrm{v}_2,\ \mathrm{v}_1, \mathrm{v}_2 \in V$$

which satisfy the rules

$$(\mathrm{v}_1 + \mathrm{v}_2) \wedge \mathrm{v}_3 = \mathrm{v}_1 \wedge \mathrm{v}_3 + \mathrm{v}_2 \wedge \mathrm{v}_3$$

$$(\lambda \cdot \mathrm{v}_1) \wedge \mathrm{v}_2 = \lambda \cdot (\mathrm{v}_1 \wedge \mathrm{v}_2)$$

$$\mathrm{v}_1 \wedge \mathrm{v}_2 = -\mathrm{v}_2 \wedge \mathrm{v}_1 \text{ (alternating)}$$

If $(\mathrm{v}_1, ..., \mathrm{v}_n)$ is a basis of V then the elements

$$\mathrm{v}_i \wedge \mathrm{v}_j,\ 1 \leq i < j \leq n$$

are a basis of $\bigwedge^2 V$. As consequence

$$dim \bigwedge^2 V = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Lemma 4.9 (Basis of T_p^2). *Let X be a Riemann surface. Consider a point $p \in X$ and a chart around p of X*

$$z : U \to V$$

with decomposition

$$z = x + i \cdot y.$$

Then each of the following elements

$$d_p x \wedge d_p y \text{ and } d_p z \wedge d_p \bar{z} = -2i \cdot d_p x \wedge d_p y$$

is a basis of the 1*-dimensional exterior product of the smooth cotangent space*

$$T_p^2 := \bigwedge^2 T_p^1$$

Definition 4.10 (Differential forms). Consider a Riemann surface X. For $j = 1,2$ define

$$T^jX := \dot{\bigcup}_{x\in X} T_x^j$$

with the canonical map

$$p_j : T^jX \to X, \text{ v} \mapsto x \text{ if v} \in T_x^j.$$

A *differential form* on an open set $Y \subset X$ of order j is a section of p_j, i.e. a map

$$\omega : Y \to T^jX \text{ with } p_j \circ \omega = id_Y.$$

In Definition 4.10 the condition

$$p_j \circ \omega = id_Y$$

means that a differential form ω of order j evaluates at each $x \in Y$ to a value

$$\omega(x) \in T_x^j$$

in the smooth cotangent space respectively in its exterior power.

If

$$z : U \to V$$

is chart of X around $p \in Y$ then any first order differential form ω on $U \cap Y$ has the form

$$\omega = f \cdot dz + g \cdot d\bar{z}$$

with two functions $f, g : U \cap Y \to \mathbb{C}$. Here dz and $d\bar{z}$ are the differential forms on U which evaluate at a point $p \in U$ to

$$d_pz \in T_p^1 \text{ respectively } d_p\bar{z} \in T_p^1.$$

Any second order differential form η on $U \cap Y$ has the form

$$\eta = h \cdot dz \wedge d\bar{z}$$

with a function $h : U \cap Y \to \mathbb{C}$. Here

$$dz \wedge d\bar{z}$$

evaluates at $p \in U$ to

$$d_p z \wedge d_p \bar{z} \in T_p^2.$$

Definition 4.11 (The sheaves of respectively smooth, holomorphic, and meromorphic differential forms). Let X be a Riemann surface.

1. A first order differential form

$$\omega : Y \to T^1 X,\ Y \subset X \text{ open},$$

is *smooth* respectively *holomorphic* respectively *meromorphic* if for any chart

$$z : U \to V$$

the restriction to $U \cap Y$ has the form

$$\omega | U \cap Y = f \cdot dz + g \cdot d\bar{z}$$

with smooth functions f, g respectively

$$\omega | U \cap Y = f \cdot dz$$

with a holomorphic respectively meromorphic function f.

2. Similarly, a second order differential form

$$\omega : Y \to T^2 X, Y \subset X \text{ open},$$

is *smooth* if for any chart

$$z : U \to V$$

the restriction to $U \cap Y$ has the form

$$\omega | U \cap Y = f \cdot dz \wedge d\bar{z}$$

with a smooth function f.

3. For $j = 1, 2$, the presheaf

$$\mathscr{E}^j(U) := \{\omega : U \to T^j U : \text{ smooth}\},\ U \subset X \text{ open},$$

with canoncial restrictions is a sheaf $\mathscr{E}^j$ on X, named the *sheaf of smooth differential forms of order j*. Analogously one obtains the subsheaves

$$\mathscr{E}^{1,0} \subset \mathscr{E}^1 \text{ and } \mathscr{E}^{0,1} \subset \mathscr{E}^1$$

with the sections

$$\mathscr{E}^{1,0}(U) := \{\omega \in \mathscr{E}^1(U) : \ \omega(x) \in T_x^{1,0} \text{ for all } x \in U\}$$

and

$$\mathscr{E}^{0,1}(U) := \{\omega \in \mathscr{E}^1(U) : \ \omega(x) \in T_x^{0,1} \text{ for all } x \in U\}.$$

One defines

$$\mathscr{E}^0 := \mathscr{E} \text{ and } \mathscr{E}^{1,1} := \mathscr{E}^2$$

as respectively the smooth structure sheaf and the sheaf of smooth differential forms of highest order.

4. The presheaf

$$\Omega^1(U) := \{\omega : U \to T^1U : \ \text{holomorphic}\},\ U \subset X \text{ open},$$

with canoncial restrictions is a sheaf Ω^1 on X, named the *sheaf of holomorphic differential forms*.

5. The presheaf

$$\mathscr{M}^1(U) := \{\omega : U \to T^1U : \ \text{meromorphic}\},\ U \subset X \text{ open},$$

with canoncial restrictions is a sheaf $\mathscr{M}^1$ on X, named the *sheaf of meromorphic differential forms*.

Definition 4.12 (Exterior derivation with respect to charts). Let X be a Riemann surface and

$$z : U \to \mathbb{C}$$

a chart of X. For $j = 0, 1$ we define $\mathbb{C}$-linear maps

$$d, d', d'' : \mathscr{E}^j(U) \to \mathscr{E}^{j+1}(U)$$

as follows:

- *j=0*: For $f \in \mathscr{E}(U)$ set

$$d'f := \partial f \cdot dz \in \mathscr{E}^{1,0}(U) \text{ and } d''f := \overline{\partial} f \cdot d\bar{z} \in \mathscr{E}^{0,1}(U)$$

  and

$$df := d'f + d''f \in \mathscr{E}(U) \text{ (Total differential)}$$

- *j=1*: For

$$\omega = g \cdot dz + h \cdot d\bar{z} \in \mathscr{E}^1(U)$$

  set

$$d'\omega := d'g \wedge dz + d'h \wedge d\bar{z} = d'h \wedge d\bar{z} = \partial h \cdot dz \wedge d\bar{z} \in \mathscr{E}^2(U)$$
$$d''\omega := d''g \wedge dz + d''h \wedge d\bar{z} = d''g \wedge dz = \overline{\partial} g \cdot d\bar{z} \wedge dz \in \mathscr{E}^2(U)$$

and
$$d\omega := d'\omega + d''\omega \in \mathscr{E}^2(U).$$

In Definition 4.12 note
$$d'g \wedge dz = 0 \text{ and } d''h \wedge d\bar{z} = 0.$$

Proposition 4.13 (The sheaf morphism exterior derivation). *Let X be a Riemann surface. The locally defined exterior derivations from Definition 4.12 are independent from the choice of the charts. For* $j = 0, 1$ *they define sheaf morphisms*
$$d, d', d'' : \mathscr{E}^j \to \mathscr{E}^{j+1}$$
with
$$d'(\mathscr{E}) \subset \mathscr{E}^{1,0} \text{ and } d''(\mathscr{E}) \subset \mathscr{E}^{0,1}$$

Proof. Consider a second chart
$$w : S \to T.$$

Using the holomorphy of the transition function we obtain
$$dz = \frac{\partial z}{\partial w} \cdot dw \text{ and } d\bar{z} = \frac{\partial \bar{z}}{\partial \overline{w}} \cdot d\overline{w},$$
compare proof of Proposition 4.7. As a consequence
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial w} \cdot \frac{\partial w}{\partial z} \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \overline{w}} \cdot \frac{\partial \overline{w}}{\partial \bar{z}}$$

1. *j=0*: Consider a smooth function $f \in \mathscr{E}(U \cap S)$. With respect to the chart z we have
$$df = \frac{\partial f}{\partial z} \cdot dz + \frac{\partial f}{\partial \bar{z}} \cdot d\bar{z}$$
As a consequence
$$df = \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial w} \cdot dw + \frac{\partial f}{\partial \overline{w}} \cdot \frac{\partial \overline{w}}{\partial \bar{z}} \cdot \frac{\partial \bar{z}}{\partial \overline{w}} \cdot d\overline{w}$$
Hence
$$df = \frac{\partial f}{\partial w} \cdot dw + \frac{\partial f}{\partial \overline{w}} \cdot d\overline{w}$$
which is the exterior derivation obtained by using the chart w.

2. *j=1*:

i) First we proof the independence for the exterior derivation d':

Because
$$d'|\mathscr{E}^{(1,0)}(U) = 0$$
it suffices to consider
$$\omega = f \cdot d\bar{z} \in \mathscr{E}^{(0,1)}(U \cap S).$$
Transforming $d\bar{z}$ we obtain with respect to the chart w
$$\omega = f \cdot \left(\frac{\partial \bar{z}}{\partial \overline{w}} \cdot d\overline{w} \right) = \left(f \cdot \frac{\partial \bar{z}}{\partial \overline{w}} \right) \cdot d\overline{w}$$
and
$$d'\omega = \frac{\partial}{\partial w} \left(f \cdot \frac{\partial \bar{z}}{\partial \overline{w}} \right) \cdot dw \wedge d\overline{w} = \frac{\partial f}{\partial w} \cdot \frac{\partial \bar{z}}{\partial \overline{w}} \cdot dw \wedge d\overline{w}$$
Here
$$f \cdot \frac{\partial^2 \bar{z}}{\partial w \partial \overline{w}} = 0$$
because
$$\frac{\partial \bar{z}}{\partial w} = 0.$$
Using
$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial w}$$
we obtain
$$d'\omega = \frac{\partial f}{\partial z} \cdot \left(\frac{\partial z}{\partial w} \cdot dw \right) \wedge \left(\frac{\partial \bar{z}}{\partial \overline{w}} \cdot d\overline{w} \right) = \frac{\partial f}{\partial z} \cdot (dz \wedge d\bar{z}),$$
which equals the definition of $d'\omega$ with respect to the chart z.

ii) The proof of the independence of d'' is analogous.

iii) The independence of
$$d = d' + d''$$
follows from part i) and ii).

Proposition 4.14 (Restriction and iteration of derivations). *Let X be a Riemann surface. The exterior derivations satisfy:*

1. Restriction*:*
$$d|\Omega^1 = 0 \text{ and } d'|\mathscr{E}^{1,0} = d''|\mathscr{E}^{0,1} = 0.$$

2. Iteration*:*

$$0 = d \circ d = d' \circ d' = d'' \circ d'' : \mathscr{E} \to \mathscr{E}^2.$$

and

$$d' \circ d'' = -d'' \circ d',$$

i.e. each mesh is anticommutative:

$$\begin{array}{ccc} \mathscr{E}^{p,q} & \xrightarrow{d'} & \mathscr{E}^{p+1,q} \\ {\scriptstyle d''}\downarrow & & \downarrow{\scriptstyle d''} \\ \mathscr{E}^{p,q+1} & \xrightarrow{d'} & \mathscr{E}^{p+1,q+1} \end{array}$$

Proof. 1. The proof follows directly from the local representation.

2. It suffices to prove the analogous claim on the level of stalks. Here the question is local and w.l.o.g. $X = \mathbb{C}$:

$$(d \circ d)(f) = d(df) =$$

$$= d\left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial^2 f}{\partial z^2}\, dz \wedge dz + \frac{\partial^2 f}{\partial \bar{z} \partial z}\, d\bar{z} \wedge dz + \frac{\partial^2 f}{\partial z \partial \bar{z}}\, dz \wedge d\bar{z} + \frac{\partial^2 f}{\partial \bar{z}^2}\, d\bar{z} \wedge d\bar{z} =$$

$$= \frac{\partial^2 f}{\partial z \partial \bar{z}} \cdot (d\bar{z} \wedge dz + dz \wedge d\bar{z}) = 0.$$

Moreover

$$(d' \circ d')(f) = d'\left(\frac{\partial f}{\partial z}\, dz\right) = \frac{\partial^2 f}{\partial z^2}\, dz \wedge dz = 0$$

and

$$(d'' \circ d'')(f) = \frac{\partial^2 f}{\partial \bar{z}^2}\, d\bar{z} \wedge d\bar{z} = 0$$

Eventually

$$0 = (d \circ d) = (d' + d'') \circ (d' + d'') = d' \circ d' + d' \circ d'' + d'' \circ d' + d'' \circ d''$$

implies

$$d' \circ d'' = -d'' \circ d',\ q.e.d.$$

Definition 4.15 (Closed and exact differential forms). Consider a Riemann surface X. For an open set $U \subset X$ and $m \in \mathbb{N}$ differential forms $\omega \in \mathscr{E}^m(U)$ with $d\omega = 0$ are named *closed* and differential forms of the form $d\omega \in \mathscr{E}^{m+1}(U)$ are named *exact*.

4.3 Residue theorem

In complex analysis an important property of a holomorphic function f with isolated singularities are the residues at the singularities of f. The residue at a singularity is defined by means of the Laurent expansion of f. On a Riemann manifold the Laurent series and also the residue depends on the choice of a chart, hence it is not invariant. The solution is to define the residue not for a function but for a differential form.

Definition 4.16 (Residue of a holomorphic differential form with an isolated singularity). Let $Y \subset X$ be an open subset of a Riemann surface, and consider a point $p \in Y$ and a differential form

$$\omega \in \Omega^1(Y \setminus \{p\}),\ p \in Y.$$

After choosing a chart of X around p

$$z : U \to V \text{ with } U \subset Y,\ z(p) = 0,$$

the differential form is given as

$$\omega|(U \setminus \{p\}) = f \cdot dz$$

with a holomorphic function

$$f \in \mathscr{O}(U \setminus \{p\}).$$

One defines the *residue* of ω at $p \in Y$ as

$$res(\omega; p) := res(f \circ z^{-1};\ 0) \in \mathbb{C}.$$

Similarly to the terms in complex analysis one classifies in the situation of Definition 4.16 the singularity of ω depending on the singularity of f as *removable*, a *pole* or an *essential singularity*. A differential form is meromorphic if all singularities are removable or poles. These definitions are independent from the choice of charts on X.

Proposition 4.17 (Residue of a differential form). *Let $Y \subset X$ be an open subset of a Riemann surface and consider a differential form*

$$\omega \in \Omega^1(Y \setminus \{p\}),\ p \in Y.$$

The value

$$res(\omega; p)$$

from Definition 4.16 is independent from the choice of a chart around p.

Proof. i) *The specific case* $\omega = dg$: Consider a holomorphic function

$$g \in \mathscr{O}(U \setminus \{p\})$$

and a chart around p

$$z : U \to V.$$

The Laurent series

$$g = \sum_{n \in \mathbb{Z}} c_n \cdot z^n$$

implies the representation

$$dg = d'g = \frac{\partial g}{\partial z} \cdot dz = \left(\sum_{n \in \mathbb{Z}} n \cdot c_n \cdot z^{n-1} \right) \cdot dz.$$

In particular

$$res(dg; p) = 0.$$

Apparently the result is independent from the choice of the chart.

ii) *The specific case* $\omega = \frac{dg}{g}$ *with* $ord(g; p) = 1$: If the function $g \in \mathscr{O}(U)$ has a zero of first order with respect to the chart z then

$$g = z \cdot h \text{ with } h \in \mathscr{O}(U),\ h(p) \neq 0.$$

Then

$$dg = h \cdot dz + z \cdot dh$$

and

$$\frac{dg}{g} = \frac{h \cdot dz + z \cdot dh}{z \cdot h} = \frac{dz}{z} + \frac{dh}{h}$$

Now $h(p) \neq 0$ implies the holomorphy $\frac{dh}{h} \in \Omega^1(U)$ and $res\left(\frac{dh}{h}; p\right) = 0$. As a consequence

$$res\left(\frac{dg}{g}; p\right) = res\left(\frac{dz}{z}; p\right) = 1$$

independently from the chosen chart.

iii) *General case*: Assume

$$\omega = f \cdot dz \text{ with } f \in \mathscr{O}(U \setminus \{p\}).$$

Consider the Laurent expansion

$$f=\sum_{n\in\mathbb{Z}}c_n\cdot z^n=\left(\sum_{n=-\infty}^{-2}c_n\cdot z^n\right)+\frac{c_{-1}}{z}+\sum_{n=0}^{\infty}c_n\cdot z^n.$$

and define the Laurent series

$$g:=\sum_{n=-\infty}^{-2}\frac{c_n}{n+1}\cdot z^{n+1}+\sum_{n=0}^{\infty}\frac{c_n}{n+1}\cdot z^{n+1}$$

Then

$$\omega=dg+c_{-1}\cdot\frac{dz}{z}$$

and

$$res(\omega;p)=0+c_{-1}=c_{-1}.$$

Both summands are independent from the choice of the chart according to part i) and ii). The result proves the independence of $res(\omega;p)$ and finishes the proof, q.e.d.

To prepare the step from the level of germs to the level of gobal objects we recall some results from topology concerning locally finite covering and paracompactness.

Definition 4.18 (Paracompactness and partition of unity). Consider a topologcial Hausdorff space X.

1. *Locally finite covering*: An open covering $\mathscr{V}$ of X is *locally finite*, if any point $x\in X$ has a neighbourhood $W\subset X$ which intersects only finitely many open sets from the covering, i.e. for only finitely many $V\in\mathscr{V}$

$$W\cap V\neq\emptyset.$$

2. *Refinement*: Consider an open covering $\mathscr{U}=(U_i)_{i\in I}$ of X. An open covering $(V_j)_{j\in J}$ of X - possibly with a different index set J - is a *refinement* of $\mathscr{U}$ if a map

$$\phi:J\to I$$

exists satisfying for all $j\in J$

$$V_j\subset U_{\phi(j)}$$

Notation:

$$\mathscr{V}<\mathscr{U},$$

3. *Paracompactness*: If each open covering of X has a locally finite refinement then X is *paracompact*.

4. *Relatively compact subset*: A subset $U\subset X$ is *relatively compact* if its closure $\overline{U}$ is compact. For two subsets $V\subset U\subset X$ the notation

$$V\subset\subset U$$

is a shorthand for

$$V \subset X \text{ relatively compact and } \overline{V} \subset U.$$

Here one first takes the closure $\overline{V}$ with respect to the topology of X, and then one requires $\overline{V} \subset U$.

5. *Shrinking*: Consider an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X. An open covering $\mathscr{V} = (V_i)_{i \in I}$ of X - with the same index set I - is a *shrinking* of $\mathscr{U}$, expressed as

$$\mathscr{V} << \mathscr{U},$$

if for each $i \in I$

$$V_i \subset\subset U_i.$$

6. *Support*: The *support* of a function

$$f : X \to \mathbb{C}$$

defined on a topological space X is defined as

$$supp\ f := \overline{\{x \in X : \ f(x) \neq 0\}}.$$

Similarly one defines the support of differential forms on open sets of a Riemann surface X.

7. *Partition of unity*: A *partition of unity* subordinate to an open covering $\mathscr{U} = (U_i)_{i \in I}$ of a Riemann surface X is a family of smooth functions

$$\phi_i : X \to [0,1] \text{ with } supp(\phi_i) \subset U_i,\ i \in I,$$

satisfying the following properties:

- The family $(\phi_i)_{i \in I}$ is locally finite, i.e. each point $x \in X$ has a neighbourhood W in X with
$$W \cap supp(\phi_i) \neq \emptyset$$
for only finitely many $i \in I$,

- and
$$\sum_{i \in I} \phi_i = 1.$$
Note that the sum is well-defined due to the condition on locally finiteness.

Consider a Hausdorff space X, a relatively compact open set $U \subset X$ and an open subset

$$V \subset U \text{ with } \overline{V} \subset U.$$

Then the closed subset $\overline{V}$ of the compact set $\overline{U}$ is compact itself, hence

$$V \subset\subset U.$$

As a consequence: If a covering $\mathscr{U} = (U_i)_{i \in I}$ of a Hausdorff space X is formed by relatively compact open subsets $U_i \subset X,\ i \in I$, then any shrinking

$$\mathscr{V} = (V_i)_{i \in I} << \mathscr{U}$$

satisfies for each $i \in I$

$$V_i \subset\subset U_i.$$

We will apply the concepts from Definition 4.18 always in the context of Riemann surfaces X. Hence we may assume that the open coverings of X under consideration are formed by relatively compact subsets of X.

Proposition 4.19 (Partition of unity). *Each Riemann surface X is paracompact, and each open covering of X has a subordinate partition of unity.*

Proof. i) *Paracompactness*: By definition X is second countable. Any topological manifold is locally compact. As a consequence X is paracompact, [35, I.8.7 Satz 2].

ii) *Shrinking theorem*: Every locally finite open covering of a paracompact space has a shrinking, [35, I.8.5 Satz 2; I.8.6 Satz 2].

iii) *Partition of unity*: Each locally finite open covering of a paracompact space has a subordinate partition of unity, [35, I.8.6 Satz 3].

Remark 4.20 (Radó's theorem on second countability). In Definition 1.2 of a Riemann surface the required second countability already follows from the other conditions. The result is due to Radó, see [8, §23].

Proposition 4.21 (Differential forms with compact support). *Let X be a Riemann surface and consider a differential form $\omega \in \mathscr{E}^1(X)$ with compact support. Then*

$$\iint_X d\omega = 0.$$

Proof. The claim is a direct application of Stokes' theorem on smooth manifolds. But one can avoid the general case of Stokes' theorem. Using a partition of unity as a tool for a "divide and conquer" method the claim follows already from the specific case of Stokes's theorem for open disks in $\mathbb{R}^2$:

We choose a finite covering

$$\mathscr{U} = (U_k)_{k=1,...n}$$

of the compact set $supp\ \omega$ with charts

$$z_k : U_k \to V_k$$

and a partition of unity $(\phi_k)_k$ subordinate to $\mathscr{U}$ with $supp\ \phi_k \subset\subset U_k$. Then

$$\omega = \omega_1 + ... + \omega_n$$

with each

$$\omega_k := \phi_k \cdot \omega \in \mathscr{E}(X),\ k = 1, ..., n,$$

having compact support

$$supp\ \omega_k \subset\subset U_k.$$

The claim reduces to compute for each $k = 1, ..., n$

$$\iint_X d\omega_k.$$

Therefore we may assume w.l.o.g. $X = \mathbb{C}$ and

$$supp\ \omega \subset D_R(0)$$

for suitable $R > 0$. Stokes' theorem for open subsets of $\mathbb{R}^2$ implies

$$\iint_{D_R(0)} d\omega = \int_{|z|=R} \omega = 0$$

because

$$\omega | \partial D_R(0) = 0,\ q.e.d.$$

Theorem 4.22 (Residue theorem). *Let X be a compact Riemann surface and let*

$$p_1, ..., p_n \in X$$

be finitely many pairwise distinct points. For any diffential form

$$\omega \in \Omega^1(X \setminus \{p_1, ..., p_n\})$$

holds

$$\sum_{k=1}^{n} res(\omega; p_k) = 0.$$

Proof. Set

$$X' := X \setminus \{p_1, ..., p_n\}.$$

The differential form $\omega \in \Omega^1(X')$ is not defined at the singularities. In general ω does not extend as a holomorphic form into the singularities. But one knows from the classical residue theorem, that the Laurent series at the singulartity can be obtained by integrating along a circuit around the singularity. Here the integrand is holomorphic. The strategy of the proof in part i) modifies ω around each singularity to a smooth form $g \cdot \omega \in \mathscr{E}(X)$ which extends by zero into the singularity.

i) *Smoothing ω at each singularity*: For each $k = 1, ..., n$ we choose a chart around p_k

$$z_k : U_k \to D_1(0)$$

and we may assume for $i \neq j$

$$U_i \cap U_j = \emptyset.$$

Moreover, for $k = 1, ..., n$ we choose a smooth function $\phi_k \in \mathscr{E}(X)$ with compact support

$$supp\ \phi_k \subset\subset U_k$$

and satisfying for a neighbourhood $U'_k \subset\subset U_k$ of p_k

$$\phi_k | U'_k = 1.$$

Consider the smooth function

$$g := 1 - (\phi_1 + ... + \phi_n) \in \mathscr{E}(X).$$

It satisfies for $k = 1, ..., n$

$$g | U'_k = 0.$$

The product

$$g \cdot \omega \in \mathscr{E}^1(X)$$

is well-defined and has compact support, because X is compact. Proposition 4.21 implies

$$\iint_X d(g \cdot \omega) = 0.$$

ii) *Extending each $d(\phi_k \cdot \omega)$ by zero into its singularity*: On

$$X' := X \setminus \{p_1, ..., p_k\}$$

the restriction $\omega | X'$ is holomorphic and therefore satisfies

$$d\omega = 0.$$

For each $k = 1, ..., n$ holds in $U'_k \cap X'$

$$\phi_k \cdot \omega = \omega$$

hence in $U'_k \cap X'$

$$0 = d(\phi_k \cdot \omega) = d\omega \in \mathscr{E}^2(U'_k \setminus \{p_k\}),$$

see Figure 4.1.

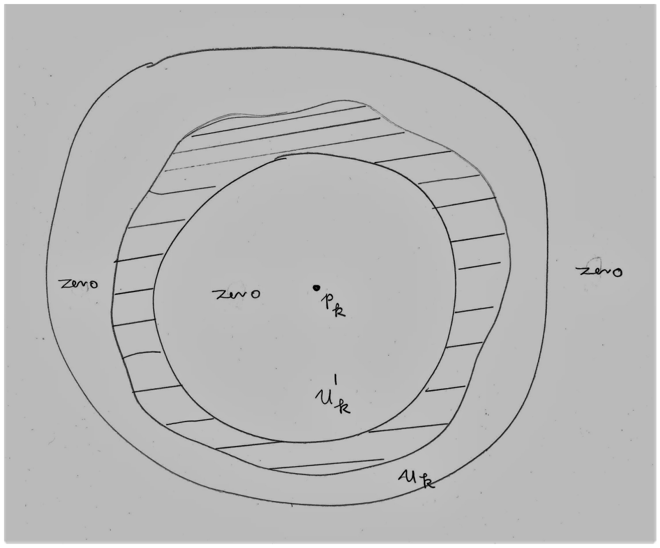


Fig. 4.1 supp $d(\phi_k \cdot \omega)$ dashed

The extension of $d(\phi_k \cdot \omega)$ by zero to the point p_k defines a global smooth differential form

$$d(\phi_k \cdot \omega) \in \mathscr{E}^2(X)$$

with compact support contained in $U_k \setminus U'_k$.

The equation on X'

$$d(g \cdot \omega) = d\omega - \sum_{k=1}^{n} d(\phi_k \cdot \omega) = -\sum_{k=1}^{n} d(\phi_k \cdot \omega)$$

implies for the extension to X the equation

$$d(g \cdot \omega) = -\sum_{k=1}^{n} d(\phi_k \cdot \omega).$$

Part i) implies

$$0 = \iint_X d(g \cdot \omega) = -\sum_{k=1}^{n} \iint_X d(\phi_k \cdot \omega).$$

iii) *Applying Stokes' theorem for open disks in the plane*: For each $k = 1, ..., n$ the differential form $d(\phi_k \cdot \omega)$ has compact support in U_k. Hence

$$\iint_X d(\phi_k \cdot \omega) = \iint_{U_k} d(\phi_k \cdot \omega)$$

and w.l.o.g. we may assume

$$U_k = D_R(0) \subset \mathbb{C}$$

such that

$$supp\ \phi_k \subset\subset D_R(0) \text{ and } \phi_k|D_\varepsilon(0) = 1$$

for suitable $0 < \varepsilon < R < \infty$. Applying sucessively the equality

$$d(\phi_k \cdot \omega) = d\omega = 0 \text{ on } U_k',$$

Stokes's theorem for open sets in the plane, and the residue theorem from complex analysis in the plane show

$$\iint_{U_k} d(\phi_k \cdot \omega) = \iint_{\varepsilon \le |z| \le R} d(\phi_k \cdot \omega) = \int_{|z|=R} \phi_k \cdot \omega - \int_{|z|=\varepsilon} \phi_k \cdot \omega =$$

$$- \int_{|z|=\varepsilon} \phi_k \cdot \omega = - \int_{|z|=\varepsilon} \omega = -2\pi i \cdot res(\omega; p_k).$$

We obtain

$$0 = - \sum_{k=1}^{n} \iint_X d(\phi_k \cdot \omega) = 2\pi i \cdot \sum_{k=1}^{n} res(\omega; p_k),\ q.e.d.$$

Remark 4.23 (Residue theorem). Consider the Riemann surface $X = \mathbb{C}$, the point $p = 0 \in X$ and the holomorphic form

$$\omega = \frac{dz}{z} \in \Omega^1(\mathbb{C}^*).$$

Then

$$res(\omega;\ p) = 2\pi i \neq 0.$$

The example shows that Theorem 4.22 needs the assumption of compactness.

Chapter 5
Dolbeault and de Rham sequences

5.1 Exactness of the Dolbeault sequence

The present section solves the partial differential equation

$$\overline{\partial} f = g \text{ with smooth source } g \in \mathscr{E}(X)$$

for disk domains

$$X = D_R(0) \subset \mathbb{C} \text{ with } 0 < R \leq \infty.$$

The result is called *Dolbeault's lemma*.

Theorem 5.1 (The inhomogeneous $\overline{\partial}$-equation with compact support). *For each smooth function $g \in \mathscr{E}(\mathbb{C})$ with compact support exists a smooth function $f \in \mathscr{E}(\mathbb{C})$with*

$$\overline{\partial} f = g.$$

Proof. The solution will be obtained by an integral formula using the Cauchy kernel. The important step of the argument is Stoke's theorem.

We define

$$f : \mathbb{C} \to \mathbb{C},\ f(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}$$

To show that the integral is well-defined for each fixed point $z \in \mathbb{C}$ we introduce polar coordinates around z

$$\zeta = z + re^{i\theta} \text{ with } d\zeta \wedge d\overline{\zeta} = -2ir\, dr \wedge d\theta$$

Hence

$$f(z) = -\frac{1}{\pi} \cdot \int_0^{2\pi} \int_0^{\infty} \frac{g(z + re^{i\theta})}{re^{i\theta}}\, r\, dr d\theta = -\frac{1}{\pi} \cdot \int_0^{2\pi} \int_0^{R} g(z + re^{i\theta}) \cdot e^{-i\theta}\, dr d\theta$$

with $R > 0$ suitable. Hence the function f is well-defined on $\mathbb{C}$. After the substitutions

$$\text{first } \zeta = z + w,\ d\zeta = dw, \text{ then eventually } w = \zeta$$

we obtain

$$f(z) = \frac{1}{2\pi i} \cdot \iint_{\mathbb{C}} \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \cdot \iint_{D_R(0)} \frac{g(z+w)}{w}\, dw \wedge d\overline{w} =$$

$$= \frac{1}{2\pi i} \cdot \iint_{D_R(0)} \frac{g(z+\zeta)}{\zeta}\, d\zeta \wedge d\overline{\zeta}$$

The integral depends smoothly on the parameter z. Hence we may interchange integration and differentiation with respect to $\overline{z}$.

$$\overline{\partial} f(z) = \frac{1}{2\pi i} \cdot \iint_{D_R(0)} \frac{\partial g(z+\zeta)}{\partial \overline{z}} \cdot \frac{1}{\zeta}\, d\zeta \wedge d\overline{\zeta} = \frac{1}{2\pi i} \cdot \lim_{\varepsilon \to 0} \iint_{A_\varepsilon} \frac{\partial g(z+\zeta)}{\partial \overline{z}} \cdot \frac{1}{\zeta}\, d\zeta \wedge d\overline{\zeta}$$

with the annulus

$$A_\varepsilon := \{\zeta \in \mathbb{C} :\ \varepsilon \le |\zeta| \le R\},\ \varepsilon > 0$$

The chain rule applied to

$$\frac{\partial g(\zeta + z)}{\partial \overline{z}} \text{ and to } \frac{\partial g(\zeta + z)}{\partial \overline{\zeta}}$$

shows for $\zeta \in A_\varepsilon$

$$\frac{\partial g(\zeta + z)}{\partial \overline{z}} \cdot \frac{1}{\zeta} = \frac{\partial g(\zeta + z)}{\partial \overline{\zeta}} \cdot \frac{1}{\zeta} = \frac{\partial}{\partial \overline{\zeta}} \left(\frac{g(\zeta + z)}{\zeta} \right)$$

because

$$\frac{\partial \zeta}{\partial \overline{\zeta}} = 0.$$

We obtain

$$\overline{\partial} f(z) = \frac{1}{2\pi i} \cdot \lim_{\varepsilon \to 0} \iint_{A_\varepsilon} \frac{\partial}{\partial \overline{\zeta}} \left(\frac{g(\zeta + z)}{\zeta} \right)\, d\zeta \wedge d\overline{\zeta} = -\lim_{\varepsilon \to 0} \iint_{A_\varepsilon}\, d\omega$$

with the differential form

$$\omega(\zeta) := \frac{1}{2\pi i} \cdot \frac{g(\zeta + z)}{\zeta}\, d\zeta \in \mathscr{E}^{1,0}(A_\varepsilon)$$

Stoke's theorem for a disk in the complex plane applies:

$$\overline{\partial} f(z) = -\lim_{\varepsilon \to 0} \iint_{A_\varepsilon}\, d\omega = -\lim_{\varepsilon \to 0} \int_{\partial A_\varepsilon}\, \omega = \lim_{\varepsilon \to 0} \int_{|\zeta| = \varepsilon}\, \omega$$

because for $|\zeta| = R$

$$\omega(\zeta) = 0$$

Using the standard parametrization of the circuit with radius $\varepsilon > 0$

$$\zeta = \varepsilon \cdot e^{i\theta} \text{ with } d\zeta = i\varepsilon \cdot e^{i\theta} d\theta = i\zeta \; d\theta$$

gives

$$\overline{\partial} f(z) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \cdot \int_0^{2\pi} g(z + \varepsilon \cdot e^{i\theta}) \; d\theta = g(z), \; q.e.d.$$

We now generalize Theorem 5.1 to the case of an arbitrary smooth source g, not necessarily having compact support. The proof will make use of exhausting a given disk by a family of relatively compact disks.

Theorem 5.2 (Dolbeault's lemma for the inhomogeneous $\overline{\partial}$-equation). *Consider a disk*

$$X := D_R(0) \textit{ with radius } 0 < R \leq \infty.$$

Then for each smooth function $g \in \mathscr{E}(X)$ exists a smooth function $f \in \mathscr{E}(X)$ with

$$\overline{\partial} f = g.$$

Proof. Because the right-hand side $g \in \mathscr{E}(X)$ does not have necessarily compact support, one cannot obtain a solution just by integrating the source function multiplied by the Cauchy kernel.

i) *Applying Dolbeault's lemma with compact support*: We construct an exhaustion of the disk $D_R(0)$ by a sequence of relatively compact disks. Therefore we choose a sequence of radii

$$0 < R_0 < ... < R_n < ... < R \text{ with } \lim_{n \to \infty} R_n = R$$

and set

$$X_n := D_{R_n}(0), \; n \in \mathbb{N}, \; X_{-1} := \emptyset$$

Then we choose for each $n \in \mathbb{N}$ a smooth function

$$\psi_n \in \mathscr{E}(X)$$

with compact support

$$supp \; \psi_n \subset\subset X_{n+1} \text{ and } \psi|X_n = 1.$$

We extend each product

$$\psi_n \cdot g$$

by zero in the complement $X \setminus X_{n+1}$ to a smooth function

$$\psi_n \cdot g \in \mathscr{E}(\mathbb{C})$$

with compact support

$$supp\ (\psi_n \cdot g) \subset\subset X_{n+1}.$$

Theorem 5.1 provides for each $n \in \mathbb{N}$ a smooth function

$$f_n \in \mathscr{E}(X) \text{ with } \overline{\partial} f_n = \psi_n \cdot g,$$

in particular on X_n

$$\overline{\partial} f_n = g.$$

ii) *Enforcing convergence by modifying the local solutions by global holomorphic functions*: Each member f_n from the family $(f_n)_{n\in\mathbb{N}}$ solves the $\overline{\partial}$-problem on X_n. But the functions do not necessarily converge to a global solution on X. Because the difference of two solutions is holomorphic we may modify each local solution by a holomorphic function to enforce convergence. We construct the modification by induction on $n \in \mathbb{N}^*$.

Step $n \geq 1$ constructs $\tilde{f}_n \in \mathscr{E}(X)$ satisfying:

- On X_n

$$\overline{\partial} \tilde{f}_n = g$$

- and

$$\|\tilde{f}_n - \tilde{f}_{n-1}\|_{X_{n-2}} \leq \frac{1}{2^{n-1}}.$$

Induction start $n = 1$: We set

$$\tilde{f}_1 := f_1.$$

By part i) on X_1

$$\overline{\partial} \tilde{f}_1 = g.$$

Induction step $n \mapsto n+1$: On X_n by induction assumption

$$\overline{\partial} \tilde{f}_n = g$$

and by part i) on X_{n+1} - and in particular on X_n -

$$\overline{\partial} f_{n+1} = g.$$

Hence on X_n

$$\overline{\partial}(f_{n+1} - \tilde{f}_n) = 0$$

which implies

$$f_{n+1} - \tilde{f}_n$$

is holomorphic on the disk X_n. We approximate the difference by one of its Taylor polynomials P with

$$\|(f_{n+1} - \tilde{f}_n) - P\|_{Xn-1} \leq \frac{1}{2^n}$$

and define

$$\tilde{f}_{n+1} := f_{n+1} - P.$$

Then

$$\|\tilde{f}_{n+1} - \tilde{f}_n\|_{Xn-1} \leq \frac{1}{2^n}.$$

On X_{n+1} we have due to part i)

$$\overline{\partial}\tilde{f}_{n+1} = \overline{\partial} f_{n+1} - \overline{\partial} P = \overline{\partial} f_{n+1} = g$$

which finishes the induction step.

The resulting family $(\tilde{f}_n)_{n\in\mathbb{N}^*}$ satisfies:

- On X_n

$$\overline{\partial}\tilde{f}_n = g$$

- and

$$\|\tilde{f}_{n+1} - \tilde{f}_n\|_{X_{n-1}} \leq \frac{1}{2^n}$$

For each fixed $z \in X$ the sequence

$$(\tilde{f}_n(z))_{n\in\mathbb{N}^*}$$

is a Cauchy sequence by construction, hence exists

$$f(z) := \lim_{n\to\infty} \tilde{f}_n(z).$$

For given $n \in \mathbb{N}^*$ we decompose

$$f = \tilde{f}_n + \sum_{k=n}^{\infty} (\tilde{f}_{k+1} - \tilde{f}_k)$$

On X_n for each $k \geq n$ the function

$$\tilde{f}_{k+1} - \tilde{f}_k$$

is holomorphic, and the sum

$$F_n := \sum_{k=n}^{\infty} (\tilde{f}_{k+1} - \tilde{f}_k)$$

is compactly convergent, hence holomorphic by Weierstrass' theorem about normal convergence. As a consequence the function

$$f = \tilde{f}_n + F_n$$

is smooth on X_n and satisfies

$$\overline{\partial} f = \overline{\partial} \tilde{f}_n + \overline{\partial} F_n = \overline{\partial} \tilde{f}_n = g.$$

Because $n \in \mathbb{N}$ is arbitrary we obtain

$$f \in \mathscr{E}(X) \text{ and } \overline{\partial} f = g, \ q.e.d.$$

Definition 5.3 (Dolbeault sequence). On a Riemann surface X the *Dolbeault sequence* is the sequence of sheaf morphisms

$$0 \to \mathscr{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$$

with the canonical injection

$$\mathscr{O} \hookrightarrow \mathscr{E}.$$

Theorem 5.4 is a consequence of Dolbeault's lemma and the Cauchy-Riemann differential equations.

Theorem 5.4 (Exactness of the Dolbeault sequence). *The Dolbeault sequence on a Riemann surface X is an exact sheaf sequence.*

Proof. Exactness of a sheaf sequence is a local statement. Hence we may assume $X = D_R(0)$ is a disk.

1. *Exactness at $\mathscr{O}$*: Apparently the injection of stalks

$$\mathscr{O}_p \hookrightarrow \mathscr{E}_p$$

is injective.

2. *Exactness at $\mathscr{E}$*: For $f \in \mathscr{O}_p$ we have

$$d''_p(f) = 0 \in \mathscr{E}^{0,1}_p.$$

For the converse direction: The inclusion

$$ker[d''_p : \mathscr{E}_p \to \mathscr{E}^{0,1}_p] \subset \mathscr{O}_p$$

is due to the Cauchy-Riemann differential equations for open subsets of $\mathbb{C}$.

3. *Exactness at $\mathscr{E}^{0,1}$*: Apparently

$$d''_p(\mathscr{E}^{0,1}_p) = 0.$$

Concerning the opposite direction consider the germ of a smooth form

$$\omega_p = g_p \, d\bar{z}_p \in \mathscr{E}^{0,1}_p.$$

We may assume that $\omega_p \in \mathscr{E}_p^{0,1}$ has a representative

$$\omega = g\ d\bar{z} \in \mathscr{E}^{0,1}(X)$$

with a smooth function $g \in \mathscr{E}(X)$. Theorem 5.2 provides a smooth function $f \in \mathscr{E}(X)$ with

$$\overline{\partial} f = g$$

As a consequence

$$d'' f = \overline{\partial} f\ d\bar{z} = g \cdot d\bar{z} = \omega,\ q.e.d.$$

One says: The Dolbeault sequence is a *resolution* of the structure sheaf $\mathscr{O}$ by sheaves of smooth differential forms.

5.2 Exactness of the de Rham sequence

Definition 5.5 (De Rham sequence). On a Riemann surface X the *de Rham sequence* is the sequence of sheaf morphisms

$$0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to 0$$

with the canonical injection

$$\mathbb{C} \hookrightarrow \mathscr{E}.$$

Theorem 5.6 (Exactness of the de Rham sequence). *The de Rham sequence on a Riemann surface X is an exact sheaf sequence.*

Proof. Exactness of a sheaf sequence on X means exactness on the induced sequence of stalks at any point $p \in X$. Hence we may assume $X = D_R(0)$ a disk and $p = 0$. Note that all germs of smooth differential forms have a representative on X.

i) *Exactness at* $\mathbb{C}$: Apparently the injection of stalks

$$\mathbb{C}_p = \mathbb{C} \hookrightarrow \mathscr{E}_p$$

is injective.

ii) *Exactness at* $\mathscr{E}$: Exterior derivation maps the germ of a constant function to zero. Vice versa, if the partial derivatives of a smooth function vanish, then the function is locally constant.

iii) *Exactness at* $\mathscr{E}^1$: Consider a smooth function $f \in \mathscr{E}(X)$. According to Proposition 4.14 we have

$$d^2 f = 0.$$

For the opposite direction consider a smooth differential form

$$\omega = f \cdot dz + g \cdot d\bar{z} \in \mathscr{E}^1(X)$$

with smooth $f, g \in \mathscr{E}(X)$ and $d\omega = 0$. The vanishing $d\omega = 0$ implies

$$\overline{\partial} f = \partial g.$$

We have to find a smooth function $F \in \mathscr{E}(X)$ solving the system of two partial differential equations

$$\partial F = f \text{ and } \overline{\partial} F = g.$$

Using that X is starlike with respect to $0 \in X$ we write down the solution F as an integral: For a given point $z \in X$ set

$$F(z) := \int_0^1 \left(f(t \cdot z) \cdot z + g(t \cdot z) \cdot \bar{z} \right) \, dt$$

The integral is well-defined because the integrand assumes its maximum on the compact interval $[0,1]$. We may interchange integration and partial derivation with respect to the parameters z and $\bar{z}$, which shows $F \in \mathscr{E}(X)$. Partial derivation of the integrand and using

$$\partial g = \overline{\partial} f$$

gives

$$\begin{aligned} \partial(f(tz) \cdot z) + \partial(g(tz) \cdot \bar{z}) &= \partial f(tz) \cdot tz + f(tz) + \partial g(tz) \cdot t\bar{z} = \\ &= \partial f(tz) \cdot tz + f(tz) + \overline{\partial} f(tz) \cdot t\bar{z}. \end{aligned}$$

Using

$$t \cdot \frac{d}{dt} f(tz) = t \cdot (\partial f(tz) \cdot z + \overline{\partial} f(tz) \cdot \bar{z})$$

shows

$$\partial F(z) = \int_0^1 \left(f(tz) + \partial f(tz) \cdot tz + \overline{\partial} f(tz) \cdot t\bar{z} \right) \, dt = \int_0^1 \left(f(tz) + t \cdot \frac{d}{dt} f(tz) \right) \, dt =$$

$$= \int_0^1 \frac{d}{dt} (t \cdot f(tz)) \, dt = f(z).$$

Analogously, one verifies

$$\overline{\partial} F = g.$$

iv) *Exactness at* $\mathscr{E}^2$: We have to show that the sheaf morphism

$$d : \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2$$

is surjective. As a corollary of Dolbeault's Theorem we will show that even the restriction

$$d|\mathscr{E}^{(1,0)} = d'' : \mathscr{E}^{(1,0)} \to \mathscr{E}^2$$

is surjective: Consider a smooth differential form

$$\omega = g \cdot d\bar{z} \wedge dz \in \mathscr{E}^2(X)$$

Theorem 5.2 provides a smooth $f \in \mathscr{E}(X)$ with

$$\overline{\partial} f = g.$$

We define

$$\eta := f \cdot dz \in \mathscr{E}^{1,0}(X).$$

Then

$$d\eta = d''\eta = \overline{\partial} f \; d\bar{z} \wedge dz = g \; d\bar{z} \wedge dz = \omega, \; q.e.d.$$

One says: The de Rham sequence is a *resolution* of the sheaf $\mathbb{C}$ of locally constant functions by sheaves of smooth differential forms.

From a topological point of view the proof in Theorem 5.6 of the exactness of the de Rham sequence uses the existence of a primitive for starlike domains and Dolbeault's solution of the $\overline{\partial}$-problem. Apparently, Theorem 5.6 is a statement about the smooth structure Σ_{smooth} and holds independently from the existence of any complex structure. In the context of smooth manifolds one proves the theorem as a consequence of the Poincaré lemma for the d-operator. The Poincaré lemma is an analogue of the Dolbeault lemma.

The whole content of the present chapter carries over to higher dimensional complex manifolds and their smooth structure. And there, in the context of higher dimensions, the results show their full strenght, see e.g., [14, Kap. II, §4].

Remark 5.7 (Poincaré Lemma). Consider a star-like domain

$$X \subset \mathbb{C} \simeq \mathbb{R}^2.$$

Then for any 1-form $\omega \in \mathscr{E}^1(X)$:

$$d\omega = 0 \text{ (closedness)} \implies \omega = d\eta \text{ for a suitable } \eta \in \mathscr{E}^1(X) \text{ (exactness).}$$

The proof results from the integral formula in the proof of Theorem 5.6.

Chapter 6
Cohomology

For a Riemann surface X the functor of sections over a fixed open set $U \subset X$

$$\underline{Sh}_X \to \underline{Ab}$$

from the category of sheaves of Abelian groups on X to the category of Abelian groups is *left exact*, i.e. for any short exact sequence of sheaf morphisms

$$0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$$

and for any open set $U \subset X$ the sequence of morphisms of Abelian groups

$$0 \to \mathscr{F}(U) \xrightarrow{f_U} \mathscr{G}(U) \xrightarrow{g_U} \mathscr{H}(U)$$

is exact. But in general, the morphism at the right-hand side

$$\mathscr{G}(U) \xrightarrow{g_U} \mathscr{H}(U)$$

is not surjective, see Remark after Proposition 2.10.

Cohomology, or right-derivation of the functor of sections, is the means to extend the exact sequence of Abelian groups above to the right-hand side by defining groups

$$H^q(U, \mathscr{F}),\ q \geq 0,$$

and obtaining a long exact sequence in the category $\underline{\text{Ab}}$.

6.1 Čech Cohomology and its inductive limit

A suitable type of cohomology for sheaves on a Riemann surface X and also on more general complex manifolds is Čech Cohomology. We now develop the basic theory.

Definition 6.1 (Cochains, cocycles, coboundaries and Čech cohomology classes). Consider a topological space X, a sheaf $\mathscr{F}$ of Abelian groups on X and an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X.

- For each $q \in \mathbb{N}$ the q-th *cochain group* of $\mathscr{F}$ with respect to $\mathscr{U}$ is the Abelian group

$$C^q(\mathscr{U},\mathscr{F}) := \prod_{(i_0 \dots i_q) \in I^{q+1}} \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_q}),\ C^{-1}(\mathscr{U},\mathscr{F}) := 0.$$

  Hence a q-cochain is a family

$$f = (f_{i_0 \dots i_q})_{i_0 \dots i_q \in I^{q+1}}$$

  of sections $f_{i_0 \dots i_q} \in \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_q})$ over the $q+1$-fold intersections

$$U_{i_0 \dots i_q} := U_{i_0} \cap \dots \cap U_{i_q}$$

  of the open sets of the covering. The group structure on $C^q(\mathscr{U},\mathscr{F})$ derives from the group structure of the factors.

- For each $q \in \mathbb{N}$ the *coboundary operator*

$$\delta := \delta^q : C^q(\mathscr{U},\mathscr{F}) \to C^{q+1}(\mathscr{U},\mathscr{F})$$

  is defined as

$$\delta f := g := (g_{i_0 \dots i_{q+1}})_{(i_0 \dots i_{q+1}) \in I^{q+2}}$$

  with the cross sum of restrictions

$$g_{i_0 \dots i_{q+1}} := \sum_{k=0}^{q+1} (-1)^k \cdot f_{i_0 \dots \hat{i}_k i_{k+1} \dots i_{q+1}} | U_{i_0 \dots i_k \dots i_{q+1}}$$

  Here $\hat{i}_k$ means to omit the index i_k.

- For each $q \in \mathbb{N}$ one defines the group of *q-cocycles*

$$Z^q(\mathscr{U},\mathscr{F}) := ker[C^q(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^q} C^{q+1}(\mathscr{U},\mathscr{F})],$$

  the group of *q-coboundaries*

$$B^q(\mathscr{U},\mathscr{F}) := im[C^{q-1}(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^{q-1}} C^q(\mathscr{U},\ \mathscr{F})],$$

and after checking

$$\delta^q \circ \delta^{q-1} = 0$$

the *q-th Čech cohomology group* of $\mathscr{F}$ with respect to the open covering $\mathscr{U}$

$$H^q(\mathscr{U},\mathscr{F}) := \frac{Z^q(\mathscr{U},\mathscr{F})}{B^q(\mathscr{U},\mathscr{F})}$$

Elements from $H^q(\mathscr{U},\mathscr{F})$ are named *q-th Čech cohomology classes* of $\mathscr{F}$ with respect to the covering $\mathscr{U}$. Two cocycles from $Z^q(\mathscr{U},\mathscr{F})$ with determine the same cohomology class in $H^q(\mathscr{U},\mathscr{F})$ are named *cohomologous*.

Remark 6.2 (Cohomology).

1. *Cocycle relation*: Mostly we will be concerned with cohomology in dimension $q = 0, 1, 2$ because a Riemann surface has real dimension $= 2$. For $q = 0, 1$ the cocycle condition has the following meaning:

- $q = 0$: A family $(f_i)_i \in C^0(\mathscr{U},\mathscr{F})$ is a 0-cocycle iff for all $i, j \in I$

$$f_j - f_i = 0 \text{ on } U_i \cap U_j,$$

i.e. if the cochain satisfies on the intersections $U_i \cap U_j$ the equality

$$f_i = f_j.$$

Because $\mathscr{F}$ is a sheaf, 0-cocyles correspond bijectively to global sections $f \in \mathscr{F}(X)$ and because

$$B^0(\mathscr{U},\mathscr{F}) = 0$$

we have

$$H^0(\mathscr{U},\mathscr{F}) = Z^0(\mathscr{U},\mathscr{F}) = \mathscr{F}(X).$$

- $q = 1$: A family $(f_{ij})_{ij} \in C^1(\mathscr{U},\mathscr{F})$ is a 1-cocycle iff for all $i, j, k \in I$

$$0 = f_{jk} - f_{ik} + f_{ij}$$

i.e. the cochain satisfies on the 3-fold intersections $U_i \cap U_j \cap U_k$ the *cocycle condition*

$$f_{ik} = f_{ij} + f_{jk}.$$

With the first cohomology group a new concept enters sheaf theory. The group

$$H^1(\mathscr{U},\mathscr{F})$$

often collects the obstructions against glueing local solutions of a problem to a global solution. Theorem 6.14 will show that on a Riemann surface all obstructions vanish in the category of smooth functions, but not in the category of holomorphic functions. A first means to classify compact Riemann surfaces will be the size of the groups $H^1(\mathscr{U},\mathscr{O})$, see Definition 7.17.

2. *Iteration of the coboundary operator*: One verifies that the composition of the coboundary operator from Definition 6.1 satisfies for each $q \in \mathbb{N}$ the equation

$$\delta^q \circ \delta^{q-1} = 0,$$

i.e. cochains and coboundary form a complex of Abelian groups. For the proof one uses that the sum in the definition of the coboundary operator is an alternating sum. For example for $q = 1$:

$$C^0(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^0} C^1(\mathscr{U},\mathscr{F}) \xrightarrow{\delta^1} C^2(\mathscr{U},\mathscr{F})$$

satisfies

$$\delta^0((f_i)_{i\in I}) = (g_{ij} := f_j - f_i)_{i,j\in I} \in C^1(\mathscr{U},\mathscr{F})$$

and

$$\delta^1((g_{ij})_{i,j\in I}) := ((h_{ijk} = g_{jk} - g_{ik} + g_{ij})_{ijk\in I}) \in C^2(\mathscr{U},\mathscr{F}).$$

As a consequence

$$h_{ijk} = (f_k - f_j) - (f_k - f_i) + (f_j - f_i) = 0.$$

Hence

$$B^q(\mathscr{U},\mathscr{F}) \subset Z^q(\mathscr{U},\mathscr{F})$$

and the quotient

$$\frac{Z^q(\mathscr{U},\mathscr{F})}{B^q(\mathscr{U},\mathscr{F})}$$

is well-defined.

3. The cohomology with respect to an open covering introduced in Definition 6.1 is named *Čech cohomology* and the corresponding objects are often written with the Čech accent like in $\check{\mathrm{H}}^n(\mathscr{U},\mathscr{F})$. We will not use this notation.

Our next aim is to remove the dependency of the cohomology on a given open covering of the Riemann surface X. We show how to abstract from the covering to obtain a cohomology theory which only depends on X and on the sheaf $\mathscr{F}$.

Definition 6.3 (Refinement of open coverings). Consider a Riemann surface X, and two open coverings $\mathscr{U} = (U_i)_{i\in I}$ and $\mathscr{V} = (V_j)_{j\in J}$ of X.

- If $\mathscr{V} < \mathscr{U}$ with respect to the refinement map

$$\tau : J \to I$$

then τ induces for any sheaf $\mathscr{F}$ on X a restriction

$$t^{\mathscr{U}}_{\mathscr{V}} : Z^q(\mathscr{U},\mathscr{F}) \to Z^q(\mathscr{V},\mathscr{F})$$

which maps the family

$$f = (f_{i_0 \dots i_q})_{i_0 \dots i_q} \text{ with sections } f_{i_0 \dots i_q} \text{ defined on } U_{i_0} \cap \dots \cap U_{i_q}$$

to the family

$$t^{\mathscr{V}}_{\mathscr{U}}(f) := g := (g_{j_0 \dots j_q})_{j_0 \dots j_q}$$

with sections defined as the restriction

$$g_{j_0 \dots j_q} := f_{\tau(j_0)\dots\tau(j_q)} | V_{j_0} \cap \dots \cap V_{j_q}$$

- The restriction map is compatible with coboundaries, hence induces a *refinement map* between the cohomology groups

$$t^{\mathscr{U}}_{\mathscr{V}} : H^q(\mathscr{U},\mathscr{F}) \to H^q(\mathscr{V},\mathscr{F}).$$

Lemma 6.4 (The refinement map in cohomology). *Consider a topological space and a refinement of coverings*

$$\mathscr{V} = (V_j)_{j\in J} < \mathscr{U} = (U_\alpha)_{\alpha \in I}$$

with respect to

$$\tau : J \to I.$$

Then for any $q \in \mathbb{N}$ the restriction

$$t^{\mathscr{U}}_{\mathscr{V}} : H^q(\mathscr{U},\mathscr{F}) \to H^q(\mathscr{V},\mathscr{F})$$

does not depend on the choice of the refinement map

$$\tau : J \to I.$$

If $q = 1$ then the restriction map is injective.

Proof. i) *Independence*: We only prove the case $q = 1$. For the general case see [14, Kap. B, §2.3] and for its proof [7, §7, Satz]. Consider a second refinement map

$$\sigma : J \to I \text{ with } V_j \subset U_{\sigma(j)},\ j \in J.$$

For a given cocycle

$$f = (f_{\alpha\beta})_{\alpha\beta} \in Z^1(\mathscr{U},\mathscr{F})$$

we have to show that the two restrictions

$$t^{\mathscr{U}}_{\mathscr{V}}(f) =: g^\tau = (g^\tau_{kl}) \in Z^1(\mathscr{V},\mathscr{F}) \text{ with } g^\tau_{kl} = f_{\tau(k)\tau(l)} \text{ on } V_k \cap V_l$$

and

$$s_{\mathscr{V}}^{\mathscr{U}}(f) =: g^{\sigma} = (g_{kl}^{\sigma}) \in Z^1(\mathscr{V}, \mathscr{F}) \text{ with } g_{kl}^{\sigma} = f_{\sigma(k)\sigma(l)} \text{ on } V_k \cap V_l$$

are cohomologous. By definition for any $k \in J$

$$V_k \subset U_{\tau(k)} \cap U_{\sigma(k)}$$

We define the cochain

$$h = (h_k)_{k \in J} \in C^0(\mathscr{V}, \mathscr{F})$$

by the restrictions on V_k

$$h_k := f_{\sigma(k)\tau(k)}$$

Then on $V_k \cap V_l$ the 1-cocycle condition implies

$$g_{kl}^{\sigma} - g_{kl}^{\tau} = f_{\sigma(k)\sigma(l)} - f_{\tau(k)\tau(l)} = (f_{\sigma(k)\sigma(l)} + f_{\sigma(l)\tau(k)}) - (f_{\sigma(l)\tau(k)} + f_{\tau(k)\tau(l)}) =$$

$$= f_{\sigma(k)\tau(k)} - f_{\sigma(l)\tau(l)} = h_k - h_l$$

Hence

$$g^{\sigma} - g^{\tau} = -\delta h$$

which proves the independence.

ii) *Injectivity*: Consider a cocycle

$$f = (f_{\alpha\beta})_{\alpha\beta} \in Z^1(\mathscr{U}, \mathscr{F})$$

and assume

$$t_{\mathscr{V}}^{\mathscr{U}}([f]) = 0 \in H^1(\mathscr{V}, \mathscr{F})$$

Hence a cochain

$$g = (g_i)_i \in C^0(\mathscr{V}, \mathscr{F})$$

exists such that on

$$V_i \cap V_j \cap U_{\alpha} \subset U_{\tau(i)} \cap U_{\tau(j)} \cap U_{\alpha}$$

$$(\delta g)_{ij} = g_j - g_i = f_{\tau(i)\tau(j)} = f_{\tau(i)\alpha} + f_{\alpha\tau(j)} = f_{\alpha\tau(j)} - f_{\alpha\tau(i)}$$

i.e.

$$f_{\alpha\tau(i)} - g_i = f_{\alpha\tau(j)} - g_j$$

Keeping α fixed and varying i shows: The family

$$(f_{\alpha\tau(i)} - g_i)_i$$

is the restriction of a section

$$h_{\alpha} \in \mathscr{F}(U_{\alpha})$$

satisfying on each $U_{\alpha} \cap V_i$

$$h_{\alpha} = f_{\alpha\tau(i)} - g_i$$

For fixed α, β and variable i we have on

$$U_\alpha \cap U_\beta \cap V_i$$

due to the cocycle condition

$$f_{\alpha\beta} = f_{\alpha\tau(i)} + f_{\tau(i)\beta} = (f_{\alpha\tau(i)} - g_i) - (f_{\beta\tau(i)} - g_i) = h_\alpha - h_\beta$$

Note $f_{\beta\tau(k)} = -f_{\tau(k)\beta}$.

As a consequence on $U_\alpha \cap U_\beta$

$$f_{\alpha\beta} = h_\alpha - h_\beta = (-h_\beta) - (-h_\alpha),$$

showing

$$f = \delta h$$

with

$$h := (-h_\alpha)_\alpha \in C^0(\mathscr{U}, \mathscr{F}).$$

Therefore

$$[f] = 0 \in H^1(\mathscr{U}, \mathscr{F}),\ q.e.d.$$

Definition 6.5 (Čech cohomology groups). Consider a topological space X and a sheaf $\mathscr{F}$ of Abelian groups on X. For each $q \in \mathbb{N}$ the inductive limit

$$H^q(X, \mathscr{F}) = \varinjlim_{\mathscr{U}} H^q(\mathscr{U}, \mathscr{F})$$

with respect to the family of open coverings of X and refinement maps

$$t^{\mathscr{U}}_{\mathscr{V}} : H^q(\mathscr{U}, \mathscr{F}) \to H^q(\mathscr{V}, \mathscr{F})$$

according to Definition 6.3 is named the q-th *Čech cohomology group* of X with values in $\mathscr{F}$.

Lemma 6.4 ensures that in Definition 6.5 the refinement map between cohomology groups does not depend on the choice of the refinement map $\mathscr{V} < \mathscr{U}$. Therefore the inductive limit is well-defined.

Remark 6.6 (Čech Cohomology). Consider a topological space X and a sheaf $\mathscr{F}$ of Abelian groups on X.

1. Consider two open coverings $\mathscr{U}$ and $\mathscr{V}$ of X. Then two cohomology classes

$$[f_{\mathscr{U}}] \in H^q(\mathscr{U}, \mathscr{F}) \text{ and } [f_{\mathscr{V}}] \in H^q(\mathscr{V}, \mathscr{F})$$

are named *equivalent with respect to refinement* if a common refinement

$$\mathscr{W} < \mathscr{V} \text{ and } \mathscr{W} < \mathscr{U}$$

exists with

$$t^{\mathscr{U}}_{\mathscr{W}}([f_{\mathscr{U}}]) = t^{\mathscr{V}}_{\mathscr{W}}([f_{\mathscr{V}}]) \in H^q(\mathscr{W}, \mathscr{F}).$$

According to the definition of the inductive limit: Two cohomology classes which refer to different open coverings are equivalent if and onl if they define the same class in $H^(X, \mathscr{F})$.

2. For locally constant sheaves like $\mathbb{Z}$ or $\mathbb{C}$ on a manifold the Čech cohomology groups from Definition 6.3 are isomorphic to the singular cohomology groups as defined by the methods of algebraic topology, see [39, Théor. 4.17]. In particular, for a compact Riemann surface X holds

$$H^2(X, \mathbb{Z}) = \mathbb{Z} \text{ and } H^2(X, \mathbb{C}) = \mathbb{C}$$

because the underlying 2-dimensional compact topological manifold is orientable.

Corollary 6.7 (Vanishing of $H^1(X, \mathscr{F})$). *Consider a topological space X and a sheaf $\mathscr{F}$. Then are equivalent:*

- $$H^1(X, \mathscr{F}) = 0$$
- *and for all open coverings $\mathscr{U}$ of X*

$$H^1(\mathscr{U}, \mathscr{F}) = 0.$$

Proof. The result follows from the injectiviy of the restriction, see Lemma 6.4,

$$t^{\mathscr{U}}_{\mathscr{V}} : H^1(\mathscr{U}, \mathscr{F}) \to H^1(\mathscr{V}, \mathscr{F})$$

for any refinement

$$\mathscr{V} < \mathscr{U}, \; q.e.d.$$

Theorem 6.8 is the particular case $q = 1$ of the general Leray theorem which states: The Čech cohomology cohomology cohomology of X with values in a sheaf $\mathscr{F}$ can be computed as the Čech cohomology with respect to an open covering $\mathscr{U}$, if $\mathscr{F}$ is acyclic with respect to $\mathscr{U}$, i.e. if $\mathscr{F}$ has no cohomology on the subsets of the covering for the dimension $q = 1$. Generalizing the theorem to arbitrary values $q \in \mathbb{N}$ is a difficult and lengthy task.

Theorem 6.8 is of fundamental importance for the explicit computation of cohomology groups. The theorem and Definition 6.9 are named in honour of Jean Leray who invented the concept of sheaves.

Theorem 6.8 (Leray). *Consider a topological space X, an open covering of X*

$$\mathscr{U} = (U_\alpha)_{\alpha \in I}$$

and a sheaf $\mathscr{F}$ on X. If the pair $(\mathscr{F}, \mathscr{U})$ satisfies for all $\alpha \in I$

$$H^1(U_\alpha, \mathscr{F}) = 0$$

then

$$H^1(X, \mathscr{F}) = H^1(\mathscr{U}, \mathscr{F}).$$

Note. The group $H^1(U_\alpha, \mathscr{F})$ denotes the first cohomology of the open set U_α with values in the sheaf $\mathscr{F}$. It is not a Čech cohomology group with respect to a covering.

Proof. The proof will show that for any refinement

$$\mathscr{V} = (V_j)_{j \in J} < \mathscr{U} = (U_\alpha)_{\alpha \in I}$$

the refinement map

$$t^{\mathscr{U}}_{\mathscr{V}} : H^1(\mathscr{U}, \mathscr{F}) \to H^1(\mathscr{V}, \mathscr{F})$$

is an isomorphism of Abelian groups.

i) The map is injective due to Lemma 6.4.

ii) For the proof of the surjectivity assume a refinement map

$$\tau : J \to I \text{ with } V_j \subset U_{\tau(j)}, j \in J.$$

Consider a cocycle

$$f = (f_{ij})_{ij} \in Z^1(\mathscr{V}, \mathscr{F})$$

We have to find a cocycle

$$F = (F_{\alpha\beta})_{\alpha\beta} \in Z^1(\mathscr{U}, \mathscr{F})$$

such that

$$t^{\mathscr{U}}_{\mathscr{V}}(F) - f = (F_{\tau(i)\tau(j)} - f_{ij})_{ij} \in B^1(\mathscr{V}, \mathscr{F})$$

i.e. the difference is even a coboundary.

- *The cocycle $(f_{ij})_{ij}$ restricts to each U_α to a coboundary*

  $$(f_{ij})_{ij} = (g_{\alpha,i} - g_{\alpha,j})_{ij} \in B^1(U_\alpha \cap \mathscr{V}, \mathscr{F})$$

  due to $H^1(U_\alpha \cap \mathscr{V}, \mathscr{F}) = 0$. And on $U_\alpha \cap U_\beta$ the family

$$(g_{\alpha,j} - g_{\beta,j})_j \in C^0(U_\alpha \cap U_\beta \cap \mathscr{V}, \mathscr{F})$$

defines a section $F_{\alpha\beta} \in \mathscr{F}(U_\alpha \cap U_\beta)$:

Choose arbitrary but fixed indices $\alpha, \beta \in I$. The open set U_α has the open covering

$$U_\alpha \cap \mathscr{V} = (U_\alpha \cap V_j)_{j \in J}.$$

The assumption of the theorem and Corollary 6.7, applied to the open set $U_\alpha \subset X$ and its covering $U_\alpha \cap \mathscr{V}$, imply

$$H^1(U_\alpha \cap \mathscr{V}, \mathscr{F}) = 0$$

Hence a cochain

$$g_\alpha = (g_{\alpha,j})_{j \in J} \in C^0(U_\alpha \cap \mathscr{V}, \mathscr{F})$$

exists such that for $i, j \in J$

– on $U_\alpha \cap V_i \cap V_j$

$$f_{ij} = g_{\alpha,j} - g_{\alpha,i}$$

– on $U_\beta \cap V_i \cap V_j$

$$f_{ij} = g_{\beta,j} - g_{\beta,i}$$

– hence on $U_\alpha \cap U_\beta \cap V_i \cap V_j$

$$g_{\alpha,j} - g_{\alpha,i} = f_{ij} = g_{\beta,j} - g_{\beta,i}$$

i. e.

$$g_{\alpha,j} - g_{\beta,j} = g_{\alpha,i} - g_{\beta,i}$$

Hence fixing $\alpha, \beta \in I$ and varying $j \in J$ shows

$$(g_{\alpha,j} - g_{\beta,j})_{j \in J} \in Z^0(U_\alpha \cap U_\beta \cap \mathscr{V}, \mathscr{F})$$

is a cocycle on $U_\alpha \cap U_\beta$. Therefore the sheaf property of $\mathscr{F}$ implies the existence of a section

$$F_{\alpha\beta} \in \mathscr{F}(U_\alpha \cap U_\beta)$$

with for all $j \in J$

$$F_{\alpha\beta} | U_\alpha \cap U_\beta \cap V_j = g_{\alpha,j} - g_{\beta,j}$$

- *The family* $(F_{\alpha\beta})_{\alpha\beta} \in C^1(\mathscr{U}, \mathscr{F})$ *is even a cocycle*:

For each $j \in J$ the cocycle relation

$$F_{\alpha\gamma} = F_{\alpha\beta} + F_{\beta\gamma}$$

is satisfied on each

$$U_\alpha \cap U_\beta \cap U_\gamma \cap V_j$$

because

$$F_{\alpha\beta} + F_{\beta\gamma} = (g_{\alpha,j} - g_{\beta,j}) + (g_{\beta,j} - g_{\gamma,j}) = g_{\alpha,j} - g_{\gamma,j} = F_{\alpha\gamma}.$$

Hence the cocycle relation is satisfied even on

$$U_\alpha \cap U_\beta \cap U_\gamma$$

i.e.

$$(F_{\alpha\beta})_{\alpha\beta} \in Z^1(\mathscr{U}, \mathscr{F}).$$

- *The cochain*

$$h := (-g_{\tau(j),j})_j \in C^0(\mathscr{V}, \mathscr{F})$$

satisfies

$$t^{\mathscr{U}}_{\mathscr{V}}(F) - f = \delta h \in B^1(\mathscr{V}, \mathscr{F}) :$$

We define the cochain

$$h = (h_j)_{j \in J} \in C^0(\mathscr{V}, \mathscr{F})$$

as

$$h_j := -g_{\tau(j),j} \in \mathscr{F}(V_j)$$

It satisfies on $V_i \cap V_j$

$$(\delta h)_{ij} = h_j - h_i = g_{\tau(i),i} - g_{\tau(j),j} = (g_{\tau(i),i} - g_{\tau(j),i}) - (g_{\tau(j),j} - g_{\tau(j),i}) =$$

$$= F_{\tau(i)\tau(j)} - f_{ij}$$

hence

$$\delta h = t^{\mathscr{U}}_{\mathscr{V}}(F) - f \in B^1(\mathscr{V}, \mathscr{F}),\ q.e.d.$$

Definition 6.9 (Leray covering). Let X be a Riemann surface. An open covering $\mathscr{U} = (U_i)_{i \in I}$ is a *Leray covering* for a sheaf $\mathscr{F}$ on X if for each $i \in I$

$$H^1(U_i, \mathscr{F}) = 0.$$

Leray's theorem 6.8 shows for a Leray covering $\mathscr{U}$ for $\mathscr{F}$

$$H^1(X, \mathscr{F}) = H^1(\mathscr{U}, \mathscr{F}).$$

Hence Čech cohomology with respect to a Leray covering equals sheaf cohomology. There is no need to compute an inductive limit.

6.2 Long exact cohomology sequence

Remark 6.10 (Left exactness of the functor $\Gamma(X,-)$). On a topological space X the covariant functor "global sections"

$$\Gamma(X,-) : \underline{Sheaf}_X \to \underline{Ab},\ \Gamma(X,\mathscr{F}) := \mathscr{F}(X),$$

is *left-exact*, i.e. for any short exact sequence of sheaves of Abelian groups on X

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

the sequence of Abelian groups

$$0 \to \Gamma(X,\mathscr{F}) \xrightarrow{\Gamma(\alpha)} \Gamma(X,\mathscr{G}) \xrightarrow{\Gamma(\beta)} \mathscr{H}(X)$$

is exact.

Here $\underline{Sheaf}_X$ denotes the category of sheaves of Abelian groups on X and $\underline{Ab}$ denotes the category of Abelian groups.

Note: For any open covering $\mathscr{U}$ of X and any sheaf $\mathscr{F}$ holds

$$\Gamma(X,\mathscr{F}) = \mathscr{F}(X) = Z^0(\mathscr{U},\mathscr{F}) = H^0(\mathscr{U},\mathscr{F}) = H^0(X,\mathscr{F}).$$

Definition 6.11 (Connecting morphism). Consider a topological space X and a short exact sequence of sheaves of Abelian groups

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0,$$

see Definition 2.8. A morphism, named *connecting morphism* of the short exact sequence,

$$\partial : H^0(X,\mathscr{H}) \to H^1(X,\mathscr{F})$$

is defined as follows: For all $x \in X$ the morphism

$$g_x : \mathscr{G}_x \to \mathscr{H}_x$$

is surjective by definition of the exactness of the sheaf sequence.

- Hence for any element

  $$h \in \mathscr{H}(X) = H^0(X,\mathscr{H})$$

  an open covering $\mathscr{U} = (U_i)_{i\in I}$ of X exists and a family

  $$g_i \in \mathscr{G}(U_i),\ i \in I,$$

satisfying for all $i \in I$
$$\beta_{U_i}(g_i) = h_i := h|U_i.$$

- For each pair $i, j \in I$ on
$$U_{ij} := U_i \cap U_j$$
the elements
$$g_{ij} := g_j - g_i \in \mathscr{G}(U_{ij})$$
satisfy
$$\beta_{U_{ij}}(g_{ij}) = h_j|U_{ij} - h_i|U_{ij} = 0.$$
Hence there is a family
$$f = (f_{ij})_{ij} \in C^1(\mathscr{U}, \mathscr{F})$$
satisfying on U_{ij}
$$\alpha_{U_{ij}}(f_{ij}) = g_{ij}.$$
For each triple $i, j, k \in I$ on
$$U_{ijk} := U_i \cap U_j \cap U_k$$
holds
$$\alpha_{U_{ijk}}(f_{jk} - f_{ik} + f_{ij}) = g_{jk} - g_{ik} + g_{ij} = 0.$$
Because $\alpha_{U_{ijk}}$ is injective, the family $f = (f_{ij})_{ij}$ is even a cocycle
$$f \in Z^1(\mathscr{U}, \mathscr{F})$$
and the class
$$[f] \in H^1(\mathscr{U}, \mathscr{F})$$
is well-defined.

- Via the canonical map induced by the inductive limit
$$\pi : H^1(\mathscr{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$
we define
$$\partial(h) := \pi([f]) \in H^1(X, \mathscr{F}).$$

Remark 6.12 (Connecting morphism).

1. The construction of ∂ in Definition 6.11 operates by "climbing stairs" according to

$$0 \longrightarrow \mathscr{F} \xrightarrow{\quad\alpha\quad} \mathscr{G} \xrightarrow{\quad\beta\quad} \mathscr{H} \longrightarrow 0$$

$$\begin{array}{ccc} (f_{ij}) \leftarrow (g_{ij} = g_j - g_i) & & \\ \uparrow & & \\ (g_i) & \longleftarrow & (h_i) \end{array}$$

2. The definition of the connecting morphism from Definition 6.11

$$\partial : H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F})$$

is independent from the choice of $\mathscr{U}$ and the choice of $(g_i)_i \in Z^1(\mathscr{U}, \mathscr{G})$, hence a well-defined morphism of Abelian groups.

3. Definition 6.11 generalizes to a family of connecting morphisms between the higher cohomology groups

$$\partial^q : H^q(X, \mathscr{H}) \to H^{q+1}(X, \mathscr{F}),\ q \in \mathbb{N},$$

with $\partial^0 := \partial$, see [7, §8 Satz 1].

Theorem 6.13 (Long exact cohomology sequence). *Let X be a topological space. A short exact sequence of sheaves of Abelian groups on X*

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0,$$

induces a long exact sequence of Abelian groups

$$0 \to H^0(X, \mathscr{F}) \xrightarrow{\alpha} H^0(X, \mathscr{G}) \xrightarrow{\beta} H^0(X, \mathscr{H}) \xrightarrow{\partial^0}$$

$$H^1(X, \mathscr{F}) \xrightarrow{\alpha} H^1(X, \mathscr{G}) \xrightarrow{\beta} H^1(X, \mathscr{H})$$

Proof. i) Exactness at

$$H^0(X, \mathscr{F}) \text{ and } H^0(X, \mathscr{G})$$

follows from the left-exactness of $\Gamma(X, -)$, see Remark 6.10.

ii) Exactness at $H^0(X, \mathscr{H})$:

- First, assume that

$$h = (h_i)_i \in Z^0(\mathscr{U}, \mathscr{H}) = H^0(X, \mathscr{H})$$

  satisfies

$$\partial^0 h = 0 \in H^1(X, \mathscr{F}).$$

By definition the element $\partial^0 h$ is represented by a cocycle

$$f = (f_{ij})_{ij} \in Z^1(\mathscr{U}, \mathscr{F}),$$

and by assumption f is a coboundary: There exists a cochain

$$F = (F_i)_i \in C^0(\mathscr{U}, \mathscr{F})$$

satisfying on $U_{ij} = U_i \cap U_j$

$$f_{ij} = F_j - F_i.$$

Hence

$$\alpha(f_{ij}) = \alpha(F_j) - \alpha(F_i).$$

By construction

$$\alpha(f_{ij}) = g_j - g_i.$$

Therefore

$$g_i - \alpha(F_i) = g_j - \alpha(F_j).$$

We obtain a global section

$$G := (g_i + \alpha(F_i))_i \in Z^0(\mathscr{U}, \mathscr{G}) = H^0(X, \mathscr{G})$$

satisfying

$$\beta(G) = (\beta(g_i))_i = (h_i)_i = h \in H^0(\mathscr{U}, \mathscr{H}) = H^0(X, \mathscr{H}).$$

- Secondly, consider an element

$$g = (g_i)_i \in Z^0(\mathscr{U}, \mathscr{G})$$

and set

$$h = \beta(g).$$

If

$$f = (f_{ij})_{ij} \in Z^1(\mathscr{U}, \mathscr{F})$$

represents $\partial^0 h$ then according to the definition of $\partial^0 h$

$$\alpha(f_{ij}) = g_j - g_i = 0$$

Injectivity of α implies $f = 0$, in particular

$$[f] = 0 \in H^1(\mathscr{U}, \mathscr{F}) \text{ and } \partial^0 h := \pi([f]) = 0 \in H^1(X, \mathscr{F}).$$

iii) *Exactness at $H^1(X, \mathscr{F})$ and at $H^1(X, \mathscr{G})$*: See [8, §15, Satz 15.12], q.e.d.

Theorem 6.13 generalizes to connecting morphisms between cohomology groups $H^q(X, -)$ of arbitrary order $q \geq 2$. The proof presents some technical diffulties. They can be solved by introducing cohomology in the broader context of presheaves and exact sequences of presheaf morphisms.

6.3 Computation of cohomology groups

Theorem 6.14 (Cohomology of the smooth structure sheaf). *The smooth structure sheaf of a Riemann surface X satisfies*

$$H^1(X,\mathscr{E}) = 0.$$

Proof. According to Corollary 6.7 we have to show

$$H^1(\mathscr{U},\mathscr{E}) = 0$$

for each open covering $\mathscr{U} = (U_i)_{i\in I}$ of X. Consider an open covering $\mathscr{U}$ and a smooth partition of unity $(\phi_i)_{i\in I}$ subordinate to $\mathscr{U}$, see Proposition 4.19.

Consider a 1-cocycle

$$f = (f_{ij})_{ij} \in Z^1(\mathscr{U},\mathscr{E}).$$

Choose an arbitrary but fixed index $i \in I$. For each $k \in I$ the product

$$\phi_k \cdot f_{ki} : U_i \cap U_k \to \mathbb{C}$$

has support in $U_i \cap U_k$ and extends by zero to a smooth function

$$\tilde{f}_{ki} \in \mathscr{E}(U_i).$$

The sum

$$F_i := \sum_{k\in I} \tilde{f}_{ki} \in \mathscr{E}(U_i)$$

is well-defined. On $U_i \cap U_j$ we have

$$F_i - F_j = \sum_{k\in I} \tilde{f}_{ki} - \sum_{k\in I} \tilde{f}_{kj} = \sum_{k\in I} \phi_k \cdot (f_{ki} - f_{kj}) =$$

$$= -\sum_{k\in I} \phi_k \cdot (f_{ik} + f_{kj}) = -\sum_{k\in I} \phi_k \cdot f_{ij} = f_{ji} \cdot \sum_{k\in I} \phi_k = f_{ji}$$

Hence

$$f = \delta F$$

with the cochain

$$F := (F_k)_{k\in I} \in C^0(\mathscr{U},\mathscr{E}),\ q.e.d.$$

Analogously to Theorem 6.14 one proves by means of a partition of unity for other sheaves of smooth objects:

$$H^1(X,\mathscr{E}^1) = H^1(X,\mathscr{E}^{0,1}) = H^1(X,\mathscr{E}^{1,0}) = H^1(X,\mathscr{E}^2) = 0$$

and with some additional work for all these sheaves also the vanishing

$$H^q(X,-) = 0,\ q \geq 1.$$

Theorem 6.15 makes precise in which way elements of a first cohomology group act obstructions.

Theorem 6.15 (The theorems of Dolbeault and de Rham). *Consider a Riemann surface X. Then*

1. Dolbeault*: The resolution of the structure sheaf $\mathscr{O}$ by the Dolbeault sequence, see Definition 5.3, induces an isomorphism of complex vector spaces*

$$H^1(X,\mathscr{O}) \simeq \frac{H^0(X,\mathscr{E}^{0,1})}{im[H^0(X,\mathscr{E}) \xrightarrow{d''} H^0(X,\mathscr{E}^{0,1})]}$$

The resolution of the sheaf Ω^1

$$0 \to \Omega^1 \to \mathscr{E}^{1,0} \xrightarrow{d''} \mathscr{E}^{1,1} \to 0$$

induces an isomorphism of complex vector spaces

$$H^1(X,\Omega^1) \simeq \frac{H^0(X,\mathscr{E}^{1,1})}{im[H^0(X,\mathscr{E}^{1,0}) \xrightarrow{d''} H^0(X,\mathscr{E}^{1,1})]}$$

The groups on the right-hand side are named the Dolbeault groups $Dolb^{0,1}(X)$ and $Dolb^{1,1}(X)$ of X respectively.

2. de Rham*: The resolution of the sheaf $\mathbb{C}$ by the de Rham sequence, see Definition 5.6, induces an isomorphism of complex vector spaces*

$$H^1(X,\mathbb{C}) \simeq \frac{ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]}{im[H^0(X,\mathscr{E}) \xrightarrow{d} H^0(X,\mathscr{E}^1)]}$$

The group on the right-hand side is named the de Rham group *$Rh^1(X)$ of X.*

In addition one has apparently the Dolbeault groups

$$H^0(X,\mathscr{O}) = ker[H^0(X,\mathscr{E}) \xrightarrow{d''} H^0(X,\mathscr{E}^{0,1})] =: Dolb^{0,0}(X)$$

and

$$H^0(X,\Omega^1) = ker[H^0(X,\mathscr{E}^{1,0}) \xrightarrow{d''} H^0(X,\mathscr{E}^{1,1})] =: Dolb^{1,0}(X)$$

Proof (Theorem 6.15).

1. The exact Dolbeault sequence, see Theorem 5.4,

$$0 \to \mathscr{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$$

induces the long exact cohomology sequence

$$0 \to H^0(X,\mathscr{O}) \to H^0(X,\mathscr{E}) \xrightarrow{d''} H^0(X,\mathscr{E}^{0,1}) \xrightarrow{\partial^0} H^1(X,\mathscr{O}) \to H^1(X,\mathscr{E}) = 0$$

which proves the claim about $\mathscr{O}$.

Exactness of the sequence of sheaves

$$0 \to \Omega^1 \to \mathscr{E}^{1,0} \xrightarrow{d''} \mathscr{E}^{1,1} \to 0$$

means exactness of the corresponding sequence of stalks at each point $x \in X$. Hence we may assume $X = \mathbb{C}$ and $x = 0 \in \mathbb{C}$.

- Exactness at Ω^1_x: We have the injection $\Omega^1_x \hookrightarrow \mathscr{E}^{1,0}_x$.
- Exactness at $\mathscr{E}^{1,0}_x$: If
$$\omega = f\ dz \in \Omega^1_x$$
then $d''\omega = 0$ because f is holomorphic. For the converse assume
$$\eta = f\ dz \in \mathscr{E}^{1,0}_x$$
with $d''\eta = 0$. Then $\overline{\partial} f = 0$, hence $\omega \in \Omega^1_x$.
- Exactness at $\mathscr{E}^{1,1}_x$: Consider a form
$$\omega = g\ d\bar{z} \wedge dz \in \mathscr{E}^{1,1}_x.$$
Theorem 5.2 provides a smooth germ $f \in \mathscr{E}_x$ satisfying
$$\overline{\partial} f = g.$$
We set
$$\eta := f\ dz \in \mathscr{E}^{1,0}_x.$$
Then
$$d''\eta = \overline{\partial} f\ d\bar{z} \wedge dz = g\ d\bar{z}\ \wedge dz = \omega.$$

The resolution of Ω^1 induces the long exact cohomology sequence

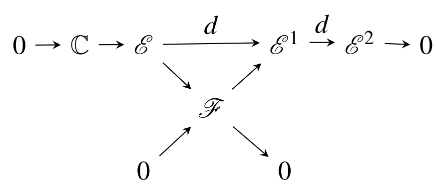
$$0 \to H^0(X,\Omega^1) \to H^0(X,\mathscr{E}^{1,0}) \xrightarrow{d''} H^0(X,\mathscr{E}^{1,1}) \xrightarrow{\partial^0} H^1(X,\Omega^1) \to H^1(X,\mathscr{E}^{1,0}) = 0$$

which proves the claim about Ω^1.

2. The de Rham sequence

$$0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to 0$$

is an exact sheaf sequence, but it is not a short sequence. Hence we split the de Rham sequence into two short exact sequences according to



by introducing the sheaf

$$\mathscr{F} := ker[\mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2] = im[\mathscr{E} \xrightarrow{d} \mathscr{E}^1]$$

Note that the kernel of a sheaf morphism is a sheaf. We obtain the two short exact sequences

$$0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{F} \to 0$$

and

$$0 \to \mathscr{F} \to \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to 0$$

- The first sheaf sequence has a long exact cohomology sequence with the segment

$$H^0(X,\mathscr{E}) \xrightarrow{d} H^0(X,\mathscr{F}) \xrightarrow{\partial^0} H^1(X,\mathbb{C}) \to H^1(X,\mathscr{E}) = 0$$

  which implies

$$H^1(X,\mathbb{C}) \simeq \frac{H^0(X,\mathscr{F})}{im[H^0(X,\mathscr{E}) \xrightarrow{d} H^0(X,\mathscr{F})]}$$

  Here

$$im[H^0(X,\mathscr{E}) \xrightarrow{d} H^0(X,\mathscr{F})] = im[H^0(X,\mathscr{E}) \xrightarrow{d} H^0(X,\mathscr{E}^1)]$$

- While the long exact cohomology sequence of the second sheaf sequence has the segment

$$0 \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)$$

which shows

$$H^0(X,\mathscr{F}) = ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)].$$

Combining the results from both cohomology sequences we obtain

$$H^1(X,\mathbb{C}) \simeq \frac{ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]}{im[H^0(X,\mathscr{E}) \xrightarrow{d} H^0(X,\mathscr{E}^1)]}$$

which finishes the proof, q.e.d.

Theorem 6.16 (Cohomology of the structure sheaf of a disk). *Consider a disk*

$$X := D_R(0) \subset \mathbb{C},\ 0 < R \leq \infty.$$

Then

$$H^1(X,\mathscr{O}) = 0.$$

Proof. The idea of the proof is to consider a holomorphic cocycle from the viewpoint of smooth functions. The cocyle splits in the smooth context due to Theorem 6.14. Then Dolbeault's Theorem applies about the solution of the inhomogenous $\overline{\partial}$-differential equation.

i) We apply Corollary 6.7. Consider an open covering

$$\mathscr{U} = (U_i)_{i\in I}$$

of X and a cocycle

$$f = (f_{ij})_{i,j\in I} \in Z^1(\mathscr{U},\mathscr{O}).$$

We consider $f \in Z^1(\mathscr{U},\mathscr{E})$ as a cocycle with values in the sheaf $\mathscr{E}$ of smooth functions. Theorem 6.14 states

$$H^1(X,\mathscr{E}) = 0.$$

Hence a cochain

$$(g_i)_{i\in I} \in C^0(\mathscr{U},\mathscr{E})$$

exists such that for all $i, j \in I$

$$f_{ij} = g_j - g_i.$$

ii) Because f_{ij} is holomorphic

$$0 = \frac{\partial f_{ij}}{\partial \overline{z}} = \frac{\partial g_j}{\partial \overline{z}} - \frac{\partial g_i}{\partial \overline{z}}$$

Hence a global function $h \in \mathscr{E}(X)$ exist which satisfies for all $i \in I$

$$h|U_i = \frac{\partial g_i}{\partial \overline{z}}.$$

iii) Theorem 5.2 provides a global function

$$g \in \mathscr{E}(X) \text{ with } \frac{\partial g}{\partial \bar{z}} = h.$$

Finally, the cochain

$$F := (F_i := g_i - (g|U_i))_{i \in I} \in C^0(\mathscr{U}, \mathscr{E})$$

satisfies for all $i \in I$

$$\frac{\partial F_i}{\partial \bar{z}} = \frac{\partial (g_i - (g|U_i))}{\partial \bar{z}} = 0,$$

which implies

$$F \in C^0(\mathscr{U}, \mathscr{O}).$$

Apparently for all $i, j \in I$

$$F_j - F_i = (g_j - (g|U_j) - (g_i - (g|U_i)) = g_j - g_i = f_{ij}$$

i.e.

$$\delta(F) = f \in B^1(\mathscr{U}, \mathscr{O}), \ q.e.d.$$

Theorem 6.16 shows: Any Riemann surface X has a Leray covering $\mathscr{U}$ for its structure sheaf $\mathscr{O}$: One takes coordinate neighbourhoods homeomorphic to a disk as elements of the covering. The theorem is the basis for the computation of the cohomology of locally free sheaves on Riemann surfaces.

Proposition 6.17 (Cohomology of the structure sheaf of $\mathbb{P}^1$). *The structure sheaf of the projective space satisfies*

$$H^1(\mathbb{P}^1, \mathscr{O}) = 0.$$

Proof. The standard covering $\mathscr{U} = (U_0, U_1)$ from Example 1.4 is a Leray covering of $\mathbb{P}^1$ due to Theorem 6.16. Hence Leray's Theorem 6.8 implies

$$H^1(\mathbb{P}^1, \mathscr{O}) = H^1(\mathscr{U}, \mathscr{O})$$

Consider a cocylce

$$(f_{01}, f_{10}) \in Z^1(\mathscr{U}, \mathscr{O})$$

Because $U_0 \cap U_1$ is biholomorph equivalent to $\mathbb{C}^*$ the holomophic function

$$f_{10} = -f_{01} \in \mathscr{O}(U_0 \cap U_1)$$

is determined by its Laurent series with respect to the coordinate $u := \phi_0$ on U_0

$$f_{10}(u) = \sum_{n \in \mathbb{Z}} c_n \cdot u^n.$$

On $U_0 \cap U_1$ we have

$$f_{10}(u) = f_0(u) + f_1(u)$$

with the definitions

$$f_0(u) := \sum_{n \in \mathbb{N}} c_n \cdot u^n \text{ and } f_1(u) := \sum_{n=\infty}^{-1} c_n \cdot u^n = \sum_{n=1}^{\infty} c_{-n} \cdot \left(\frac{1}{u}\right)^n$$

To split the function $\hat{f}_0$ we introduce the two holomorphic functions

$$g_0 : U_0 \to \mathbb{C},\ g_0(x) := -f_0(u(x)),$$

and

$$g_1 : U_1 \to \mathbb{C},\ g_1(x) := f_1\left(\frac{1}{\mathrm{v}(x)}\right) = \sum_{n=1}^{\infty} c_{-n} \cdot \mathrm{v}(x)^n$$

Then for all $x \in U_0 \cap U_1$ holds

$$f_{01}(x) = -g_0(x) + g_1(x)$$

i.e. on $U_0 \cap U_1$ holds the splitting

$$f_{01} = -g_0 + g_1$$

Hence

$$Z^1(\mathscr{U}, \mathscr{O}) = B^1(\mathscr{U}, \mathscr{O})$$

which implies

$$H^1(\mathscr{U}, \mathscr{O}) = 0 = H^1(\mathbb{P}^1, \mathscr{O}),\ q.e.d.$$

Theorem 6.18 generalizes Proposition 6.17.

Theorem 6.18 (Cohomology of the twisted sheaves on $\mathbb{P}^1$). *The cohomology of the twisted sheaves on* $\mathbb{P}^1$

$$\mathscr{L} := \mathscr{O}(k),\ k \in \mathbb{Z},$$

satisfies

$$dim_{\mathbb{C}}\, H^1(\mathbb{P}^1, \mathscr{L}) = \begin{cases} 0 & k > -2 \\ 1 + (-k-2) & k \le -2 \end{cases}$$

In particular

$$dim_{\mathbb{C}}\, H^1(\mathbb{P}^1, \mathscr{L}) = dim_{\mathbb{C}}\, H^0(\mathbb{P}^1, \mathscr{L}^\vee \otimes_{\mathscr{O}} \Omega^1).$$

Proof. The standard covering

$$\mathscr{U} = (U_0, U_1)$$

is a Leray covering of $\mathbb{P}^1$ for $\mathscr{L}$. We choose coordinates

$$u := \frac{z_1}{z_0} \text{ and } \mathrm{v} := \frac{z_0}{z_1}$$

with the transformation

$$u = \frac{1}{\mathrm{v}}$$

Hence

$$H^1(\mathscr{U}, \mathscr{L}) = H^1(\mathbb{P}^1, \mathscr{L}).$$

We have

$$Z^1(\mathscr{U}, \mathscr{L}) \simeq \mathscr{L}(U_{01}) \simeq \mathscr{O}(U_{01}) \simeq \mathscr{O}(\mathbb{C}^*).$$

On one hand,

$$Z^1(\mathscr{U}, \mathscr{L}) = \left\{ \sum_{n=-\infty}^{\infty} c_n \cdot u^n : \text{ convergent Laurent series} \right\}.$$

On the other hand,

$$B^1(\mathscr{U}, \mathscr{L}) = \{ s_0 - s_1 \in \mathscr{L}(U_{01}) : (s_0, s_1) \in \mathscr{L}(U_0) \times \mathscr{L}(U_1) \}$$

Using on U_{01} the u-coordinate we obtain coboundaries as the following holomorphic functions

$$B^1(\mathscr{U}, \mathscr{L}) \simeq \left\{ \sum_{n=0}^{\infty} c_n \cdot u^n - u^k \cdot \sum_{n=-\infty}^{0} d_n \cdot u^n : \text{ convergent Laurent series} \right\} \subset Z^1(\mathscr{U}, \mathscr{L})$$

From

$$H^1(\mathbb{P}^1, \mathscr{L}) = \frac{Z^1(\mathscr{U}, \mathscr{L})}{B^1(\mathscr{U}, \mathscr{L})}$$

results

$$dim_{\mathbb{C}}\, H^1(\mathbb{P}^1, \mathscr{L}) = \begin{cases} 0 & k > -2 \\ 1 + (-k-2) & k \leq -2 \end{cases}$$

Recalling from Exercise 29 the isomorphy

$$\Omega^1 \simeq \mathscr{O}(-2)$$

shows

$$\mathscr{L}^\vee \otimes_{\mathscr{O}} \Omega^1 = \mathscr{L}^\vee \otimes_{\mathscr{O}} \mathscr{O}(-2) = \mathscr{O}(-k-2).$$

The dimension of

$$H^0(\mathbb{P}^1, \mathscr{O}(-k-2))$$

has been computed in Example 2.11 and confirms

$$dim_{\mathbb{C}}\, H^1(\mathbb{P}^1, \mathscr{L}) = dim\, H^0(\mathbb{P}^1, \mathscr{O}(-k-2)) = dim\, H^0(\mathbb{P}^1, \mathscr{L}^\vee \otimes_{\mathscr{O}} \Omega^1),\ q.e.d.$$

Theorem 6.17 is a particular case of Serre's duality theorem for invertible sheaves, see Theorem 10.28.

Čech cohomology and cohomology theory for complex manifolds of arbitrary finite dimension respectively complex spaces is the content of [7, §7 and §8] and [14, Kap. A and Kap. B].

The book [17, Chap. VI] introduces sheaf cohomology by using sheaf resolutions by fine sheaves: Chap. VI, Sect. B, Theorem 4 proves the uniqueness of the cohomology theory for paracompact Hausdorff spaces. The book [12] is devoted to sheaf theory and provides different methods to obtain a cohomology theory. A general reference for sheaf theory is [3].

Part II
Compact Riemann Surfaces

Chapter 7
The finiteness theorem

On a compact Riemann surface X the structure sheaf satisfies

$$\mathit{dim}\, H^0(X, \mathscr{O}) = 1.$$

The result is a consequence of the open mapping theorem for non-constant holomorphic functions. The finiteness theorem is a far reaching generalization: For all invertible sheaves $\mathscr{L}$ on X and all $q \in \mathbb{N}$ holds

$$\mathit{dim}\, H^q(X, \mathscr{L}) < \infty.$$

This finiteness result does not generalize to the sheaf $\mathscr{M}$ of meromorphic functions: We have seen that $\mathscr{M}(\mathbb{P}^1)$ is a pure transcendental field extension of $\mathbb{C}$, hence

$$\mathit{dim}\, H^0(\mathbb{P}^1, \mathscr{M}) = \infty.$$

If not stated otherwise, all vector spaces in this chapter are complex vector spaces and all dimension formulas refer to their complex vector space dimension.

7.1 Topological vector spaces of holomorphic functions

The present section combines functional analysis and complex analysis. Our first aim is to provide certain groups of holomorphic functions and holomorphic cochains with the structure of a topological vector space. In general these spaces are infinite-dimensional vector spaces. Therefore one has to provide them with the additional structure of a topological vector space. One of the strictest structures of this kind are Hilbert spaces. More general structures are Fréchet spaces.

Cohomology groups are cokernels. Hence an element of a cohomology group is an equivalence class. First, one has to define a topology on cocycles and on the subgroup of coboundaries. Secondly, one has to obtain a well-defined topology on the quotient. All these topologies are generally obtained by means of a suitable atlas of the manifold. Therefore the final step is to verify that the induced vector space topology is independent from the chosen atlas.

Definition 7.1 (Fréchet space).

1. A *topological vector space* is a vector space such that addition and scalar multiplication are continuous functions. We assume that the base field is $\mathbb{C}$, provided with its Euclidean topology.

2. A *seminorm* on a vector space V is a map

$$p : V \to \mathbb{R}_+$$

satisfiying:

i) For all $\lambda \in \mathbb{C}$ and for all $\mathrm{v} \in V$

$$p(\lambda \cdot \mathrm{v}) = |\lambda| \cdot p(\mathrm{v})$$

ii) For all $\mathrm{v}_1, \mathrm{v}_2 \in V$

$$p(\mathrm{v}_1 + \mathrm{v}_2) \le p(\mathrm{v}_1) + p(\mathrm{v}_2).$$

A seminorm p is a *norm* if in addition

$$p(\mathrm{v}) = 0 \iff \mathrm{v} = 0.$$

3. A topological vector space V is a *Fréchet space* if V is a complete Hausdorff space and the topology is defined by a countable family $(p_n)_{n\in\mathbb{N}}$ of seminorms, i.e. the finite intersections of sets

$$V(j,\varepsilon) := \{\mathrm{v} \in V : \ p_j(\mathrm{v}) < \varepsilon\}, \ \varepsilon > 0,$$

form a neighbourhood basis of $0 \in V$.

Apparently the concept of Fréchet spaces generalizes the concept of Banach spaces by replacing a fixed norm by a countable family of seminorms. A sequence

$$(f_\nu)_\nu \in \mathbb{N}$$

in a Fréchet space V is a *Cauchy sequence* if for each neighbourhood of zero $W \subset V$ exists $N \in \mathbb{N}$ such that for all $\nu, \ \mu \ge N$

$$f_\nu - f_\mu \in W.$$

Each Fréchet space V is metrizable, e.g. by the metric

$$d(f,g) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(f-g)}{1+p_n(f-g)},\ f,g \in V.$$

Definition 7.2 (Topology of compact convergence). Consider an open set $U \subset \mathbb{C}$. On the vector space $\mathscr{O}(U)$ of holomorphic functions on U the *topology of compact convergence* is defined as follows: A sequence

$$(f_\nu)_{\nu \in \mathbb{N}},$$

of holomorphic functions $f_\nu \in \mathscr{O}(U),\ \nu \in \mathbb{N}$, is convergent towards $f \in \mathscr{O}(U)$ if and only if for each compact $K \subset U$

$$\lim_{\nu \to \infty} f_\nu | K = f | K$$

as the limit of uniform convergence.

Proposition 7.3 (Fréchet space $\mathscr{O}(U)$). *Consider an open set $U \subset \mathbb{C}$. The vector space $\mathscr{O}(U)$ of holomorphic functions on U provided with the topology of compact convergence is a Fréchet space.*

Proof. One chooses an exhaustion $(U_n)_{n \in \mathbb{N}}$ of U by relatively compact subsets

$$U_n \subset\subset U_{n+1},\ n \in \mathbb{N},$$

and defines the seminorms

$$p_n : \mathscr{O}(U) \to \mathbb{R}_+,\ p_n(f) := \|f\|_{U_n} := sup\ \{|f(z)| :\ z \in U_n\}$$

The Hausdorff property follows from the equivalence

$$f = 0 \iff p_n(f) = 0 \text{ for all } n \in \mathbb{N}.$$

Completeness of $\mathscr{O}(U)$ follows from Weierstrass' convergence theorem, see [41, Ch. 3], q.e.d.

The vector space of complex square-integrable functions $L^2(U,\mathbb{C})$ is a Hilbert space with respect to the Hermitian scalar product

$$< -,- >: L^2(U,\mathbb{C}) \times L^2(U,\mathbb{C}) \to \mathbb{C},\ < f,g > := \int_U f(z) \cdot \overline{g(z)}\ dx \wedge dy$$

For each $f \in L^2(U,\mathbb{C})$ denote by

$$\|f\|_{L^2(U)} := \sqrt{< f,f >}$$

the induced norm.

Definition 7.4 (Square-integrable holomorphic functions). For an open set $U \subset \mathbb{C}$ denote by

$$L^2(U,\mathcal{O}) := \left\{ f \in \mathcal{O}(U) : \|f\|^2_{L^2(U)} < \infty \right\} \subset L^2(U,\mathbb{C})$$

the subspace of *square-integrable holomorphic functions*.

Proposition 7.5 (Estimating L^2-norm by sup-norm). *For an open set $U \subset \mathbb{C}$ and $f \in L^2(U,\mathbb{C})$ holds*

$$\|f\|_{L^2(U)} \leq \sqrt{vol(U)} \cdot \|f\|_U \leq \infty.$$

Here

$$vol(U) := \iint_U dx \wedge dy.$$

Lemma 7.6 (Orthogonal basis). *Consider a point $a \in \mathbb{C}$ and the disk*

$$D := D_R(a),\ 0 < R \leq \infty.$$

1. *The family of monomials*

$$\phi_n(z) := (z-a)^n,\ n \in \mathbb{N},$$

is an orthogonal basis in $L^2(D,\mathcal{O})$ with

$$\|\phi_n\|^2_{L^2(D)} = \pi R^2 \cdot \frac{R^{2n}}{n+1}$$

2. *For $f \in L^2(D,\mathcal{O})$ with Taylor series*

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

the coefficients of the Taylor series equal the Fourier coefficients with respect to the orthogonal basis, i.e.

$$c_n = \frac{< f, \phi_n >}{\|\phi_n\|^2_{L^2(D)}}$$

In particular

$$\|f\|^2_{L^2(D)} = \pi R^2 \cdot \sum_{n=0}^{\infty} R^{2n} \cdot \frac{|c_n|^2}{n+1}$$

Proof. 1. i) One computes the values

$$<\phi_n, \phi_m> \in \mathbb{C}$$

by introducing polar coordinates.

ii) To verify completeness of the orthogonal family we have to show for each $f \in L^2(D, \mathscr{O})$:

$$<f, \phi_m> = 0 \text{ for all } m \in \mathbb{N} \implies f = 0.$$

The holomorphic function $f \in \mathscr{O}(D)$ expands into a Taylor series with center a and radius of convergence $\geq R$

$$f(z) = \sum_{n=0}^{\infty} c_n \cdot (z-a)^n$$

Consider a radius $0 < r < R$ and set

$$D_r := D_r(a) \subset\subset D$$

The Taylor series of f is uniformly convergent on D_r. For each $m \in \mathbb{N}$

$$\iint_{D_r} f \cdot \overline{\phi}_m \, dx \wedge dy = \iint_{D_\rho} \sum_{n=0}^{\infty} c_n \cdot (z-a)^n \cdot (\overline{z-a})^m \, dx \wedge dy =$$

$$= \sum_{n=0}^{\infty} c_n \cdot \iint_{D_r} (z-a)^n \cdot (\overline{z-a})^m \, dx \wedge dy = \sum_{n=0}^{\infty} c_n \cdot \int_0^{2\pi} \int_0^r \rho^{n+m+1} \cdot e^{i\theta(n-m)} \, d\rho \wedge d\theta =$$

$$= 2\pi \cdot c_m \cdot \int_0^r \rho^{2m+1} \, d\rho = \pi r^2 \cdot c_m \cdot \frac{r^{2m}}{m+1}$$

The Hölder estimate implies

$$\iint_D |f \cdot \phi_m| \, dx \wedge dy = <f, \phi_m> \leq \|f\|_{L^2(D)} \cdot \|\phi_m\|_{L^2(D)} < \infty$$

Hence

$$\lim_{r \to R} \iint_{D_r} f \cdot \overline{\phi}_m \, dx \wedge dy = \iint_D f \cdot \overline{\phi}_m \, dx \wedge dy$$

which implies

$$<f, \phi_m> = \iint_D f \cdot \overline{\phi}_m \, dx \wedge dy = \lim_{r \to R} \pi r^2 \cdot c_m \cdot \frac{r^{2m}}{m+1} = c_m \cdot \pi R^2 \cdot \frac{R^{2m}}{m+1}.$$

The assumption

$$<f, \phi_m> = 0$$

for all $m \in \mathbb{N}$ implies: For all $m \in \mathbb{N}$

$$c_m = 0$$

hence $f=0$.

2. The Parseval equation applied to the orthogonal family $(\phi_n)_{n\in\mathbb{N}}$ implies the formula for $\|f\|^2_{L^2(D)}$, q.e.d.

Proposition 7.7 shows: If a holomorphic function f on an open set is square-integrable, then its L^2-norm majorizes for any compact subset the maximum norm of f. Corollary 7.8 then concludes that $L^2(U,\mathscr{O})$ is a Hilbert space.

Proposition 7.7 (Estimating sup-norm by L^2-norm after shrinking). *Consider an open set $U\subset\mathbb{C}$ and denote for each $r>0$ by*

$$U_r := \{z\in U:\ D_r(z)\subset U\}$$

the subset of points with boundary distance at least $=r$. Then each $f\in L^2(U,\mathscr{O})$ satisfies the estimate:

$$\|f\|_{U_r} \le \frac{1}{r\cdot\sqrt{\pi}}\cdot\|f\|_{L^2(U)}$$

Proof. Consider the Taylor expansion of F with center a point $a\in U_r$

$$f(z)=\sum_{n=0}^{\infty} c_n\cdot(z-a)^n$$

Lemma 7.6 implies

$$|f(a)|^2=|c_0|^2\le\frac{1}{r^2\pi}\cdot\|f\|^2_{L^2(D_r(a))}$$

Hence

$$|f(a)|\le\frac{1}{r\sqrt{\pi}}\cdot\|f\|_{L^2(D_r(a))}\le\frac{1}{r\sqrt{\pi}}\cdot\|f\|_{L^2(U)}$$

and

$$\|f\|_{U_r}\le\frac{1}{r\sqrt{\pi}}\cdot\|f\|_{L^2(U)},\ q.e.d.$$

Corollary 7.8 (The Hilbert space $L^2(U,\mathscr{O})$). *Consider an open set $U\subset\mathbb{C}\simeq\mathbb{R}^2$ and the complex vector space of square-integrable holomorphic functions on U*

$$L^2(U,\mathscr{O}) := \left\{f\in\mathscr{O}(U):\|f\|^2_{L^2(U)}<\infty\right\}.$$

The subspace of holomorphic square-integrable functions

$$L^2(U,\mathscr{O})\subset L^2(U,\mathbb{C})$$

is a closed subspace of the Hilbert space $L^2(U,\mathbb{C})$, hence itself a Hilbert space with respect to the induced scalar product.

Proof. Consider a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of holomorphic square-integrable functions and assume the existence of a square-integrable function $f \in L^2(U,\mathbb{C})$ with

$$f = \lim_{\nu\to\infty} f_\nu \text{ i.e. } \lim_{\nu\to\infty} \|f_\nu - f_\nu\|_{L^2(U)} = 0$$

We have to show that the limit f is holomorphic. Because holomorphy is a local property it suffices to prove that the restriction of f to suitable open subsets of U is holomorphic. For each subset $U_r \subset U$ with $D_r(z) \subset U$ for all $z \in U_r$ Proposition 7.7 implies

$$\|f - f_\nu\|_{U_r} \leq \frac{1}{r\sqrt{\pi}} \cdot \|f - f_\nu\|_{L^2(U_r)}$$

Weierstrass' convergence theorem implies the holomorphy of the restriction $f|U_r$, q.e.d.

The Hilbert space $L^2(U,\mathscr{O})$ is named a *Bergmann space*.

7.2 Hilbert spaces of holomorphic cochains

We now carry over the Hilbert space topology on square integrable holomorphic functions to cochains and cocyles of Čech-cohomology. In order to apply the results from Section 7.1 we choose on a given Riemann surface X a finite family of charts which map biholomorphically onto disks in the plane.

Definition 7.9 (Square integrable cochains). Consider a Riemann surface X and a finite family of charts on X

$$\phi_i : U_i^* \xrightarrow{\simeq} D_i(0) \text{ disk},\ i = 1,...,n.$$

For a family

$$\mathscr{U} = (U_i)_{i=1,...,n}$$

of open subsets

$$U_i \subset U_i^*,\ i = 1,...,n,$$

set

$$Y := \bigcup_{i=1}^{n} U_i \subset X.$$

Then consider the Čech-cochains of Y with respect to the open covering $\mathscr{U}$:

- For $\eta = (f_i) \in C^0(\mathscr{U}, \mathscr{O})$ set

$$\|\eta\|^2_{L^2(\mathscr{U})} := \sum_{i=1}^{n} \|f_i\|^2_{L^2} \in \mathbb{R}^+ \cup \{\infty\}$$

- For $\xi = (f_{ij}) \in C^1(\mathscr{U}, \mathscr{O})$ set

$$\|\xi\|^2_{L^2(\mathscr{U})} := \sum_{i,j=1}^{n} \|f_{i,j}\|^2_{L^2} \in \mathbb{R}^+ \cup \{\infty\}$$

Here

$$\|f_i\|_{L^2} := \|f_i \circ \phi_i^{-1}\|_{L^2(\phi_i(U_i))}$$

and

$$\|f_{i,j}\|_{L^2} := \|f_{i,j} \circ \phi_i^{-1}\|_{L^2(\phi_i(U_i \cap U_j))}$$

We define the complex vector spaces of *square integrable cochains* on Y with respect to $\mathscr{U}$ as

$$C^0_{L^2}(\mathscr{U}, \mathscr{O}) := \{\eta \in C^0(\mathscr{U}, \mathscr{O}) : \ \|\eta\|^2_{L^2(\mathscr{U})} < \infty\}$$

and

$$C^1_{L^2}(\mathscr{U}, \mathscr{O}) := \{\xi \in C^1(\mathscr{U}, \mathscr{O}) : \ \|\xi\|^2_{L^2(\mathscr{U})} < \infty\}$$

Note. Definition 7.9 does not presuppose that $(U_i^*)_{i \in I}$ covers all of of X.

Lemma 7.10 (Square integrable cocycles).

1. *The vector spaces of square integrable cochains*

$$C^0_{L^2}(\mathscr{U}, \mathscr{O}) := \{\eta \in C^0(\mathscr{U}, \mathscr{O}) : \ \|\eta\|^2_{L^2(\mathscr{U})} < \infty\} \subset C^0(\mathscr{U}, \mathscr{O})$$

and

$$C^1_{L^2}(\mathscr{U}, \mathscr{O}) := \{\xi \in C^1(\mathscr{U}, \mathscr{O}) : \ \|\xi\|^2_{L^2(\mathscr{U})} < \infty\} \subset C^1(\mathscr{U}, \mathscr{O})$$

are Hilbert spaces.

2. *Their subspaces of cocycles*

$$Z^0_{L^2}(\mathscr{U}, \mathscr{O}) := Z^0(\mathscr{U}, \mathscr{O}) \cap C^0_{L^2}(\mathscr{U}, \mathscr{O})$$

and

$$Z^1_{L^2}(\mathscr{U}, \mathscr{O}) := Z^1(\mathscr{U}, \mathscr{O}) \cap C^1_{L^2}(\mathscr{U}, \mathscr{O})$$

are closed, hence Hilbert spaces too.

The open mapping theorem for Hilbert spaces is the main ingredient from functional analysis to prove the finiteness theorem 7.16.

Remark 7.11 (Open mapping theorem of functional analysis). Any surjective continuous linear map

$$f : H_1 \to H_2$$

between two Hilbert spaces is an open map. One proves the result in the more general category of Banach spaces by using Baire's category theorem, see [21, Satz 9.1]. The result generalizes to Fréchet spaces [31, Theor. 2.11].

In the following we often consider pairs of open coverings $\mathscr{U}$ and $\mathscr{V}$ of a topological space, which form a shrinking

$$\mathscr{V} << \mathscr{U}.$$

Recall from Definition 4.18 that both coverings of a shrinking have the same index set I, and for all $i \in I$ holds

$$V_i \subset\subset U_i.$$

Results about the cohomology of a Riemann surface X can be proven by referring to suitable fine coverings of X. Lemma 7.12 shows how to extend a cocycle without changing its cohomology class with respect to families

$$\mathscr{W} << \mathscr{V} << \mathscr{U}$$

We work with the Hilbert spaces of square integrable holomorphic cochains. The relative compactness of pairs of open sets

$$V \subset\subset U$$

is used to conclude that holomorphic functions on U become bounded when restricted to the compact closure

$$\overline{V} \subset U.$$

Boundedness in the *sup*-norm then allows to estimate the L^2-norm of the restriction. We show by using the Dolbeault lemma: Any cocycle $\xi \in Z^1_{L^2(\mathscr{V})}(\mathscr{V}, \mathscr{O})$ extends to a cocycle $\zeta \in Z^1_{L^2(\mathscr{U})}(\mathscr{U}, \mathscr{O})$ such that with respect to $\mathscr{W}$

$$\zeta = \xi + \delta\eta$$

with a cochain

$$\eta \in C^0_{L^2(\mathscr{W})}(\mathscr{W}, \mathscr{O}).$$

In addition, the L^2-norms of ζ and η depend continuously on the L^2-norm of ξ. Lemma 7.12 will be used for the induction step in the proof of Proposition 7.14.

Lemma 7.12 (Extending cocycles and restricting cohomology classes). *Consider a Riemann surface X and a finite family of charts on X*

$$\phi_i : U_i^* \xrightarrow{\simeq} D_1(0),\ i = 1, ..., n.$$

Assume families of open subsets

$$\mathscr{W} << \mathscr{V} << \mathscr{U} << \mathscr{U}^*.$$

Then a constant $C > 0$ with the following property exists: For each $\xi \in Z^1_{L^2}(\mathscr{V}, \mathscr{O})$ exist

- *a cocycle $\zeta \in Z^1_{L^2}(\mathscr{U}, \mathscr{O})$ and*
- *a cochain $\eta \in C^0_{L^2}(\mathscr{W}, \mathscr{O})$*

satisfying with respect to $\mathscr{W}$

$$\zeta = \xi + \delta\eta$$

in particular

$$[\zeta|\mathscr{W}] = [\xi|\mathscr{W}] \in H^1(\mathscr{W}, \mathscr{O})$$

and

$$max\{\|\zeta\|_{L^2(\mathscr{U})},\ \|\eta\|_{L^2(\mathscr{W})}\} \leq C \cdot \|\xi\|_{L^2(\mathscr{V})}$$

Proof. We set

$$V := |\mathscr{V}| := \bigcup_{i=1}^{n} V_i,\ W := |\mathscr{W}|,\ U^* := |\mathscr{U}^*| := \bigcup_{i=1}^{n} U_i^*$$

i) *Extend the cocycle from $\mathscr{V}$ to $\mathscr{U}$*: First, we consider the smooth category and split

$$\xi = (\xi_{ij})_{i,j=1,...,n} \in Z^1_{L^2}(\mathscr{V}, \mathscr{O}) \subset Z^1(\mathscr{V}, \mathscr{E})$$

as

$$\xi_{ij} = g_j - g_i$$

with a smooth cochain $(g_i)_{i=1,...,n} \in C^0(\mathscr{V}, \mathscr{E})$. Because

$$d''\xi_{ij} = 0$$

we have on $V_i \cap V_j$

$$d''g_i = d''g_j$$

and obtain a global differential form

$$\omega \in \mathscr{E}^{0,1}(V)$$

satisfying for all $i = 1, ..., n$

$$\omega|V_i = d''g_i.$$

Because

$$W \subset\subset V$$

we may choose a function $\psi \in \mathscr{E}(X)$ with

$$supp\ \psi \subset V \text{ and } \psi|W = 1.$$

We obtain an extension

$$\psi \cdot \omega \in \mathscr{E}^{0,1}(U^*).$$

For each $i = 1, ..., n$ on the coordinate neighbourhood U_i^* Dolbeault's Lemma, Theorem 5.2, provides a smooth function $h_i \in \mathscr{E}(U_i^*)$ with

$$d''h_i = \psi \cdot \omega | U_i^*.$$

As a consequence, on the intersections $U_i^* \cap U_j^*$ the functions

$$F_{ij} := h_j - h_i$$

are holomorphic. Because

$$\mathscr{U} \subset\subset \mathscr{U}^*$$

estimating the L^2-norm against the sup-norm according to Proposition 7.5 provides a cocycle

$$\zeta := (F_{ij}|U_i \cap U_j)_{i,j=1,...,n} \in Z^1_{L^2}(\mathscr{U}, \mathscr{O}).$$

ii) *Construct a coboundary on $\mathscr{W}$*: For each $i = 1, ..., n$ we have on W_i

$$d''h_i = \psi \cdot \omega = \omega = d''g_i,$$

which implies the holomorphy of

$$h_i - g_i.$$

Because

$$\mathscr{W} << \mathscr{V} << \mathscr{U}^*$$

the estimate from Proposition 7.5 assures

$$\eta := (h_i - g_i)_{i=1,...,n} \in C^0_{L^2}(\mathscr{W}, \mathscr{O}).$$

We have on $W_i \cap W_j$

$$F_{ij} - \xi_{ij} = (h_j - h_i) - (g_j - g_i) = (h_j - g_j) - (h_i - g_i),$$

hence on $\mathscr{W}$

$$\zeta - \xi = \delta\eta.$$

iii) *Estimate the L^2-norms*:The Cartesian product of Hilbert spaces

$$H := Z^1_{L^2}(\mathscr{U}, \mathscr{O}) \times Z^1_{L^2}(\mathscr{V}, \mathscr{O}) \times C^0_{L^2}(\mathscr{W}, \mathscr{O})$$

is a Hilbert space with induced norm

$$\|(\zeta,\xi,\eta)\|_{L^2} := \sqrt{\|\zeta\|^2_{L^2(\mathscr{U})} + \|\xi\|^2_{L^2(\mathscr{V})} + \|\eta\|^2_{L^2(\mathscr{W})}}$$

Its subspace

$$A := \{(\zeta,\xi,\eta) \in H : \ \zeta = \xi + \delta\eta \text{ on } \mathscr{W}\} \subset H$$

is closed because the restriction as well as the coboundary map are continuous. Hence A is a Hilbert space itself. The canonical projection

$$pr_2 : A \to Z^1_{L^2}(\mathscr{V},\mathscr{O}),\ (\zeta,\xi,\eta) \mapsto \xi,$$

is linear and continuous. It is surjective according to part i) and ii). The open mapping theorem, see Remark 7.11, implies: The map pr_2 is also open. Hence a constant $C > 0$ exists such that any

$$\xi \in Z^1_{L^2}(\mathscr{V},\mathscr{O})$$

has under pr_2 an inverse image

$$x = (\zeta,\xi,\eta) \in A$$

with

$$\|x\|_{L^2} \le C \cdot \|\xi\|_{L^2(\mathscr{W})},\ q.e.d.$$

Lemma 7.13 prepares the proof of Proposition 7.14. It formalizes the result: Those holomorphic functions on an open set $U \subset \mathbb{C}$ with a high fraction of their L^2-norm concentrated near the boundary of U form a finite-dimensional subspace of $L^2(U,\mathscr{O})$. Hence Lemma 7.13 indicates the point where certain subspaces of holomorphic sections with finite co-dimension are identified.

Lemma 7.13 (Finite codimension). *Consider a pair of relatively compact open subsets*

$$W \subset\subset U \subset \mathbb{C}$$

Then for each $\varepsilon > 0$ a closed subspace $A \subset L^2(U,\mathscr{O})$ exists with finite codimension such that for all $f \in A$

$$\|f\|_{L^2(W)} \le \varepsilon \cdot \|f\|_{L^2(U)}$$

Proof. i) *Topology*: Because $\overline{W}$ is compact and $\overline{W} \subset U$ we may choose a radius $r > 0$ and a finite set

$$P := \{a_1,...,a_k\} \subset W$$

such that

$$W \subset \bigcup_{a \in P} D_{r/2}(a_j) \subset\subset \bigcup_{a \in P} D_r(a_j) \subset U$$

We choose $n \in \mathbb{N}$ such that

$$\frac{k}{2^{n+1}} \le \varepsilon.$$

Consider
$$A := \{f \in L^2(U,\mathcal{O}) :\ ord(f;\ a) \geq\ n \text{ for all } a \in P\},$$
the closed subspace of all functions which vanish at least of order n at each point of P. Then
$$A \subset L^2(U,\mathcal{O})$$
has codimension at most $k \cdot n$.

ii) *Estimate*: For each fixed $a \in P$ and $f \in A$ we consider the Taylor series of f with center a
$$f(z) = \sum_{\nu=n}^{\infty} c_\nu \cdot (z-a)^\nu$$
For any $0 < \rho \leq r$ Lemma 7.6 implies
$$\|f\|^2_{L^2(D_\rho(a))} = \sum_{\nu=n}^{\infty} \frac{\pi \cdot \rho^{2\nu+2}}{\nu+1} \cdot |c_\nu|^2$$
In particular for $r/2 < r$
$$\|f\|^2_{L^2(D_{r/2}(a))} \leq \frac{1}{2^{2n+2}} \cdot \sum_{\nu=n}^{\infty} \frac{\pi \cdot r^{2\nu+2}}{\nu+1} \cdot |c_\nu|^2 \leq \frac{1}{2^{2n+2}} \cdot \|f\|^2_{L^2(D_r(a))} \leq \frac{1}{2^{2n+2}} \cdot \|f\|^2_{L^2(U)}$$
or
$$\|f\|_{L^2(D_{r/2}(a))} \leq \frac{1}{2^{n+1}} \cdot \|f\|_{L^2(U)}$$
Because $a \in P$ is arbitrary and
$$W \subset \bigcup_{k=1}^{n} D_{r/2}(a)$$
we get
$$\|f\|_{L^2(W)} \leq \frac{k}{2^{n+1}} \cdot \|f\|_{L^2(U)} \leq \varepsilon \cdot \|f\|_{L^2(U)},\ q.e.d.$$

Applying Lemma 7.13, Proposition 7.14 identifies certain closed subspaces of $Z^1(\mathcal{U},\mathcal{O})$ with finite co-dimension, which can be neglected for the cohomology in the final proof of the finiteness theorem.

Proposition 7.14 (Restricting cohomology along relatively compact coverings). *Consider a Riemann surface X and a finite family $\mathcal{U}^* = (U_i^*)_{i=1,...,n}$ of charts on X*
$$\phi_i : U_i^* \xrightarrow{\simeq} D_1(0),\ i = 1,...,n.$$
Assume families of open subsets of X

$$\mathscr{W} << \mathscr{U} << \mathscr{U}^*.$$

Then the image of the canonical restriction

$$H^1(\mathscr{U},\mathscr{O}) \to H^1(\mathscr{W},\mathscr{O})$$

is finite-dimensional.

Proof. i) *Providing a distinguished finite-dimensional subspace of* $Z^1_{L^2}(\mathscr{U},\mathscr{O})$: We insert a further family

$$\mathscr{W} << \mathscr{V} << \mathscr{U}.$$

Lemma 7.12, applied to the triple of families, provides a constant $C > 0$ as scaling factor for the extension from $\mathscr{V}$ to $\mathscr{U}$. We fix

$$\varepsilon := \frac{1}{2} \cdot \frac{1}{C}$$

as the scaling factor for the restriction from $\mathscr{U}$ to $\mathscr{V}$. Lemma 7.13 provides a closed subspace

$$A \subset Z^1_{L^2}(\mathscr{U},\mathscr{O})$$

with finite codimension such that for all $\alpha \in A$

$$\|\alpha\|_{L^2(\mathscr{V})} \leq \varepsilon \cdot \|\alpha\|_{L^2(\mathscr{U})}.$$

Let

$$S := A^{\perp} \subset Z^1_{L^2}(\mathscr{U},\mathscr{O})$$

be the orthogonal complement of A. It is finite-dimensional. Each induction step in the subsequent part of the proof will use the orthogonal decomposition

$$Z^1_{L^2}(\mathscr{U},\mathscr{O}) = A \overset{\perp}{\oplus} S$$

Note: We use the fact that in a Hilbert space each closed subspace has an orthogonal complement; an analogoue does not hold in general Banach spaces much less in Fréchet spaces.

We claim: The restriction of $Z^1(\mathscr{U},\mathscr{O})$ to $\mathscr{W}$ is cohomologuous with the restriction of the finite-dimensional subspace S, i.e. for each cocycle

$$\xi \in Z^1(\mathscr{U},\mathscr{O})$$

exist

- a cocycle $\sigma \in S \subset Z^1_{L^2}(\mathscr{U},\mathscr{O})$ and
- a cochain $\eta \in C^0(\mathscr{W},\mathscr{O})$

such that on $\mathscr{W}$

$$\sigma = \xi + \delta\eta, \ i.e.\ [\sigma] = [\xi] \in H^1(\mathscr{W}, \mathscr{O}).$$

ii) *Inductive construction of* σ *and* η: Choose an arbitrary but fixed

$$\xi \in Z^1(\mathscr{U}, \mathscr{O})$$

The estimate from Proposition 7.5 allows to set

$$M := \|\xi\|_{L^2(\mathscr{V})} < \infty$$

because $\mathscr{V} << \mathscr{U}$ is a shrinking. Hence Lemma 7.12 applies to the restriction

$$\xi|\mathscr{V} \in Z^1_{L^2}(\mathscr{V}, \mathscr{O}):$$

There exist

$$\zeta_0 \in Z^1_{L^2}(\mathscr{U}, \mathscr{O}) \text{ and } \eta_0 \in C^0_{L^2}(\mathscr{W}, \mathscr{O})$$

such that on $\mathscr{W}$

$$\zeta_0 = \xi + \delta\eta_0$$

and

$$max\left\{\|\zeta_0\|_{L^2(\mathscr{U})}, \ \|\eta_0\|_{L^2(\mathscr{W})}\right\} \le C \cdot \|\xi\|_{L^2(\mathscr{V})} = CM$$

The orthogonal decomposition splits

$$\zeta_0 =: \alpha_0 + \sigma_0 \text{ with unique } \alpha_0 \in A \text{ and } \sigma_0 \in S.$$

Then

$$\alpha_0 + \sigma_0 = \xi + \delta\eta_0$$

Our aim is to decrease step by step the error term $\alpha_0 \in A$ by modifying σ_0 to σ and η_0 to η, such that in the limit the error term α vanishes. The idea to decrease the sucessive error terms α_ν: Each step makes a round trip comprising

- the restriction from $\mathscr{U}$ to $\mathscr{V}$
- and the extension from $\mathscr{V}$ to $\mathscr{U}$

The round trip reduces the error term by a factor at least $1/2$.

We will obtain σ and η as convergent series

$$\sigma = \sum_{\nu=0}^{\infty} \sigma_\nu, \ \sigma_\nu \in S \subset Z^1_{L^2}(\mathscr{U}, \mathscr{O}),$$

and

$$\eta = \sum_{\nu=0}^{\infty} \eta_\nu, \ \eta_\nu \in C^0_{L^2}(\mathscr{W}, \mathscr{O}).$$

By induction on $\nu \in \mathbb{N}$ we verify: There exist elements

$$\zeta_\nu \in Z^1_{L^2}(\mathscr{U},\mathscr{O}),\ \eta_\nu \in C^0_{L^2}(\mathscr{W},\mathscr{O}), \alpha_\nu \in A \subset Z^1_{L^2}(\mathscr{U},\mathscr{O}),\ \sigma_\nu \in S \subset Z^1_{L^2}(\mathscr{U},\mathscr{O}),$$

satisfying:

- On $\mathscr{W}$
$$\zeta_\nu = \alpha_{\nu-1} + \delta\eta_\nu,$$
- on $\mathscr{U}$
$$\zeta_\nu = \alpha_\nu + \sigma_\nu$$
- and the estimate
$$max\{\|\zeta_\nu\|_{L^2(\mathscr{U})}, \|\eta_\nu\|_{L^2(\mathscr{W})}\} \leq \frac{1}{2^\nu}\cdot CM$$

Start of induction $\nu = 0$: With
$$\alpha_{-1} := \xi$$
the start of induction has been constructed above.

For the induction step $(\leq \nu) \mapsto \nu+1$:

1. By induction assumption the splitting
$$\zeta_\nu = \alpha_\nu + \sigma_\nu$$
and the estimate for ζ_ν imply the estimate
$$\|\alpha_\nu\|_{L^2(\mathscr{U})} \leq \frac{1}{2^\nu}\cdot CM$$
Lemma 7.13 implies for the restriction of $\alpha_\nu \in A$ to $\mathscr{V}$
$$\|\alpha_\nu\|_{L^2(\mathscr{V})} \leq \varepsilon\cdot\|\alpha_\nu\|_{L^2(\mathscr{U})} \leq \frac{1}{2^{\nu+1}}\cdot M$$
2. Lemma 7.12 applied to
$$\alpha_\nu|\mathscr{V} \in Z^1_{L^2}(\mathscr{V},\mathscr{O})$$
provides elements
$$\zeta_{\nu+1} \in Z^1_{L^2}(\mathscr{U},\mathscr{O}) \text{ and } \eta_{\nu+1} \in C^0_{L^2}(\mathscr{W},\mathscr{O})$$
satisfying with respect to $\mathscr{W}$
$$\zeta_{\nu+1} = \alpha_\nu + \delta\eta_{\nu+1}$$
3. Splitting the extension $\zeta_{\nu+1}$ due to the ortogonal decomposition
$$\zeta_{\nu+1} = \alpha_{\nu+1} + \sigma_{\nu+1} \in A \overset{\perp}{\oplus} S = Z^1_{L^2}(\mathscr{U},\mathscr{O})$$

provides the new error term $\alpha_{\nu+1}$

4. By induction assumption for $j \leq \nu$ all constructed elements satisfy

$$\zeta_j = \alpha_j + \sigma_j,$$

which adds up to

$$\alpha_\nu + \sum_{j=0}^{\nu} \sigma_j = \xi + \delta(\sum_{j=0}^{\nu} \eta_j)$$

The estimate from Lemma 7.12 implies the estimate

$$max\{\|\zeta_{\nu+1}\|_{L^2(\mathcal{U})}, \|\eta_{\nu+1}\|_{L^2(\mathcal{W})}\} \leq C \cdot \|\alpha_\nu\|_{L^2(\mathcal{V})} \leq \frac{1}{2^{\nu+1}} \cdot CM$$

This finishes the induction step.

iii) *Convergence of the solution*: Thanks to the estimate from part ii)

$$max\{\|\zeta_\nu\|_{L^2(\mathcal{U})}, \|\eta_\nu\|_{L^2(\mathcal{W})}\} \leq \frac{1}{2^\nu} \cdot CM$$

and the apparent estimate

$$max\left\{\|\alpha_\nu\|_{L^2(\mathcal{U})}, \|\sigma_\nu\|_{L^2(\mathcal{U})}\right\} \leq \|\zeta_\nu\|_{L^2(\mathcal{U})}$$

the two series

$$\sigma := \sum_{\nu=0}^{\infty} \sigma_\nu \in S \subset Z^1_{L^2}(\mathcal{U},\mathcal{O}) \text{ and } \eta := \sum_{\nu=0}^{\infty} \eta_\nu \in C^0_{L^2}(\mathcal{W},\mathcal{O})$$

are convergent and

$$\lim_{\nu\to\infty} \alpha_\nu = 0.$$

Hence on $\mathcal{W}$

$$\sigma = \xi + \delta\eta, \; q.e.d.$$

7.3 Finiteness of $dim\,H^1(X,\mathcal{O})$ and applications

Due to the preparations referring to square integrable cocycles from Section 7.2 we are now ready to prove the finiteness theorem 7.16.

Proposition 7.15 (Finite-dimensional restriction of cohomology along relatively-compact pairs). *Consider a Riemann surface X and a pair of relatively-compact open subsets*

$$Y_1 \subset\subset Y_2 \subset X.$$

Then the image of the restriction map

$$H^1(Y_2, \mathscr{O}) \to H^1(Y_1, \mathscr{O})$$

has finite dimension.

Proof. Because $\overline{Y}_1$ is compact and $\overline{Y}_1 \subset Y_2$ we can find a finite family of charts on X

$$\phi_i : U_i^* \xrightarrow{\simeq} D_1(0),\ i = 1, ..., n,$$

and families

$$\mathscr{W} << \mathscr{U} << \mathscr{U}^*$$

satisfying:

- $$Y_1 \subset \hat{Y}_1 := \bigcup_{i=1}^{n} W_i \subset\subset \hat{Y}_2 := \bigcup_{i=1}^{n} U_i \subset Y_2.$$
- and for all $i = 1, ..., n$ the sets
$$\phi_i(U_i), \phi_i(W_i) \subset \mathbb{C}$$
are disks.

The coverings $\mathscr{U}$ and $\mathscr{W}$ are Leray covers of respectively $\hat{Y}_2$ and $\hat{Y}_1$ for the structure sheaf $\mathscr{O}$, see Theorem 6.16. Hence Leray's Theorem 6.8 implies

$$H^1(\hat{Y}_2, \mathscr{O}) = H^1(\mathscr{U}, \mathscr{O}) \text{ and } H^1(\hat{Y}_1, \mathscr{O}) = H^1(\mathscr{W}, \mathscr{O}).$$

Proposition 7.14 implies that the restriction

$$H^1(\hat{Y}_2, \mathscr{O}) \to H^1(\hat{Y}_1, \mathscr{O})$$

has finite-dimensional image. The restriction factorizes as

$$H^1(Y_2, \mathscr{O}) \to H^1(\hat{Y}_2, \mathscr{O}) \to H^1(\hat{Y}_1, \mathscr{O}) \to H^1(Y_1, \mathscr{O}).$$

Hence it has finite-dimensional image, q.e.d.

As a corollary to Proposition 7.15 we obtain the fundamental finiteness Theorem 7.16 for compact Riemann surfaces.

Theorem 7.16 (Finiteness). *For a compact Riemann surface X*

$$\dim_{\mathbb{C}} H^1(X, \mathscr{O}) < \infty.$$

Proof. We apply Proposition 7.15 for the special case

$$X = Y_1 = Y_2,\ q.e.d.$$

Definition 7.17 (Genus). The *genus* of a compact Riemann surface X is defined as

$$g(X) := dim_{\mathbb{C}}\ H^1(X,\mathscr{O}).$$

Due to Proposition 6.17 holds

$$g(\mathbb{P}^1) = 0.$$

Proposition 7.18 is the first example of the principle that finiteness of the holomorphic cohomology implies the existenc of a meromorphic object with suitable properties. Here the finitesness of the cohomology of the structure sheaf implies the existence of a meromorphic function with suitable properties.

Proposition 7.18 (Existence of meromorphic functions). *Let X be a Riemann surface and $Y \subset\subset X$ a relatively compact open subset. Then for any point $p \in Y$ exists a meromorphic function $f \in \mathscr{M}(Y)$ with a single pole, located at p.*

Proof. Proposition 7.15 implies that the image of the restriction is finite-dimensional

$$dim\ im[H^1(X,\mathscr{O}) \to H^1(Y,\mathscr{O})] =: k < \infty$$

Consider a chart of X around p

$$z : U_0 \to D_R(0)$$

Setting $U_1 := X \setminus \{p\}$ defines an open covering

$$\mathscr{U} = (U_0, U_1)$$

of X. We consider the commutative diagram of horizontal restrictions with respect to the refinement, and with vertical projections to the inductive limit

$$\begin{array}{ccc} H^1(\mathscr{U},\mathscr{O}) & \longrightarrow & H^1(\mathscr{U}\cap Y,\mathscr{O}) \\ \big\downarrow & & \big\downarrow \pi \\ H^1(X,\mathscr{O}) & \longrightarrow & H^1(Y,\mathscr{O}) \end{array}$$

Lemma 6.4 implies that the map

$$\pi : H^1(\mathscr{U} \cap Y, \mathscr{O}) \to H^1(Y, \mathscr{O})$$

is injective. Hence for the upper horizontal restriction

$$dim\ im[H^1(\mathscr{U}, \mathscr{O}) \to H^1(\mathscr{U} \cap Y, \mathscr{O})] \leq k.$$

On

$$U_0 \cap U_1 = U_0 \setminus \{p\} =: U_0^*$$

the holomorphic functions

$$1/z^j \in \mathscr{O}(U_0^*),\ j = 1, ..., k+1,$$

represent $k+1$ cocycles

$$\zeta_j \in Z^1(\mathscr{U}, \mathscr{O}) \text{ with } (\zeta_j)_{01} = 1/z^j$$

The finiteness condition implies that the classes of the cocycles become linearly dependent when restricted to Y: There exist complex numbers $c_1, ..., c_{k+1}$, not all zero, and a cochain

$$\eta = (f_0, f_1) \in C^0(\mathscr{U} \cap Y, \mathscr{O})$$

such that on $U_0^* \cap Y$

$$\sum_{j=1}^{k+1} c_j \cdot \zeta_j = \delta\eta.$$

As a consequence on $U_0^* \cap Y$ holds

$$\sum_{j=1}^{k+1} c_j \cdot (1/z^j) = f_1 - f_0.$$

The cocycle

$$(f_0 + \sum_{j=1}^{k+1} c_j \cdot (1/z^j), f_1) \in Z^0(\mathscr{U} \cap Y, \mathscr{M})$$

defines a meromorphic function

$$f \in \mathscr{M}(Y)$$

with a single pole at $p \in U_0 \cap Y$, q.e.d.

Note that Proposition 7.18 does not specify the order of the pole at p. The proof only bounds the order by $k+1$. In case of the complex torus

$$X = Y = \mathbb{C}/\Lambda$$

holds the isomorphiy of sheaves

$$\mathscr{O} \simeq \Omega^1.$$

The residue theorem, Theorem 4.22, implies: There is no meromorphic function on the torus with exactly one pole, and this pole having the order = 1.

Corollary 7.19 (Existence of global meromorphic functions). *Let X be a compact Riemann surface. Then for any point $p \in X$ exists a meromorphic function $f \in \mathscr{M}(X)$ with a single pole, located at p. In particular the field*

$$\mathscr{M}(X) \neq \mathbb{C}$$

is an infinite-dimensional complex vector space.

Proof. The proof follows from Proposition 7.18 with $Y := X$, q.e.d.

Chapter 8
Riemann-Roch theorem

The theorem of Riemann-Roch is the fundamental result about the dimension of the cohomology of a distinguished class of sheaves on a compact Riemann surface X. The present chapter studies the theorem for sheaves $\mathscr{O}_D$ attached to a divisor D. The basis for all calculations is the finiteness result for $dim\ H^1(X, \mathscr{O})$ from Chapter 7.

Chapter 9 will refine the Riemann-Roch theorem by Serre's duality theorem. After introducing line bundles L and the corresponding invertible sheaves we then show that any invertible sheaf has the form $\mathscr{O}_D$ for a suitable divisor D on X. Hence both theorems hold for the class of locally free sheaves of rank 1. The most general domain of validity of both theorems is the class of coherent $\mathscr{O}$-modules, which covers in particular all locally free sheaves of arbitrary finite rank. These sheaves arise from vector bundles on X. But we will not cover this case.

8.1 Divisors

A divisor formalizes a set of poles and zeros of a given order for meromorphic functions or differential forms on a Riemann surface X. E.g., a single pole at a point $x \in X$, with order $k \in \mathbb{N}$, is formalized by the map

$$D : X \to \mathbb{Z},\ D(x) = \begin{cases} -k & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

Similarly, an arbitrary set of poles is specified.

Definition 8.1 (Divisor). Consider a Riemann surface X.

1. A *(Weil) divisor* D on an open set $U \subset X$ is a map

$$D : U \to \mathbb{Z}$$

with support

$$supp\ D := \{x \in U :\ D(x) \neq 0\}$$

a discrete set, closed in U. Note. A discrete set $A \subset U$ is closed in U iff it has no accumulation point in U.

The set $Div(U)$ of all divisors on U is in a canonical way an additive Abelian group.

2. A point $p \in X$ defines the *point divisor* $P \in Div(X)$ with

$$P : X \to \mathbb{Z},\ x \mapsto \begin{cases} 1 & x = p \\ 0 & otherwise \end{cases}$$

3. For two divisors $D_1, D_2 \in Div(U)$ one defines

$$D_1 \leq D_2$$

if for all $x \in U$

$$D_1(x) \leq D_2(x).$$

In particular, $D \geq 0$ iff $D(x) \geq 0$ for all $x \in U$.

A divisor D is named *effective* or *non-negative* if $D \geq 0$. Note. Apparently each divisor $D \in Div(U)$ can be written as the difference

$$D = D_1 - D_2$$

of two effective divisors $D_1, D_2 \in Div(U)$.

4. For compact X each divisor $D \in Div(X)$ has finite support $supp\ D$. Hence the *degree* of the divisor

$$deg\ D := \sum_{x \in supp\ D} D(x) \in \mathbb{Z}$$

is well-defined.

Definition 8.2 (Divisor of meromorphic functions and differential forms). Consider a Riemann surface X.

1. The divisor of a meromorphic function $f \in \mathscr{M}^*(X)$, denoted $div\ f$ or (f), is the divisor

$$div\ f : X \to \mathbb{Z},\ (div\ f)(x) := ord(f;\ x)$$

These divisors are named *principal divisors*.

2. Two divisors $D_1, D_2 \in Div(X)$ are *equivalent*, denoted

$$D_1 \sim D_2$$

if $D_1 - D_2$ is a principal divisor. The quotient group by the subgroup of principal divisors

$$Cl(X) := Div(X)/\{D \in \text{Div}(X) : \text{ D principal}\}$$

is named the *divisor class group* of X.

3. For a meromorphic differential form $\omega \in \mathscr{M}^1(X)$, with $\omega|U \neq 0$ for all $U \subset X$ open, the divisor

$$div\ \omega : X \to \mathbb{Z}$$

is defined locally: For a given point $p \in X$ one chooses a chart around p

$$z : U \to V.$$

On U one has the local representation

$$\omega|U = f \cdot dz$$

with a meromorphic function $f \in \mathscr{M}^*(U)$. One defines

$$(div\ \omega)(p) := (div\ f)(p)$$

The definition is independent from the choice of the chart because a holomorphic coordinate transformation does not change the order of a pole.

Note that we do not define the divisor of a meromorphic function or of a differential form if they vanish identically in the neighbourhood of a point. For a compact Riemann surface X Corollary 3.24 implies

$$deg(f) = 0$$

for any meromorphic function $f \in \mathscr{M}^*(X)$. Hence the degree induces a group homomorphism with the same name

$$deg : Cl(X) \to \mathbb{Z}.$$

Every divisor $D \in Div(X)$ on a Riemann surface X singles out a subsheaf

$$\mathscr{O}_D \subset \mathscr{M}$$

of meromorphic functions on X: One considers all meromorphic functions f with

$$(f) \geq -D$$

For an effective divisor D, these are meromorphic functions with poles of no higher order as defined by D. If $D \leq 0$, these are holomorphic functions having zeros at least of the order defined by D.

The sheaves $\mathscr{O}_D$ will play a dominant role in the study of compact Riemann surfaces X. The cohomology of $\mathscr{O}_D$ is the subject matter of the Riemann-Roch theorem in the present chapter and of Serre's duality theorem in Chapter 9. We shall then see that the sheaves $\mathscr{O}_D$ are exactly the invertible sheaves of holomorphic sections on line bundles on X.

Definition 8.3 (The sheaves of multiples of a divisor). Consider a Riemann surface X.

1. For a divisor $D \in Div(X)$ the presheaf of multiples of the divisor $-D$

$$\mathscr{O}_D(U) := \{f \in \mathscr{M}(U) : \; ord(f;x) \geq -D(x) \text{ for all } x \in U\}, \; U \subset X \text{ open},$$

with the canonical restriction of meromorphic functions is an $\mathscr{O}$-module sheaf, denoted $\mathscr{O}_D$. It is named the *sheaf of meromorphic functions which are multiples of* $-D$, for short the *sheaf of multiples of* $-D$.

One defines the $\mathscr{O}$-module sheaf

$$\Omega^1_D := \Omega^1 \otimes_{\mathscr{O}} \mathscr{O}_D.$$

Then

$$\Omega^1_D(U) := \{\omega \in \mathscr{M}^1(U) : \; ord(\omega;\, x) \geq -D(x) \text{ for all } x \in U\}, \; U \subset X \text{ open}.$$

2. For two divisors $D_1, D_2 \in Div(X)$ with

$$D_1 \leq D_2$$

one has a canonical inclusion of sheaves

$$\mathscr{O}_{D_1} \hookrightarrow \mathscr{O}_{D_2}.$$

The quotient sheaf, i.e. the sheafification of the presheaf

$$U \mapsto \mathscr{O}_{D_2}(U)/\mathscr{O}_{D_1}(U), \; U \subset X \text{ open},$$

is denoted

$$\mathscr{H}^{D_2}_{D_1} := \mathscr{O}_{D_2}/\mathscr{O}_{D_1}.$$

It fits into the short exact sequence of sheaf morphisms

$$0 \to \mathscr{O}_{D_1} \to \mathscr{O}_{D_2} \to \mathscr{H}^{D_2}_{D_1} \to 0$$

Note. Concerning the minus sign in Definition of $\mathscr{O}_D$ textbooks are not uniform: The sheaf $\mathscr{O}_D$ equals the sheaf $\mathscr{O}(\mathfrak{d})$ from [16] with

$$\mathfrak{d} = -D.$$

Very useful is the vanishing result from Proposition 8.4.

Proposition 8.4 (Divisors of negative degree). *Consider a compact Riemann surface X and a divisor $D \in Div(X)$ with deg $D < 0$. Then*

$$H^0(X, \mathscr{O}_D) = 0.$$

Proof. The divisor of a non-zero meromorphic function $f \in H^0(X, \mathscr{O}_D)$ satisfies

$$(f) \geq -D > 0$$

As a consequence, the principal divisor (f) has positive degree, a contradiction to Corollary 3.24, which implies $deg\ (f) = 0$, q.e.d.

Definition 8.5 (Sheaf of divisors). Consider a Riemann surface X. The presheaf of additive Abelian groups

$$U \mapsto Div(U),\ U \subset X \text{ open},$$

with the canonical restriction of maps is a sheaf on X, denoted $\mathscr{D}$ and named the *sheaf of divisors*.

Proposition 8.6 (Divisor sequence). *On any Riemann surface X the following sequence of sheaves of Abelian groups is exact*

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \xrightarrow{div} \mathscr{D} \to 0$$

Here the sheaves of Abelian groups $\mathscr{O}^*$ and $\mathscr{M}^*$ are considered multiplicatively, while $\mathscr{D}$ is considered additively.

Proof. The sheaf morphism

$$\mathscr{M}^*(U) \xrightarrow{div_U} \mathscr{D}(U) = Div(U), f \mapsto div\ f,\ U \subset X \text{ open},$$

is well-defined. It is surjective because each divisor - having discrete support - is locally the divisor of a meromorphic function, q.e.d.

The connecting morphism of the divisor sequence from Proposition 8.6

$$\partial : H^0(X,\mathscr{D}) \to H^1(X,\mathscr{O}^*)$$

is obtained as follows: Represent a given divisor

$$D \in Div(X) = H^0(X,\mathscr{D})$$

with respect to a suitable open covering $\mathscr{U} = (U_i)_{i\in I}$ of X by a 0-cochain of meromorphic functions

$$(f_i)_{i\in I} \in C^0(\mathscr{U},\mathscr{M}^*)$$

with

$$D|U_i = div\ f_i,\ i \in I.$$

Then

$$\partial D \in H^1(\mathscr{U},\mathscr{O}^*)$$

is represented by the 1-cocycle of holomorphic functions without zeros

$$g = (g_{ij})_{i,j\in I} \in Z^1(\mathscr{U},\mathscr{O}^*) \text{ with } g_{ij} := \frac{f_j}{f_i}$$

and

$$\partial D = [g] \in H^1(X,\mathscr{O}^*).$$

Proposition 8.7 (Cohomologyy of the sheaf of divisors). *The sheaf $\mathscr{D}$ of divisors on a Riemann surface X satisfies*

$$H^1(X,\mathscr{D}) = 0$$

Proof. i) *Integer valued partition of unity*: Consider an arbitrary open covering $\mathscr{U}$ of X. Second countability of X implies the existence of a countable refinement

$$\mathscr{V} = (V_n)_{n\in\mathbb{N}} < \mathscr{U}$$

Proposition 4.19 implies the existence of a shrinking

$$\mathscr{W} << \mathscr{V}$$

and a partition of unity $(\phi_n)_{n\in\mathbb{N}}$ subordinate to $\mathscr{W} = (W_n)_{n\in\mathbb{N}}$, i.e. satisfying

$$supp(\phi_n) \subset W_n$$

For each $x \in X$:

- For at least one $n \in \mathbb{N}$ holds $x \in W_n$

- There exists an open neighbourhood W of x with
$$W \cap W_n \neq \emptyset$$
for only finitely many $n \in \mathbb{N}$

For each $n \in \mathbb{N}$ define
$$\phi_n : X \to \mathbb{Z},\ \phi_n(x) := \begin{cases} 1 & x \in W_n, \text{ but } x \notin W_k \text{ for } k < n \\ 0 & \text{otherwise} \end{cases}$$
For each $x \in X$ we have
$$\sum_{k \in \mathbb{Z}} \phi_k(x) = \phi_n(x) = 1,\ x \in W_n \text{ but } x \notin W_k \text{ for } k < n$$
Then $(\phi_n)_{n \in \mathbb{N}}$ is an integer valued partition of unity subordinate to $\mathscr{W}$.

ii) *Splitting by means of an integer valued partition of unity*: The proof is similar to the proof of Theorem 6.14. For a given open covering $\mathscr{U}$ we choose a countable, locally-finite refinement
$$\mathscr{V} = (V_i)_{i \in \mathbb{Z}} < \mathscr{U}$$
with a subordinate integer valued partition of unity $(\phi_i)_{i \in \mathbb{Z}}$, see part i). Consider a cocycle
$$f = (f_{ij})_{ij} \in Z^1(\mathscr{V}, \mathscr{D})$$
Choose an arbitrary but fixed index $i \in \mathbb{Z}$. For each $k \in \mathbb{Z}$ the product
$$\phi_k \cdot f_{ki} : V_i \cap V_k \to \mathbb{Z}$$
has support in $V_k \cap V_i$ and extends to a function
$$\tilde{f}_{ki} : V_i \to \mathbb{Z},$$
i.e.
$$\tilde{f}_{ki} \in \mathscr{D}(V_i).$$
The sum
$$F_i := \sum_{k \in \mathbb{Z}} \tilde{f}_{ki} \in \mathscr{D}(V_i)$$
is well-defined. On $V_i \cap V_j$ we have
$$F_j - F_i = \sum_{k \in \mathbb{Z}} \tilde{f}_{kj} - \sum_{k \in \mathbb{Z}} \tilde{f}_{ki} = \sum_{k \in \mathbb{Z}} \phi_k \cdot (f_{kj} - f_{ki}) = -\sum_{k \in \mathbb{Z}} \phi_k \cdot f_{ji} = f_{ij} \cdot \sum_{k \in \mathbb{Z}} \phi_k = f_{ij}$$
Hence
$$f = \delta F$$

with the cochain

$$F := (F_k)_{k\in\mathbb{Z}} \in C^0(\mathscr{V},\mathscr{D}),\ q.e.d.$$

8.2 The Euler characteristic of the sheaves $\mathscr{O}_D$

The hard part of the proof of the Riemann-Roch theorem for the sheaves $\mathscr{O}_D$ on compact Riemann surfaces is the finiteness theorem which has been proved in Theorem 7.16. The subsequent computation of the Euler characteristic of the sheaf $\mathscr{O}_D$, attached to a divisor $D \in Div(X)$, is a simple reduction. It starts with the consideration of effective divisors.

The sheaf $\mathscr{H}_{D_1}^{D_2}$ is the means to compare the cohomology of two divisors $D_1 \le D_2$. Lemma 8.8 states the cohomological properties $\mathscr{H}_{D_1}^{D_2}$. Together with Lemma 8.9 it prepares the proof of Theorem 8.10.

Lemma 8.8 (Comparing two divisors). *Let X be a Riemann surface and $D_1, D_2 \in Div(X)$ two divisors with*

$$D_1 \le D_2.$$

1. Then

$$H^1(X,\mathscr{H}_{D_1}^{D_2}) = 0$$

2. For compact X holds

$$dim_{\mathbb{C}}\, H^0(X,\mathscr{H}_{D_1}^{D_2}) = deg\ D_2 - deg\ D_1.$$

Proof. The set

$$S := \{x \in X : D_1(x) \ne D_2(x)\}$$

is a discrete set, closed in X.

1. A given class from $H^1(X,\mathscr{H}_{D_1}^{D_2})$ can be represented by a cocycle from

$$Z^1(\mathscr{U},\mathscr{H}_{D_1}^{D_2})$$

with a suitable open covering $\mathscr{U}$ of X. We choose a refinement

$$\mathscr{V} = (V_i)_{i\in I} < \mathscr{U}$$

such that each point $s \in S$ is contained in V_i for exactly one index $i \in I$. As a consequence, for $i \ne j \in I$

$$\mathscr{H}_{D_1}^{D_2}(V_i \cap V_j) = 0$$

which imples
$$Z^1(\mathscr{V}, \mathscr{H}_{D_1}^{D_2}) = 0.$$
We obtain for the inductive limit
$$H^1(X, \mathscr{H}_{D_1}^{D_2}) = 0.$$

2. For compact X the set S is finite and
$$H^0(X, \mathscr{H}_{D_1}^{D_2}) = \prod_{s \in S} (\mathscr{O}_{D_2})_s / (\mathscr{O}_{D_1})_s$$
Hence
$$dim_{\mathbb{C}}\ H^0(X, \mathscr{H}_{D_1}^{D_2}) = \sum_{s \in S} dim_{\mathbb{C}}\ (\mathscr{O}_{D_2})_s / (\mathscr{O}_{D_1})_s =$$
$$= \sum_{s \in S} (D_2(s) - D_1(s)) = deg\ D_2 - deg\ D_1.$$
Here we used that the quotient of stalks
$$(\mathscr{O}_{D_2})_s / (\mathscr{O}_{D_1})_s$$
is isomorphic to the space of all Laurent series of the form
$$\sum_{n=-D_2(s)}^{-D_1(s)-1} c_n \cdot z^n.$$
The latter is a vector space with the finite dimension
$$D_2(s) - D_1(s),\ q.e.d.$$

Lemma 8.9 (Comparing the cohomology of the multiples of two divisors). *Consider a Riemann surface X and two divisors*
$$D_1 \leq D_2$$
on X. Then the inclusion
$$\mathscr{O}_{D_1} \hookrightarrow \mathscr{O}_{D_2}$$
induces a surjective morphism
$$H^1(X, \mathscr{O}_{D_1}) \to H^1(X, \mathscr{O}_{D_2})$$
with injective dual
$$i_{D_1}^{D_2} : H^1(X, \mathscr{O}_{D_2})^{\vee} \to H^1(X, \mathscr{O}_{D_1})^{\vee}.$$

Proof. The claim follows from the long exact cohomology sequence

$$0 \to H^0(X,\mathscr{O}_{D_1}) \to H^0(X,\mathscr{O}_{D_2}) \to H^0(X,\mathscr{H}_{D_1}^{D_2}) \xrightarrow{\partial}$$

$$H^1(X,\mathscr{O}_{D_1}) \to H^1(X,\mathscr{O}_{D_2}) \to 0 = H^1(X,\mathscr{H}_{D_1}^{D_2})$$

of the sheaf sequence from Definition 8.3 and the application of Lemma 8.8, q.e.d.

Theorem 8.10 (Riemann Roch theorem for the sheaves $\mathscr{O}_D$). *Consider a compact Riemann surface X with genus $g(X)$ and a divisor $D \in Div(X)$. Then:*

1. *The complex vector spaces*

$$H^0(X,\mathscr{O}_D) \text{ and } H^1(X,\mathscr{O}_D)$$

are finite-dimensional.

2. *The* Euler characteristic *of $\mathscr{O}_D$*

$$\chi(\mathscr{O}_D) := dim\, H^0(X,\mathscr{O}_D) - dim\, H^1(X,\mathscr{O}_D)$$

has the numerical value

$$\chi(\mathscr{O}_D) = 1 - g(X) + deg\, D \in \mathbb{Z}$$

Proof. The proof rests on the Finiteness Theorem 7.16 for the zero divisor $D=0$ with its multiple the structure sheaf $\mathscr{O} = \mathscr{O}_D$. The long exact cohomology sequence reduces the case of a general divisor to the specific case $D=0$.

i) *Effective divisor $D \geq 0$*: Lemma 8.9 implies

$$0 \to H^0(X,\mathscr{O}) \to H^0(X,\mathscr{O}_D) \to H^0(X,\mathscr{H}_0^D) \to H^1(X,\mathscr{O}) \to H^1(X,\mathscr{O}_D) \to 0$$

Due to the compactness of X Theorem 1.9 implies

$$dim\, H^0(X,\mathscr{O}) = 1,$$

and Theorem 7.16 implies

$$g(X) := dim\, H^1(X,\mathscr{O}) < \infty$$

Lemma 8.8 implies

$$dim\, H^0(X,\mathscr{H}_0^D) = deg\, D < \infty$$

As a consequence, the long exact sequence above implies

$$dim\, H^1(X,\mathscr{O}_D) < \infty$$

and eventually

$$dim\, H^0(X,\mathscr{O}_D) < \infty$$

Computing the alternate cross sum of the dimension of the finite dimensional vector spaces of the exact sequence gives

$$0 = 1 - dim\, H^0(X, \mathscr{O}_D) + deg\, D - g(X) + dim\, H^1(X, \mathscr{O}_D)$$

or

$$\chi(\mathscr{O}_D) = 1 - g(X) + deg\, D$$

ii) *General case* $D \in Div(X)$: We decompose D as the difference of two effective divisors

$$D = D_1 - D_2,\ D_1, D_2 \geq 0.$$

Then $D \leq D_1$. Lemma 8.9 implies the exact sequence

$$0 \to H^0(X, \mathscr{O}_D) \to H^0(X, \mathscr{O}_{D_1}) \to H^0(X, \mathscr{H}_D^{D_1}) \to H^1(X, \mathscr{O}_D) \to H^1(X, \mathscr{O}_{D_1}) \to 0$$

Part i) and Lemma 8.8 imply

$$dim\, H^0(X, \mathscr{O}_D),\ dim H^1(X, \mathscr{O}_D),\ dim\, H^0(X, \mathscr{H}_D^{D_1}) < \infty.$$

Therefore also

$$dim\, H^0(X, \mathscr{O}_{D_1}),\ dim\, H^1(X, \mathscr{O}_{D_1}) < \infty$$

Computing the alternate cross sum gives

$$0 = dim\, H^0(X, \mathscr{O}_D) - \underline{dim\, H^0(X, \mathscr{O}_{D_1})} +$$

$$+ (deg\, D_1 - deg\, D) - dim\, H^1(X, \mathscr{O}_D) + \underline{dim\, H^1(X, \mathscr{O}_{D_1})}$$

Due to part i) applied to D_1

$$dim\, H^0(X, \mathscr{O}_{D_1}) - dim\, H^1(X, \mathscr{O}_{D_1}) = 1 - g(X) + deg\, D_1$$

As a consequence

$$0 = dim\, H^0(X, \mathscr{O}_D) - 1 + g(X) - deg\, D_1 + (deg\, D_1 - deg\, D) - dim\, H^1(X, \mathscr{O}_D)$$

or

$$\chi(\mathscr{O}_D) = 1 - g(X) + deg\, D,\ q.e.d.$$

Theorem 8.10 shows: The value

$$\chi(\mathscr{O}_D) - deg\, D = 1 - g(X)$$

is an invariant of the Riemann surface X, independent from the divisor D.

Chapter 9
Serre duality

The Riemann-Roch theorem on a Riemann surface X computes the holomorphic Euler characteristic

$$\chi(\mathscr{O}_D) = dim\, H^0(X, \mathscr{O}_D) - dim\, H^1(X, \mathscr{O}_D).$$

Serres duality theorem replaces the first cohomology group

$$dim\, H^1(X, \mathscr{O}_D)$$

by the 0-th cohomology group

$$dim\, H^0(X, \omega_{-D})$$

with

$$\omega_{-D} = \mathscr{O}_{-D} \otimes_{\mathscr{O}} \omega$$

and ω a distinguished sheaf on X. Recall as a particular case Theorem 6.18 about the twisted sheaves

$$\mathscr{L} = \mathscr{O}(k)$$

on $X = \mathbb{P}^1$:

$$dim\, H^1(\mathbb{P}^1, \mathscr{L}) = dim\, H^0(\mathbb{P}^1, \mathscr{L}^\vee \otimes_{\mathscr{O}} \Omega^1).$$

9.1 Dualizing sheaf and residue map

The theorem is based on the dualizing sheaf ω and its residue map defined on $H^1(X, \omega)$. To define the residue map we have to consider also meromorphic differential forms. Therefore we first consider the cohomology of meromorphic functions and differential forms.

Theorem 9.1 (Cohomology of the sheaves $\mathscr{M}$ and $\mathscr{M}^1$). *For a compact Riemann surface X*

$$H^1(X,\mathscr{M}) = H^1(X,\mathscr{M}^1) = 0.$$

Proof. i) *Meromorphic functions*: A given class $\xi \in H^1(X,\mathscr{M})$ is represented with respect to a suitable open covering $\mathscr{U} = (U_i)_{i\in I}$ by a cocycle

$$(f_{ij})_{ij} \in Z^1(\mathscr{U},\mathscr{M}).$$

After shrinking $\mathscr{U}$ we may assume that for each $i, j \in I$ the meromorphic function

$$f_{ij} \in \mathscr{M}(U_i \cap U_j)$$

has only finitely many poles. We choose a refinement

$$\mathscr{V} = (V_\alpha)_{\alpha\in A} < \mathscr{U}$$

with the refinement map

$$\tau : A \to I$$

such that for each pair $i,\ j \in I$ each pole of

$$f_{ij} \in \mathscr{M}(U_i \cap U_j)$$

is contained in V_α for exactly one $\alpha \in A$. As a consequence

$$f_{\tau(\alpha)\tau(\beta)}|V_\alpha \cap V_\beta$$

has no poles, i.e.

$$(f_{ij})_{ij} \in Z^1(\mathscr{U},\mathscr{M})$$

is mapped under the restriction

$$t^{\mathscr{U}}_{\mathscr{V}} : Z^1(\mathscr{U},\mathscr{M}) \to Z^1(\mathscr{V},\mathscr{M})$$

to the subspace of holomorphic cocycles

$$Z^1(\mathscr{V},\mathscr{O}) \subset Z^1(\mathscr{V},\mathscr{M}).$$

As a consequence, the embedding

$$\mathscr{O} \hookrightarrow \mathscr{M}$$

induces in the direct limit a surjective map

$$H^1(X,\mathscr{O}) \to H^1(X,\mathscr{M})$$

The finiteness of $dim_{\mathbb{C}}\, H^1(X,\mathscr{O})$ implies

$$dim_{\mathbb{C}}\, H^1(X,\mathscr{M}) < \infty.$$

Because is $H^1(X,\mathscr{M})$ is also vector space over the field $H^0(X,\mathscr{M})$, and the latter is an infinite-dimensional $\mathbb{C}$-vector space due to Corollary 7.19, we conclude

$$H^1(X,\mathscr{M}) = 0.$$

ii) *Meromorphic differential forms*: According to Corollary 7.19 we may choose a non-zero global, meromorphic function $f \in \mathscr{M}(X)$ and set

$$\eta := df \in \mathscr{M}^1(X).$$

The map

$$\mathscr{M} \to \mathscr{M}^1,\ g \mapsto g \cdot \eta,$$

is an isomorphism of sheaves with inverse

$$\mathscr{M}^1 \to \mathscr{M},\ \zeta \mapsto \zeta/\eta.$$

Here the quotient

$$q := \zeta/\eta \in H^0(X,\mathscr{M})$$

has to be computed locally with respect to a chart

$$z : U \to V$$

If

$$\eta = f_\eta \cdot dz \text{ and } \zeta = f_\zeta \cdot dz$$

then

$$q|U := \frac{f_\zeta}{f_\eta} \in \mathscr{M}(U).$$

Hence

$$H^1(X,\mathscr{M}) \simeq H^1(X,\mathscr{M}^1)$$

with the first group vanishing due to part i), q.e.d.

Lemma 9.2 is a consequence of the Riemann-Roch theorem. It provides non-zero sections for divisors with large degree.

Lemma 9.2 (Growth of $dim\ H^0(X,\Omega^1_D)$**).** *For each compact Riemann surface X exists a numerical constant* $k_0 \in \mathbb{Z}$ *such that for all divisors* $D \in Div(X)$

$$dim\ H^0(X,\Omega^1_D) \geq k_0 + deg\ D.$$

Proof. We choose a non-constant global, meromorphic function $f \in \mathscr{M}(X)$ according to Corollary 7.19 and set

$$\eta := df \in \mathscr{M}^1(X).$$

Denote by

$$K := div\ \eta \in Div(X)$$

the divisor of η and set

$$k_0 := 1 - g(X) + deg\ K$$

The map

$$\mathscr{O}_{D+K} \to \Omega^1_D,\ f \mapsto f \cdot \eta,$$

is an isomorphism of sheaves. Theorem 8.10 implies

$$dim\ H^0(X, \Omega^1_D) = dim\ H^0(X, \mathscr{O}_{D+K}) =$$

$$= dim\ H^1(X, \mathscr{O}_{D+K}) + 1 - g(X) + deg(D+K) \geq deg\ D + k_0,\ q.e.d.$$

Definition 9.3 (Dualizing sheaf). Let X be a Riemann surface. The sheaf

$$\omega := \Omega^1$$

of holomorphic differential forms is the *dualizing sheaf* of X.

The next step is to define on a compact Riemann surface X a residue map

$$res : H^1(X, \omega) \to \mathbb{C}.$$

Therefore we relate elements from $H^1(X, \omega)$ to meromorphic differential forms and consider the residue of these forms. The construction is based on the following results:

- The injection
$$\Omega^1 \hookrightarrow \mathscr{M}^1,$$
- the vanishing
$$H^1(X, \mathscr{M}^1) = 0$$
- and the residue theorem applied to *Mittag-Leffler distributions of differential forms*.

The Mittag-Leffler problem from complex analysis in the plane asks for meromorphic functions with given principal parts on a domain $G \subset \mathbb{C}$. It is well-known that the problem is solvable for $G = \mathbb{C}$. The concept of a principal part does not carry over literally to a Riemann surface X because the Laurent expansion of a meromophic function at a pole depends on the choice of a chart around the pole. To obtain on X an absolute notion of the principal part one has to replace meromophic functions by meromorphic differential forms. The formal means is the Mittag-Leffler distribution of differential forms.

Definition 9.4 (Mittag-Leffler distribution of differential forms). Let X be a Riemann surface.

1. Consider an open covering $\mathscr{U}$ of X and a meromorphic 0-cochain $\mu \in C^0(\mathscr{U}, \mathscr{M}^1)$ with holomorphic coboundary
$$\delta\mu = (\mu_j - \mu_i)_{ij} \in Z^1(\mathscr{U}, \omega).$$
For a given point $p \in X$ the *residue* of μ at p, defined as
$$res(\mu;\ p) := res(\mu_i;\ p) \text{ for } i \in I \text{ with } p \in U_i,$$
is independent from $i \in I$ because the coboundary $\delta\mu$ is holomorphic.

2. A *Mittag-Leffler distribution of differential forms* on X is a pair
$$ML = (\mathscr{U}, \mu)$$
with an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X and a meromorphic 0-cochain
$$\mu \in C^0(\mathscr{U}, \mathscr{M}^1)$$
with holomorphic coboundary
$$\delta\mu \in Z^1(\mathscr{U}, \omega).$$
For any point $p \in X$ the *residue* of the Mittag-Leffler distribution at p is defined as
$$res(ML; p) := res(\mu;\ p).$$

3. For compact X a Mittag-Leffler distribution ML has only finitely many poles, because the poles of a meromorphic function have no accumulation point. In particular, there are only finitely many points $p \in X$ with
$$res(ML;\ p) \neq 0.$$
The complex number
$$res\ ML := \sum_{p \in X} res(ML;\ p) \in \mathbb{C}$$
is the *residue of the Mittag-Leffler distribution ML.*

Definition 9.5 (Residue map). Let X be a compact Riemann surface. Then define the *residue map* on X
$$res : H^1(X, \omega) \to \mathbb{C}$$
as follows: Any class

$$\eta \in H^1(X,\omega)$$

can be represented by a cocycle

$$(\eta_{ij})_{ij} \in Z^1(\mathscr{U},\omega)$$

with respect to a suitable open covering $\mathscr{U} = (U_i)_{i\in I}$ of X. Because Theorem 9.1 implies

$$H^1(\mathscr{U},\mathscr{M}^1) = 0,$$

the injection

$$\omega = \Omega^1 \hookrightarrow \mathscr{M}^1$$

provides a Mittag-Leffler distribution

$$ML(\eta) = (\mathscr{U},\mu),\ \mu \in C^0(\mathscr{U},\mathscr{M}^1),$$

satisfying

$$\delta\mu = (\eta_{ij})_{ij} \in Z^1(\mathscr{U},\omega)$$

in particular

$$[\delta\mu] = \eta \in H^1(X,\omega).$$

Set

$$res\ \eta := res\ ML(\eta).$$

Lemma 9.6 (Independence of the residue map). *On a compact Riemann surface X the value $res(\eta) \in \mathbb{C}$ of the residue map in Definition 9.5 is independent from the choice of the Mittag-Leffler distribution $ML(\eta)$.*

Proof. The proof is based on the residue theorem.

Assume that a given element $\eta \in H^1(X,\omega)$ is represented by two cocyles, possibly with respect to two different open coverings. Passing to a common refinement $\mathscr{U}$ of the open coverings we may assume for $k = 1,2$ two representatives of η

$$\eta^k \in C^1(\mathscr{U},\mathscr{M}^1)$$

and Mittag-Leffler distributions

$$(\mathscr{U},\ \mu^k) \text{ with } \delta\mu^k = \eta^k.$$

The meromorphic cochain

$$\mu := \mu^1 - \mu^2 \in C^0(\mathscr{U},\mathscr{M}^1)$$

satisfies

$$[\delta\mu] = [\delta\mu^1 - \delta\mu^2] = [\eta^1 - \eta^2] = \eta - \eta = 0 \in H^1(\mathscr{U},\omega).$$

Hence there exists a holomorphic cochain

$$\sigma \in C^0(\mathscr{U}, \omega) \text{ with } \delta\mu = \delta\sigma \in B^1(\mathscr{U}, \omega).$$

First, the holomorphy of σ allows to compute the residue as

$$res(\mu) = res(\mu - \sigma).$$

Secondly,

$$\delta(\mu - \sigma) = 0$$

shows that the cochain

$$\mu - \sigma \in C^0(\mathscr{U}, \mathscr{M}^1)$$

is even a 0-cocyle, i.e. a global meromorphic form

$$\mu - \sigma \in Z^0(\mathscr{U}, \mathscr{M}^1) = \mathscr{M}^1(X).$$

The Residue Theorem 4.22 implies

$$0 = res(\mu - \sigma) = res(\mu) \text{ i.e. } res(\mu^1) = res(\mu^2),\ q.e.d.$$

Definition 9.5 defines the residue map

$$res : H^1(X, \omega) \to \mathbb{C}$$

in a form which at once explains the name: The final value $res(\eta)$ is the sum of finitely many *local* values, which derive as the residues from the singularities of the meromorphic representation of the 1-class $\eta \in H^1(X, \omega)$. This form of $res(\eta)$ will be used in Section 9.2 for the investigation of Serre duality.
There is a second representation of $res(\eta)$, this time obtained by integrating a *global* object, the smooth Dolbeault class of η. Proposition 9.7 proves the equivalence of both representations. This result shows at once by applying Stokes' theorem that the residue as defined by Definition 9.5 is independent from the choice of a Mittag-Leffler distribution. Hence Proposition 9.7 is also a substitute for Lemma 9.6.

The main tool in the proof of Proposition 9.7 is again the residue theorem.

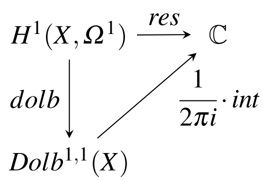
Proposition 9.7 (Residue map via integration of Dolbeault class). *Consider a compact Riemann surface X and the Dolbeault isomorphism from Theorem 6.15*

$$dolb : H^1(X, \Omega^1) \xrightarrow{\simeq} Dolb^{1,1}(X) = \frac{H^0(X, \mathscr{E}^{1,1})}{im[d'' : H^0(X, \mathscr{E}^{1,0}) \to H^0(X, \mathscr{E}^{1,1})]}$$

and the integration

$$int : Dolb^{1,1} \to \mathbb{C},\ [\zeta] \mapsto \iint_X \zeta$$

Then the residue map renders commutative the following diagram



Note that the integral in Proposition 9.7 is well-defined: Due to Stokes' theorem it depends only on the class $[\zeta]$.

Proof. The proof uses the Dolbeault resolution from Theorem 6.15

$$0 \to \Omega^1 \to \mathscr{E}^{1,0} \xrightarrow{d''} \mathscr{E}^{1,1} \to 0$$

of the sheaf Ω^1 and the connecting morphism

$$H^0(X,\mathscr{E}^{1,1}) \xrightarrow{\partial} H^1(X,\Omega^1) \to 0$$

of the corresponding long exact coholomology sequence. Consider an element

$$\eta \in H^1(X,\Omega^1)$$

represented with respect to a suitable open covering $\mathscr{U}$ of X by a cocycle

$$(\eta_{ij})_{ij} \in Z^1(\mathscr{U},\Omega^1).$$

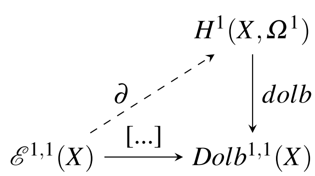
i) *Connecting morphism as lift of Dolbeault morphism*: Due to the definition of the connecting morphism any $\zeta \in H^0(X,\mathscr{E}^{1,1})$ with

$$dolb(\eta) = [\zeta] \in Dolb^{1,1}(X)$$

satisfies

$$\partial\zeta = \eta$$

and vice versa.



ii) *Constructing an inverse image of the connecting morphism*: Due to

$$H^1(X,\mathscr{E}^{1,0}) = 0$$

exists a smooth cochain

$$\sigma = (\sigma_i)_{i\in I} \in C^0(\mathscr{U}, \mathscr{E}^{1,0})$$

satisfying for each $i, j \in I$

$$\eta_{ij} = \sigma_j - \sigma_i.$$

Because

$$d\eta_{ij} = d''\eta_{ij} = 0$$

we have

$$d\sigma_j = d\sigma_i.$$

Hence there exists a global form

$$\zeta \in H^0(X, \mathscr{E}^{1,1})$$

satisfying for each $i \in I$

$$\zeta | U_i = d\sigma_i.$$

iii) *A Mittag-Leffler distribution of* η: Choose a Mittag-Leffler distribution $ML(\eta)$ of the given class η

$$ML(\eta) = (\mathscr{U}, \mu)$$

The 0-cochain

$$\mu = (\mu_i)_i \in C^0(\mathscr{U}, \mathscr{M}^1)$$

satisfies for $i, j \in I$

$$\mu_j - \mu_i = \eta_{ij}.$$

We study the finite pole set of μ

$$P = \{a_1, ..., a_n\} \subset X$$

and the residues at the different poles. Set

$$X' := X \setminus P.$$

For each $i, j \in I$ we have on $U_i \cap U_j \cap X'$

$$\mu_j - \mu_i = \eta_{ij} = \sigma_j - \sigma_i$$

or

$$\sigma_i - \mu_i = \sigma_j - \mu_j.$$

Hence a smooth global form

$$\rho \in \mathscr{E}^{1,0}(X')$$

exists, which satisfies for each $i \in I$ on $X' \cap U_i$

$$\rho = \sigma_i - \mu_i.$$

Hence for each $i \in I$

$$\zeta|X' \cap U_i = d(\sigma_i - \mu_i) = d\sigma_i \text{ because } d(\mu_i|X' \cap U_i) = 0,$$

or globally in X'

$$\zeta = d\rho.$$

For each $k = 1,..,n$ we now make a local study around the pole $a_k \in P$. There exists a chart of X around a_k

$$z_k : W_k \to D_1(0) \text{ the unit disk,}$$

and we may assume an index $i(k) \in I$ with $V_k \subset U_{i(k)}$. Moreover for $j \neq k$

$$V_j \cap V_k = \emptyset.$$

We choose a smooth function

$$\psi_k \in \mathscr{E}(X)$$

with

$$supp\ \psi_k \subset V_k \text{ and } \psi|V_k' = 1$$

for an open neighbourhood $V_k' \subset\subset V_k$ of a_k. Set

$$g := 1 - \sum_{k=1}^{n} \psi_k \in \mathscr{E}(X).$$

Then

$$g \cdot \rho \in \mathscr{E}^{1,0}(X')$$

vanishes in a neighbourhood of each point from P and extends by zero into the pole. Therefore it can be considered as a global smooth form

$$g \cdot \rho \in \mathscr{E}^{1,0}(X)$$

and Stokes's theorem implies

$$\iint_X d(g \cdot \rho) = 0$$

For each $k = 1,..,n$ the restriction on $V_k' \setminus \{a_k\}$

$$d(\psi_k \cdot \rho) = d\rho = d\sigma_k = d\sigma_{i(k)} - \mu_{i(k)} = d\sigma_{i(k)}$$

extends to a smooth form on V_k. Due to the vanishing of

$$\psi_k \cdot \rho \text{ on } X' \setminus supp\ \psi_k$$

the form extends even to a smooth global 2-form

$$d(\psi_k \cdot \rho) \in \mathscr{E}^2(X).$$

We have

$$\rho = g \cdot \rho + \sum_{k=1}^{n} \psi_k \cdot \rho \text{ and } \zeta = d\sigma$$

hence

$$\zeta = d\sigma = d(g \cdot \rho) + \sum_{k=1}^{n} d(\psi_k \cdot \rho)$$

As a consequence by using Stokes' theorem

$$(int \circ dolb)(\eta) = \iint_X \zeta = \sum_{k=1}^{n} \left(\iint_X d(\psi_k \cdot \rho) \right) = \sum_{k=1}^{n} \left(\iint_{V_k} d(\psi_k \cdot \sigma_{i(k)}) - \psi_k \cdot \mu_{i(k)} \right)$$

For each $k = 1, ..., n$ Stokes theorem implies for the first summand

$$\iint_{V_k} d(\psi_k \cdot \sigma_{i(k)}) = 0.$$

While for the second summand the residue theorem in the complex plane gives

$$\frac{1}{2\pi i} \cdot (int \circ dolb)(\eta) = -\frac{1}{2\pi i} \cdot \sum_{k=1}^{n} \iint_{V_k} \psi_k \cdot \mu_{i(k)} = \sum_{k=1}^{n} res(\psi_k \cdot \mu_{i(k)};\ a_k) =$$

$$= \sum_{k=1}^{n} res(\mu_{i(k)};\ a_k) = res\ ML(\eta),\ q.e.d.$$

9.2 The dual pairing of the residue form

Definition 9.8 (Residue form). Let X be a Riemann surface and $D \in Div(X)$ a divisor. Consider the sheaf morphism

$$\omega_{-D} \times \mathscr{O}_D \simeq (\omega \otimes_{\mathscr{O}} \mathscr{O}_{-D}) \times \mathscr{O}_D \to \omega$$

defined for small $U \subset X$ open as multiplication

$$\omega_{-D}(U) \times \mathscr{O}_D(U) \to \omega(U),\ (\eta \otimes g, h) \mapsto (gh) \cdot \eta$$

The sheaf morphism induces a bilinear map

$$H^0(X, \omega_{-D}) \times H^1(X, \mathscr{O}_D) \to H^1(X, \omega).$$

It is defined in Čech cohomology with respect to an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X as

$$H^0(\mathscr{U}, \omega_{-D}) \times H^1(\mathscr{U}, \mathscr{O}_D) \to H^1(\mathscr{U}, \omega)$$
$$(\zeta, (f_{ij})_{ij}) \mapsto (\zeta \cdot f_{ij})_{ij}$$

Its composition with the residue map from Definition 9.5 defines the bilinear *residue form*

$$(-,-)_D := [H^0(X, \omega_{-D}) \times H^1(X, \mathscr{O}_D) \to H^1(X, \omega) \xrightarrow{res} \mathbb{C}]$$

Remark 9.9 (Dual pairing). Consider a field $\mathbb{K}$ and two finite-dimensional $\mathbb{K}$-vector spaces V, W with a bilinear map

$$r : V \times W \to \mathbb{K}.$$

Then the following properties are equivalent:

- The map r is a *dual pairing*, i.e.
  - Injectivity of the map
  $$V \to W^\vee,\ \mathrm{v} \mapsto r(\mathrm{v}, -),$$
  - and injectivity of the map
  $$W \to V^\vee,\ w \mapsto r(-, w),$$
- The induced map
  $$i_V : V \to W^\vee,\ v \mapsto r(v, -),$$
  is an isomorphism.

Proof. We have the equivalences

$$(r(\mathrm{v}, -) = 0 \implies \mathrm{v} = 0) \iff (V \to W^\vee,\ \mathrm{v} \mapsto r(\mathrm{v}, -),\ \text{injective})$$

and

$$(r(-, w) = 0 \implies w = 0)$$
$$\iff (W \to V^\vee,\ w \mapsto r(-, w),\ \text{injective})$$
$$\iff (V \to W^\vee,\ \mathrm{v} \mapsto r(\mathrm{v}, -),\ \text{surjective}),\ q.e.d.$$

In the following we apply Remark 9.9 with

$$V = H^0(X, \omega_{-D}) \text{ and } W = H^1(X, \mathscr{O}_D)$$

and r the residue form.

Theorem 9.10 (Serre duality). *Let X be a compact Riemann surface and $D \in Div(X)$ a divisor. The residue form*

$$(-,-)_D := [H^0(X, \omega_{-D}) \times H^1(X, \mathscr{O}_D) \to H^1(X, \omega) \xrightarrow{res} \mathbb{C}]$$

is a dual pairing. In particular, it induces an isomorphism with the dual space

$$H^0(X, \omega_{-D}) \simeq H^1(X, \mathscr{O}_D)^\vee$$

and hence in particular

$$dim\ H^0(X, \omega_{-D}) = dim\ H^1(X, \mathscr{O}_D).$$

The proof of Theorem 9.10 has to show that the map

$$(.,-)_D : H^0(X, \omega_D) \to H^1(X, \mathscr{O}_D)^\vee\ \eta \mapsto (\eta, -)_D,$$

is an isomorphism. Injectivity of i_V will be proved in Proposition 9.11, surjectivity in Proposition 9.13.

Proposition 9.11 (Residue form: Injectivity). *Consider a compact Riemann surface X and a divisor $D \in Div(X)$. Then the linear map*

$$(.,-)_D : H^0(X, \omega_{-D}) \to H^1(X, \mathscr{O}_D)^\vee, \eta \mapsto (\eta, -)_D,$$

is injective. In particular

$$dim\ H^0(X, \omega_{-D}) < \infty$$

Proof. That $H^0(X, \omega_{-D})$ has finite dimension follows from Theorem 8.10 as soon as the claim of injectivity has been proved. Therefore we have to show: For each

$$\eta \in H^0(X, \omega_{-D}),\ \eta \neq 0,$$

exists

$$\xi \in H^1(X, \mathscr{O}_D)$$

such that the residue form evaluates to

$$(\eta, \xi)_D \neq 0.$$

i) *Construction of* ξ: We construct an element $\xi \in H^1(X, \mathscr{O}_D)$ such that the product

$$\eta\xi \in H^1(X, \omega)$$

has a Mittag-Leffler distribution

$$ML(\eta\xi) = (\mathscr{U}, \mu)$$

satisfying

$$(\eta,\xi)_D := res\ \mu = res(\mu;\ p) = 1$$

for an arbitrary point

$$p \in X \setminus supp\ D.$$

We choose a point $p \in X$ with $D(p) = 0$. The divisor D vanishes also in a neighbourhood of p in X. With respect to a chart around p

$$z : U_0 \to D_1(0) \text{ with } D|U_0 = 0$$

we have

$$\eta|U_0 = f \cdot dz$$

with a non-zero meromorphic function

$$f \in \mathscr{O}_{-D}(U_0).$$

Because

$$(div\ f)(p) \geq D(p) = 0$$

the function f is even holomorphic in an open neighbourhood of p. W.l.o.g. f has no zeros in

$$U_0^* := U_0 \setminus \{p\},$$

but possibly $f(p) = 0$. As a consequence the reciprocal functions

$$\frac{1}{zf} \in \mathscr{O}(U_0^*)$$

has no zeros and satisfies

$$div\left(\frac{1}{zf}\right) = 0 \geq -D|U_0^* = 0$$

We set

$$U_1 := X \setminus \{p\}.$$

With respect to the open covering

$$\mathscr{U} := (U_0, U_1) \text{ with } U_0 \cap U_1 = U_0^*.$$

we define the cochain

$$(\xi_{ij})_{ij} \in Z^1(\mathscr{U}, \mathscr{O}_D)$$

setting

$$\xi_{10} := \frac{1}{zf} \in \mathscr{O}_D(U_0^*).$$

Then

$$\xi := [(\xi_{ij})_{ij}] \in H^1(X, \mathscr{O}_D)$$

ends the construction.

ii) *Computing the residue*: With

$$\eta \in H^0(X, \omega_{-D}) = Z^0(\mathscr{U}, \omega_{-D})$$

the product cochain

$$\eta\xi \in H^1(X, \omega)$$

is represented by the cocycle $((\eta\xi)_{ij})_{ij} \in Z^1(\mathscr{U}, \omega)$ with

$$(\eta\xi)_{10} = f dz \cdot \frac{1}{zf} = \frac{dz}{z} \in \omega(U_0^*)$$

It splits by the meromorphic cochain

$$\mu = (\mu_0 := \frac{dz}{z},\ \mu_1 := 0) \in C^0(\mathscr{U}, \mathscr{M}).$$

Hence the Mittag-Leffler distribution

$$ML(\eta\xi) := (\mathscr{U}, \mu)$$

satisfies

$$res\ ML(\eta\xi) = res\ \mu = res\left(\frac{dz}{z};\ p\right) = 1,\ q.e.d.$$

Lemma 9.12 prepares the proof of Proposition 9.13. It shows under which condition the residue form $(\cdot, -)_{D_2}$ is surjective for a divisor D_2 if the residue form $(\cdot, -)_{D_1}$ is surjective for a smaller divisor $D_1 \le D_2$.

Lemma 9.12 (Comparing the residue forms of two divisors). *Let X be a compact Riemann surface. Consider two divisors $D_1, D_2 \in Div(X)$ with*

$$D_1 \le D_2.$$

They induce the injection

$$\mathscr{O}_{D_1} \hookrightarrow \mathscr{O}_{D_2}$$

and the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(X,\mathscr{O}_{D_2})^\vee & \xrightarrow{i^{D_2}_{D_1}} & H^1(X,\mathscr{O}_{D_1})^\vee \\ & & \uparrow{\scriptstyle (.,-)_{D_2}} & & \uparrow{\scriptstyle (.,-)_{D_1}} \\ 0 & \longrightarrow & H^0(X,\omega_{-D_2}) & \longrightarrow & H^0(X,\omega_{-D_1}) \end{array}$$

with horizontal maps induced by Lemma 8.9.

If two elements

$$\lambda \in H^1(X,\mathscr{O}_{D_2})^\vee \text{ and } \eta \in H^0(X,\omega_{-D_1})$$

satisfy

$$i^{D_2}_{D_1}(\lambda) = (\eta,-)_{D_1}$$

then

$$\eta \in H^0(X,\omega_{-D_2}) \text{ and } \lambda = (\eta,-)_{D_2},\ i.e.$$

$$\begin{array}{ccc} \lambda & \longrightarrow & i^{D_2}_{D_1}(\lambda) = (\eta,-)_{D_1} \\ \vdots\uparrow & & \uparrow \\ \eta & \dashrightarrow & \eta \end{array}$$

Proof. The horizontal maps in the commutative diagram above are injective due to Lemma 8.9. By assumption

$$div\ \eta \geq D_1.$$

The proof has to "lift" the section $\eta \in H^0(X,\omega_{-D_1})$ to $H^0(X,\omega_{-D_2})$. We show by indirect proof

$$div\ \eta \geq D_2 :$$

Assume the existence of a point $p \in X$ with

$$(div\ \eta)(p) < D_2(p).$$

We choose a chart around p

$$z : U_0 \to D_1(0)$$

obtaining the local representation

$$\eta|U_0 = f \cdot dz \text{ with } f \in \mathscr{M}(U_0),\ div\ f \geq D_1|U_0.$$

W.l.o.g. on $U_0^* := U_0 \setminus \{p\}$ holds

- $$D_1 = D_2 = 0$$

- and $f|U_0^* \in \mathscr{O}^*(U_0^*)$.

Set $U_1 := X \setminus \{p\}$ and consider for the open covering of X

$$\mathscr{U} := (U_0, U_1)$$

the cochain

$$\zeta := (f_0, f_1) \in C^0(\mathscr{U}, \mathscr{M})$$

with

$$f_0 := \frac{1}{zf} \text{ and } f_1 := 0.$$

The estimate

$$(div\ \eta)(p) = (div\ f)(p) < D_2(p)$$

implies

$$\left(div\ \frac{1}{zf}\right)(p) = -(div\ f)(p) - 1 \geq -D_2(p),$$

hence

$$\zeta \in C^0(\mathscr{U}, \mathscr{O}_{D_2}).$$

Because in U_0^*

$$f_0 - f_1 = \frac{1}{zf} \in \mathscr{O}^*(U_0^*)$$

we have

$$\delta\zeta \in Z^1(\mathscr{U}, \mathscr{O}) = Z^1(\mathscr{U}, \mathscr{O}_{D_1}) = Z^1(\mathscr{U}, \mathscr{O}_{D_2}).$$

For $k = 1, 2$ we take the respective cohomology classes

$$\xi_k := [\delta\zeta] \in H^1(X, \mathscr{O}_{D_k}).$$

On one hand, by construction

$$\xi_2 = 0$$

because $\delta\zeta \in B^1(\mathscr{U}, \mathscr{O}_{D_2})$, and therefore

$$(\eta, \xi_1)_{D_1} = i_{D_1}^{D_2}(\lambda)(\xi_1) = \lambda(\xi_2) = 0.$$

On the other hand,

$$(\eta, \xi_1)_{D_1} = res(\eta\zeta) = res\left(\frac{dz}{z};\ 0\right) = 1 \neq 0,$$

a contradiction. The contradiction proves

$$div(\eta) \geq D_2, \text{ i.e. } \eta \in H^0(X, \omega_{-D_2}).$$

From

$$i_{D_1}^{D_2}(\lambda) = (\eta, -)_{D_1} = i_{D_1}^{D_2}((\eta, -)_{D_2})$$

follows

$$\lambda = (\eta, -)_{D_2}$$

because $i_{D_1}^{D_2}$ is injective, q.e.d.

Proposition 9.13 (Residue form: Surjectivity). *Consider a compact Riemann surface X and a divisor $D \in Div(X)$. Then the linear map*

$$(.,-)_D : H^0(X, \omega_{-D}) \to H^1(X, \mathscr{O}_D)^\vee, \eta \mapsto (\eta, -),$$

is surjective.

Proof. Consider an arbitrary but fixed non-zero linear functional

$$\lambda \in H^1(X, \mathscr{O}_D)^\vee.$$

We have to find an element

$$\eta_0 \in H^0(X, \omega_{-D}) \text{ with } \lambda = (\eta_0, -)_D.$$

Therefore we will apply Lemma 9.12 and consider divisors $D' \in Div(X)$ with

$$D' \leq D.$$

The Riemann-Roch theorem estimates the dimension of the cohomology of certain sheaves $\mathscr{O}_E$ and ω_E with divisors $E \in Div(X)$ derived from D.

i) *The divisors $D_n = D - nP$*: We choose a point $p \in X$ and the corresponding point divisor $P \in Div(X)$. For arbitrary natural numbers $n \in \mathbb{N}$ we consider the divisors

$$D_n := D - nP \in Div(X).$$

Note

$$D_n \leq D$$

Corollary 7.19 implies for large $n \in \mathbb{N}$

$$H^0(X, \mathscr{O}_{nP}) \neq \{0\}.$$

Any

$$\psi \in H^0(X, \mathscr{O}_{nP}),\ \psi \neq 0,$$

defines by multiplication an injection

$$m_\psi : \mathscr{O}_{D_n} \hookrightarrow \mathscr{O}_D,\ f \mapsto \psi \cdot f$$

If

$$A := div\ \psi \in Div(X) \text{ and } D' := D_n - A \in Div(X)$$

then due to $D_n \le D$ the inclusion factorizes as

$$[\mathscr{O}_{D_n} \hookrightarrow \mathscr{O}_D] = [\mathscr{O}_{D_n} \xrightarrow{m_\psi} \mathscr{O}_{D'} \hookrightarrow \mathscr{O}_D]$$

with the sheaf isomorphism

$$\mathscr{O}_{D_n} \xrightarrow{\simeq m_\psi} \mathscr{O}_{D'}$$

Lemma 8.9 implies the injectivity of

$$H^1(X, \mathscr{O}_D)^\vee \to H^1(X, \mathscr{O}_{D'})^\vee \simeq H^1(X, \mathscr{O}_{D_n})^\vee,\ \lambda \mapsto \lambda \circ m_\psi.$$

ii) *The particular case* $D_n, n \gg 0$: Consider the map

$$\alpha : H^0(X, \mathscr{O}_{nP}) \to H^1(X, \mathscr{O}_{D_n})^\vee,\ \psi \mapsto \lambda \circ m_\psi$$

The map α is injective, because

$$\lambda \circ m_\psi = 0$$

implies $\psi = 0$ by the injectivity of

$$H^1(X, \mathscr{O}_D)^\vee \to H^1(X, \mathscr{O}_{D_n})^\vee$$

from part i) and because of $\lambda \neq 0$.

Set as shorthand

$$\beta := (\cdot, -)_{D_n} : H^0(X, \omega_{-D_n}) \to H^1(X, \mathscr{O}_{D_n})^\vee$$

The map β is injective due to Proposition 9.11. Consider the diagram

$$\begin{array}{ccc} H^0(X, \mathscr{O}_{nP}) & \xrightarrow{\alpha} & H^1(X, \mathscr{O}_{D_n})^\vee \\ & & \uparrow \beta \\ & & H^0(X, \omega_{-D_n}) \end{array}$$

We claim: For sufficiently large $n \in \mathbb{N}$ there exists

$$(\psi, \eta) \in H^0(X, \mathscr{O}_{nP}) \times H^0(X, \omega_{-D_n}),$$

such that

$$\lambda \circ m_\psi = \alpha(\psi) = \beta(\eta) = (\eta, -)_{D_n} \in H^1(X, \mathscr{O}_{D_n})^\vee.$$

Therefore we have to show

$$\alpha(H^0(X, \mathscr{O}_{nP})) \cap \beta(H^0(X, \omega_{-D_n})) \neq \{0\} \subset H^1(X, \mathscr{O}_{D_n})^\vee$$

or

$$dim\ H^0(X,\mathscr{O}_{nP}) + dim\ H^0(X,\omega_{-D_n}) > dim\ H^1(X,\mathscr{O}_{D_n})^\vee.$$

Consider $n \in \mathbb{N}$.

- On one hand the Riemann-Roch theorem 8.10 implies

$$dim\ H^0(X,\mathscr{O}_{nP}) \geq 1 - g(X) + n$$

- On the other hand, Lemma 9.2 provides a constant $k_0 \in \mathbb{N}$ such that

$$dim\ H^0(X,\omega_{-D_n}) \geq k_0 - deg\ D_n = k_0 - (deg\ D - n) = n + (k_0 - deg\ D)$$

- If $n > deg\ D$ then

$$deg\ D_n = (deg\ D) - n < 0$$

 and Proposition 8.4 implies

$$H^0(X,\mathscr{O}_{D_n}) = 0.$$

 Then the Riemann-Roch theorem 8.10 implies

$$\chi(\mathscr{O}_{D_n}) = -dim\ H^1(X,\mathscr{O}_{D_n}) = 1 - g + deg\ D_n = 1 - g + (deg\ D) - n$$

 i.e.

$$dim\ H^1(X,\mathscr{O}_{D_n})^\vee = dim\ H^1(X,\mathscr{O}_{D_n}) = n + (g - 1 - deg\ D)$$

Summing up, we obtain for $n > deg\ D$

$$dim\ H^0(X,\mathscr{O}_{nP}) + dim\ H^0(X,\omega_{-D_n}) \geq$$

$$(1 - g(X) + n) + (n + k_0 - deg\ D) = 1 - g(X) + k_0 - deg\ D + 2n$$

and

$$dim\ H^1(X,\mathscr{O}_{D_n})^\vee = dim\ H^1(X,\mathscr{O}_{D_n}) = -1 + g(X) - deg\ D + n.$$

As a consequence for $n \gg 0$

$$dim\ H^0(X,\mathscr{O}_{nP}) + dim\ H^0(X,\omega_{-D_n}) > dim\ H^1(X,\mathscr{O}_{D_n})^\vee,$$

which implies

$$H^0(X,\mathscr{O}_{nP}) \cap H^0(X,\omega_{-D_n}) \neq \{0\}.$$

We obtain elements

$$\psi \in H^0(X,\mathscr{O}_{nP}) \text{ and } \eta \in H^0(X,\omega_{-D_n})$$

with

$$\lambda \circ m_\psi = (\eta, -)_{D_n}$$

iii) *Reduction to the general case* $D \geq D_n$: Part ii) shows: For $n \gg 0$ exists a pair

$$(\psi, \eta) \in H^0(X, \mathscr{O}_{nP}) \times H^0(X, \omega_{-D_n})$$

satisfying

$$\lambda \circ m_\psi = (\eta, -)_{D_n}.$$

If

$$A := div\ \psi \text{ and } D' := D_n - A \leq D$$

then multiplication by ψ also defines sheaf isomorphisms

$$\omega_{-D'} \xrightarrow{m_\psi} \omega_{-D_n}$$

We obtain an element

$$\eta_0 := \frac{\eta}{\psi} \in H^0(X, \omega_{-D'})$$

with

$$\lambda = (\eta_0, -)_{D'}$$

The commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(X, \mathscr{O}_D)^\vee & \xrightarrow{i^D_{D'}} & H^1(X, \mathscr{O}_{D'})^\vee & \xrightarrow{\simeq\, m_\psi} & H^1(X, \mathscr{O}_{D_n})^\vee \\
 & & {\scriptstyle (.,-)_D} \uparrow & & {\scriptstyle (.,-)_{D'}} \uparrow & & {\scriptstyle (.,-)_{D_n}} \uparrow \\
0 & \longrightarrow & H^0(X, \omega_{-D}) & \longrightarrow & H^0(X, \omega_{-D'}) & \xrightarrow{\simeq\, m_\psi} & H^0(X, \omega_{-D_n})
\end{array}$$

and Lemma 9.12 imply $\eta_0 \in H^0(X, \omega_{-D})$:

$$\begin{array}{ccccc}
\lambda & \longrightarrow & \lambda & \longrightarrow & \psi \cdot \lambda \\
\uparrow & & \uparrow & & \uparrow \\
\eta_0 & \dashrightarrow & \eta_0 & \longrightarrow & \eta
\end{array}$$

Hence

$$\lambda = (\eta_0, -)_D,\ q.e.d.$$

Proof of Theorem 9.10. The claim follows from the Propositions 9.11 and 9.13, when taking into account Remark 9.9 and the finiteness result from the Riemann-Roch Theorem 8.10, q.e.d.

Remark 9.14 (Serre duality).

1. Consider a compact Riemann surface X and choose a fixed non-zero meromorphic form

$$\eta \in H^0(X, \mathscr{M}^1)$$

with *canonical divisor*

$$K := div\ \eta \in Div(X).$$

Then η defines a sheaf isomorphism

$$\Omega^1 \xrightarrow{\simeq} \mathscr{O}_K,\ \zeta \mapsto \frac{\zeta}{\eta}.$$

In particular

$$\omega_K = \Omega^1_{-K} \simeq \mathscr{O}.$$

If we apply Serre duality

$$(\cdot,-)_D : H^0(X, \omega_{-D}) \xrightarrow{\simeq} H^1(X, \mathscr{O}_D)^\vee$$

with the divisor

$$D := K,$$

then we obtain

$$H^0(X, \omega_{-D}) = H^0(X, \omega_{-K}) = H^0(X, \mathscr{O}) \xrightarrow{\simeq} H^1(X, \mathscr{O}_K)^\vee = H^1(X, \Omega^1)^\vee$$

2. The proof of Serre's duality theorem 9.10 rests on the Riemann-Roch theorem. In addition, it uses meromorphic differential forms and the vanishing

$$H^1(X, \mathscr{M}^1) = 0.$$

The original proof of Serre [33] is different. Serre uses fine resolutions by the sheaves of smooth forms and Fréchet topologies on their vector spaces of sections with distributions as dual spaces, see also [16, Chap. VI].

9.3 Applications of Serre duality and Riemann-Roch theorem

Corollary 9.15 (Cohomology of the dualizing sheaf). *Let X be a compact Riemann surface with dualizing sheaf ω. Then*

$$dim\ H^1(X, \omega) = 1$$

and

$$res : H^1(X, \omega) \to \mathbb{C}$$

is an isomorphism. Moreover

$$dim\ H^0(X, \omega) = g(X)$$

is the genus of X.

Proof. Remark 9.14 proves

$$H^0(X,\mathscr{O}) \simeq H^1(X,\omega)^\vee$$

which implies

$$dim\ H^1(X,\omega) = dim\ H^0(X,\mathscr{O}) = 1.$$

The residue map

$$res : H^1(X,\omega) \to \mathbb{C}$$

is non-zero, hence an isomorphism. In addition

$$g(X) := dim\ H^1(X,\mathscr{O}) = dim\ H^0(X,\omega) = dim\ H^0(X,\Omega^1),\ q.e.d.$$

Corollary 9.16 (Genus of the torus). *The complex torus* $T = \mathbb{C}/\Lambda$ *has the genus*

$$g(T) = 1.$$

Proof. The torus T is covered by complex charts $z_i : U_i \to V_i,\ i \in I$, with

$$z_i = z_j + \lambda_{ij},\ \lambda_{ij} \in \Lambda \text{ locally constant.}$$

Therefore on $U_i \cap U_j$

$$dz_i = dz_j$$

and there exists a global form $dz \in \Omega^1(T)$ without zeros. The sheaf morphism

$$\mathscr{O} \to \Omega^1,\ f \mapsto f \cdot dz$$

is an isomorphism. Therefore

$$\omega \simeq \Omega^1 \simeq \mathscr{O}.$$

Corollary 9.15 implies

$$g(T) = dim\ H^0(T,\Omega^1) = H^0(T,\mathscr{O}) = 1,\ q.e.d.$$

Proposition 9.17 (Degree of a canonical divisor). *On a compact Riemann surface* X *for all non-zero meromorphic forms* $\eta \in H^0(X,\mathscr{M}^1)$ *the* canonical divisors

$$K := div\ \eta \in Div(X)$$

are equivalent, and have degree

$$deg\ K = 2g(X) - 2.$$

Proof. i) *Degree of non-zero sections of* $H^0(X,\mathscr{M}^1)$: Remark 9.14 shows

$$\Omega^1 \simeq \mathscr{O}_K.$$

Hence Theorem 9.10 and Theorem 8.10 apply to the sheaf Ω^1. For two non-zero meromorphic sections from $\mathscr{M}^1(X)$ the quotient

$$\frac{\eta_1}{\eta_2} \in \mathscr{M}^1(X)$$

is a well-defined non-zero, global meromorphic function, which can be seen using charts. Note that the inverse of a non-zero meromorphic function is again meromorphic. Hence Corollary 3.24 implies

$$deg\ \frac{\eta_1}{\eta_2} = 0 \text{ or } deg\ \eta_1 = deg\ \eta_2,$$

independent from the choice of the meromorphic sections.

ii) *Riemann-Roch and Serre duality*: Due to part i) the Riemann-Roch theorem implies

$$\chi(\Omega^1) = \chi(\mathscr{O}_K) = 1 - g(X) + deg\ K$$

Serre duality implies, see Remark 9.14,

$$H^0(X,\Omega^1) = H^1(X,\mathscr{O})^\vee,$$

hence by definition of $g(X)$

$$\chi(\Omega^1) = -\chi(\mathscr{O}) = dim\ H^1(X,\mathscr{O}) - dim\ H^0(X,\mathscr{O}) = g(X) - 1$$

As a consequence

$$g(X) - 1 = 1 - g(X) + deg\ K$$

or

$$2g(X) - 2 = deg\ K,\ q.e.d.$$

Theorem 9.18 (Riemann-Hurwitz formula). *Consider a non-constant holomorphic map*

$$f : X \to Y$$

between to compact Riemann surfaces. Denote by

$$n(f) := card\ X_y$$

the cardinality of the fibres of f, which is independent from $y \in Y$. For each $x \in X$ denote by

$$v(f;\ x)$$

the multiplicity *of f at x, by*

$$b(f;\,x) := v(f;\,x) - 1$$

the branching order *of f at x. Then the* total branching order *of f*

$$b(f) := \sum_{x \in X} b(f;\,x)$$

and $n(f)$ relate to the genus of X and Y as

$$g(X) = 1 + \frac{b(f)}{2} + n(f) \cdot (g(Y) - 1)\,.$$

The idea of the proof is to link the numerical characteristics of X and Y by the degree of a non-zero global section $\eta \in H^0(Y, \mathscr{M}^1)$ and its pullback $f^*\eta \in H^0(X, \mathscr{M}^1)$. Then introducing local coordinates allows to calculate the

Proof. We choose a non-zero meromorphic form

$$\eta \in H^0(Y, \mathscr{M}^1).$$

and consider its pullback

$$f^*\eta \in H^0(X, \mathscr{M}^1).$$

Proposition 9.17 implies

$$deg(div\ \eta) = 2g(Y) - 2 \text{ and } deg(div\ f^*\eta) = 2g(X) - 2$$

For a given pair $y \in Y$ and $x \in Y_y$ we choose charts

$$z : U(x) \to V(x) \text{ and } w : U(y) \to V(y)$$

around $x \in X$ and $y \in Y$ respectively, such that

$$U(x) = f^{-1}(U(y))$$

and the composition

$$w \circ f \circ z^{-1}$$

has the form

$$w = z^k$$

with

$$k := \mathrm{v}(f;\,x)$$

the multiplicity of f at x, see Definition 3.21 and Proposition 1.6. If

$$\eta|U(y) = g\ dw \text{ and } dw = d(z^k) = k \cdot z^{k-1}\ dz$$

then

$$f^*\eta = k \cdot z^{k-1} \cdot g(z^k)\ dz.$$

We now vary $x \in X_y$ and $y \in Y$, obtaining

$$ord(f^*\eta;\ x) := ord(k \cdot z^{k-1} \cdot g(z^k);\ x) = b(f;\ x) + \mathrm{v}(f;\ x) \cdot ord(\eta;\ y)$$

Because independently from $y \in Y$

$$n(f) = \sum_{x \in X_y} \mathrm{v}(f;\ x)$$

we obtain by applying $\sum_{x \in X_y}$

$$\sum_{x \in X_y} ord(f^*\eta;\ x) = \sum_{x \in X_y} (b(f;\ x) + \mathrm{v}(f;\ x) \cdot ord(\eta;\ y))$$

and by applying in addition $\sum_{y \in Y}$

$$deg(div\ f^*\eta) = \sum_{x \in X} ord(f^*\eta;\ x) = \sum_{y \in Y} \sum_{x \in X_y} ord(f^*\eta;\ x) =$$

$$= \sum_{y \in Y} \left(\sum_{x \in X_y} ord(f^*\eta;\ x) \right) = \left(\sum_{x \in X} b(f;\ x) \right) + \left(n(f) \cdot \sum_{y \in Y} ord(\eta;\ y) \right) =$$

$$= b(f) + n(f) \cdot deg\ (div\ \eta).$$

Hence

$$2g(X) - 2 = b(f) + n(f) \cdot (2g(Y) - 2),\ q.e.d.$$

Recall from Corollary 3.23 that a non-constant holomorphic map between two Riemann surfaces is surjective if its domain is compact.

Corollary 9.19 (Holomorphic maps between compact Riemann surfaces).

1. Any compact Riemann surface X has a surjective holomorphic map

$$f: X \to \mathbb{P}^1$$

2. Each holomorphic map

$$f: \mathbb{P}^1 \to X$$

with X a compact Riemann surface X of genus $g(X) \geq 1$ is constant.

Proof. 1. For any compact Riemann surface X exist non-constant holomorphic maps

$$f : X \to \mathbb{P}^1$$

because the latter are the meromorphic functions on X, see Theorem 1.10 and Corollary 7.19 implies the existence of non-constant meromorphic functions on X.

2. For a non-constant holomorphic map

$$f : X \to Y$$

the Riemann-Hurwitz formula from Theorem 9.18 implies for non-constant f

$$-2 = b(f) + n(f) \cdot (2g(X) - 2) \geq b(f) + n(f)$$

a contradiction, q.e.d.

Chapter 10
Vector bundles and line bundles

10.1 Vector bundles

A vector bundle of rank $k \in \mathbb{C}$ on a topological space X is a family of local products

$$U_i \times \mathbb{C}^k$$

with respect to an open covering $(U_i)_{i \in I}$ of X, which glues on the intersections $U_i \cap U_j$ while maintaining the vector space structure. Definition 10.1 gives the formal definition and introduces the relevant concepts.

Definition 10.1 (Vector bundle). Consider a topological space X.

1. A continuous complex *vector bundle of rank* $k \in \mathbb{N}$ on X is a topological space E together with a continuous map

   $$p : E \to X$$

   satisfying the following properties:

   - Each point $x \in X$ has an open neighbourhood $U \subset X$ and a homeomophism

     $$\phi_U : p^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{C}^k,$$

     named a *linear chart*, such that the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times \mathbb{C}^k \\ & {\scriptstyle p|p^{-1}(U)} \searrow \quad \swarrow {\scriptstyle pr_U} & \\ & U & \end{array}$$

- For each two linear charts

$$\phi_i := \phi_{U_i},\ i = 1,2 \text{ with } U_{12} := U_1 \cap U_2 \neq \emptyset$$

exists a continous *matrix-valued function*

$$g_{12} : U_{12} \to GL(k,\mathbb{C})$$

such that the *transition function*

$$\phi_{12} := \phi_1 \circ \phi_2^{-1} : U_{12} \times \mathbb{C}^k \to U_{12} \times \mathbb{C}^k$$

satisfies for all $(x,\mathrm{v}) \in U_{12} \times \mathbb{C}^k$

$$\phi_{12}(x,\mathrm{v}) = (x, g_{12}(x) \cdot \mathrm{v}).$$

2. For a continuous vector bundle $p : E \to X$ and a covering $\mathscr{U} = (U_i)_{i \in I}$ of X by open sets U_i with linear charts

$$\phi_i : p^{-1}(U_i) \xrightarrow{\simeq} U_i \times \mathbb{C}^k$$

the family $(\phi_i)_{i \in I}$ is an *atlas* of the vector bundle.

3. If X is a Riemann surface than a continuous complex vector bundle

$$p : E \to X$$

is *holomorphic* if it has an atlas $(\phi_i)_{i \in I}$ with holomorphic matrix-valued functions

$$g_{ij} : U_i \cap U_j \to GL(k,\mathbb{C}) \subset \mathbb{C}^{k^2},\ i,j \in I.$$

Here a tuple of functions is holomorphic iff each component is holomorphic.

4. A vector bundle of rank $k = 1$ is a *line bundle*.

Remark 10.2 (Fibres of a vector bundle). Consider a complex vector bundle of rank $= k$

$$p : E \to X$$

on a topological space X. For each $x \in X$ a linear chart

$$\phi_U : p^{-1}(U) \to U \times \mathbb{C}^k$$

with $x \in U$ defines on the fibre $E_x := p^{-1}(x)$ the structure of a complex vector space by the restriction

$$p_U|E_x : E_x \xrightarrow{\simeq} \{x\} \times \mathbb{C}^k :$$

If $z_1, z_2 \in E_x,\ \lambda \in \mathbb{C}$, and

$$\phi_U(z_i) = (x, \mathrm{u}_i) \in \{x\} \times \mathbb{C}^k,\ i = 1, 2,$$

then

$$z_1 + z_2 := \phi_U^{-1}(x, \mathrm{u}_1 + \mathrm{u}_2) \in E_x$$

and

$$\lambda \cdot z_1 := \phi_U^{-1}(x, \lambda \cdot \mathrm{u}_1) \in E_x.$$

The vector space structure on E_x does not depend on the choice of the linear chart ϕ_U: If

$$\phi_V : p^{-1}(V) \to V \times \mathbb{C}^k$$

with $x \in V$ is a second linear chart, then for $j = 1, 2$

$$\phi_V(z_j) = (x, \mathrm{v}_j) = (x, g_{VU} \cdot u_j).$$

As a consequence

$$\phi_V^{-1}(x, \mathrm{v}_1 + \mathrm{v}_2) = \phi_V^{-1}(x, g_{U,V}(x) \cdot \mathrm{u}_1 + g_{U,V}(x) \cdot \mathrm{u}_2) =$$

$$= \phi_V^{-1}(x, g_{U,V}(x) \cdot (\mathrm{u}_1 + \mathrm{u}_2)) = \phi_U^{-1}(x, u_1 + u_2)$$

And similarly for the scalar multiplication. In particular:

$$\mathrm{u}_1 = 0 \in \mathbb{C}^k \implies v_1 = g_{U,V}(x) \cdot u_1 = 0 \in \mathbb{C}^k,$$

i.e. also the zero vector in the fibre E_x is well-defined, independent from the choice of a linear chart.

While the vector space structure on E_x is uniquely determined, there is no canonical isomorphism $E_x \simeq \mathbb{C}^k$.

For a line bundle on X the matrix functions

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^* = GL(1, \mathbb{C})$$

satisfy a cocycle relation and therefore define a class in $H^1(X, \mathscr{O}^*)$.

Proposition 10.3 (The matrix-functions of vector bundles and line bundles).

1. Let X be a topological space, consider a vector bundle

$$p : E \to X$$

of rank $= k$, and let

$$(\phi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^k)_{i \in I}$$

be an atlas of L. The corresponding family

$$(g_{ij} : U_i \cap U_j \to GL(k, \mathbb{C}))_{ij}$$

of matrix-valued functions satisfies on the threefold intersections

$$U_{ijk} := U_i \cap U_j \cap U_k \subset X$$

the relations

$$g_{ik} = g_{ij} \cdot g_{jk} : U_{ijk} \to \mathbb{C}^k.$$

2. *As a consequence, the atlas of a line bundle defines the cocycle*

$$g = (g_{ij})_{i,j \in I} \in Z^1(\mathscr{U}, \mathscr{C}^*).$$

The cohomology class

$$[g] \in H^1(X, \mathscr{C}^*)$$

does not depend on the choice of the atlas, it is uniquely determined by the line bundle.

3. *Analogously, a holomorphic line bundle on a Riemann surface X defines a class*

$$[g] \in H^1(X, \mathscr{O}^*).$$

And conversely, any class from

$$H^1(X, \mathscr{C}^*) \text{ or } H^1(X, \mathscr{O}^*)$$

defines respectively a continuous or holomorphic line bundle on X.

Proof. ad 3) Represent a given class from $H^1(X, \mathscr{O}^*)$ by a cocyle

$$g = (g_{ij})_{ij} \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

with respect to an open covering $\mathscr{U} = (U_i)_{i \in I}$. On the disjoint set

$$\dot{\bigcup}_{i \in I} U_i \times \mathbb{C}$$

consider the relation

$$(x, \mathrm{v}_i) \in U_i \times \mathbb{C} \sim (x, \mathrm{v}_j) \in U_j \times \mathbb{C} \iff \mathrm{v}_i = g_{ij}(x) \cdot \mathrm{v}_j$$

The cocyle condition implies that the relation is an equivalence relation. Define

$$L := (\dot{\bigcup}_{i \in I} U_i \times \mathbb{C}) / \sim$$

with the quotient topology. Then the canonical map

$$p : L \to X,\ [(x, v_i)] \mapsto x$$

is continuous. The family

$$p^{-1}(U_i) = [U_i \times \mathbb{C}] \xrightarrow{x} U_i \times \mathbb{C} \text{ induced by } id : U_i \times \mathbb{C} \mapsto U_i \times \mathbb{C}$$

is a holomorphic atlas of L, because the matrix-functions of its transition functions are the holomorphic functions

$$g_{ij} : U_i \cap U_j \to \mathbb{C}*$$

are holomorphic.

Due to Proposition 10.3 we will often identify a line bundle with its class from $H^1(X, \mathscr{O}^*)$, or even with a representing cocycle from $Z^1(\mathscr{U}, \mathscr{O}^*)$ with respect to the open covering $\mathscr{U}$ of an atlas of the line bundle.

Definition 10.4 (Sheaf of sections of a vector bundle). Consider a topological space X and a continuous vector bundle of rank $= k$

$$p : E \to X.$$

1. Then for any open $U \subset X$ the complex vector space continuous sections

$$\mathscr{C}_E(U) := \{s : U \to E :\ s \text{ continuous and } p \circ s = id_U\}$$

with the canonical restrictions for $V \subset U$ defines a sheaf $\mathscr{C}_E$, named the *sheaf of continuous sections of* E.

2. If X is a Riemann surface and the vector bundle is holomorphic then a continuous section

$$s : W \to E$$

on an open set $W \subset X$ is holomorphic, if each point $x \in W$ has an open neighbourhood $U \subset$W and a linear chart

$$\phi : p^{-1}(U) \to U \times \mathbb{C}^k$$

of the vector bundle such that all components of the map

$$\phi \circ s = (id, s_1, ..., s_k) : U \to U \times \mathbb{C}^k$$

are holomorphic. The presheaf

$$\mathscr{O}_E(U) := \{s : U \to E :\ s \text{ holomorphic and } p \circ s = id_U\},\ U \subset X \text{ open}$$

with the canonical restriction of sections defines a sheaf $\mathscr{O}_E$, named the *sheaf of holomorphic sections of* E. For a line bundle L one often writes

$$\mathscr{L} := \mathscr{O}_L.$$

In both cases addition and scalar multiplication of sections is done poinwise. One adds and multiplies the values of sections due to the vector space structure on the fibres of the vector bundle.

Proposition 10.5 (Local representation of sections of a vector bundle). *Consider a vector bundle*

$$p : E \to X$$

and an atlas

$$\left(\phi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^k\right)_{i \in I}$$

with respectively continuous or holomorphic matrix-valued functions

$$g_{ij} : U_i \cap U_j \to GL(k, \mathbb{C}) \subset \mathbb{C}^{k^2},\ i, j \in I.$$

1. *For an open set $U \subset X$ the vector spaces of global sections*

$$\mathscr{C}_E(U) \text{ or } \mathscr{O}_E(U)$$

correspond bijectively to the families of continuous or holomorphic maps

$$(s_i : U \cap U_i \to \mathbb{C}^k)_{i \in I}$$

satisfying on $U \cap U_i \cap U_j$

$$s_i = g_{ij} \cdot s_j,$$

i.e. for all $x \in U \cap U_i \cap U_j$

$$s_i(x) = g_{ij}(x) \cdot s_j(x) \in \mathbb{C}^k.$$

2. *If X is a Riemann surface and the vector bundle is holomorphic, then the family*

$$\mathscr{M}_E(U_i) := \{s_i := (s_{i,1}, ..., s_{i,k}) :\ s_{i,1} ... s_{i,k} \in \mathscr{M}(U_i)\},\ i \in I,$$

with

$$s_i = g_{ij} \cdot s_j,\ i, j \in I,$$

and the canonical restrictions is a $\mathscr{B}$-sheaf with $\mathscr{B}$ a basis of open sets $U \subset X$ with corresponding linear charts ϕ_U of E. The family extends to $\mathscr{M}_E$, the sheaf of meromorphic sections of *E on X, see Proposition 2.17.*

10.2 Line bundles and Chern classes

The present sections deals with the following concepts to investigate a Riemann surface X:

- *Holomorphic line bundle*: A line bundle on X is a map

$$p : L \to X$$

which has over small open sets $U \subset X$ a product structure over U

$$\phi_U : p^{-1}(U) \xrightarrow{\simeq} U \times \mathbb{C}.$$

Over the intersections $U_1 \cap U_2$ the respective product structures transform holomorphically into each other. Hence the fibres $p^{-1}(x)$ are complex lines varying holomorphically with the base point $x \in X$.

- *Invertible sheaf*: An invertible sheaf $\mathscr{L}$ on X is a generalization of the structure sheaf $\mathscr{O}$ of X: On small open sets $U \subset X$ the restricted sheaves

$$\mathscr{L}|U \text{ and } \mathscr{O}|U$$

are isomorphic, see Definition 2.15.

- *Cocycle of the sheaf $\mathscr{O}^*$*: The group of cocycles $Z^1(\mathscr{U}, \mathscr{O}^*)$ for an open covering $\mathscr{U}$ of X and the multiplicatively cohomology group $H^1(X, \mathscr{O}^*)$.

- *Divisor*: A divisor D on X is a discrete and closed subset of points of X with prescribed integer multiples. Each integer is considered the order of a pole or a zero at the corresponding point, see Definition 8.1.

We will show that on a compact Riemann surface these concepts are nothing but different views onto one and the same mathematical object. The equivalence of cocycles, line bundles and invertible sheaves is just a formal computation. For the equivalence of cocycles and line bundles see Proposition 10.3, part 3). If not stated otherwise we will therefore identify both concepts in the following. But the equivalence of divisors and line bundles is a non-trivial result. The proof will be completed by Theorem 10.23.

On the projective space $\mathbb{P}^1$ the twisted sheaves $\mathscr{O}(k),\ k \in \mathbb{Z}$, are the sheaves of sections of line bundles. Each is characterized by the integer $k \in \mathbb{Z}$. The concept of Chern classes - or more precise Chern integers - generalizes the attachment of integers to line bundles on any compact Riemann surface X. But different from the case $X = \mathbb{P}^1$ for general X the Chern number does not determine the line bundle. Nevertheless, the Chern number is the only invariant which enters into the Riemann-Roch theorem for the Euler number of the line bundle.

Proposition 10.6 (Line bundles and invertible sheaves). *Consider a Riemann surface X. For any a holomorphic line bundle*

$$p : L \to X$$

the sheaf $\mathscr{L}$ of holomorphic sections of the line bundle is an invertible sheaf. Vice versa, any invertible sheaf on X is the sheaf of holomorphic sections of a line bundle on X.

Proof. i) Consider a holomorphic line bundle

$$p : L \to X.$$

For any open set $U \subset X$ from an atlas of the line bundle we have

$$\mathscr{L}|U \simeq \mathscr{O}|U.$$

ii) Consider an invertible sheaf $\mathscr{L}$ on X and assume an open covering $\mathscr{U} = (U_i)_{i\in I}$ of X with sheaf isomorphisms

$$\phi_i : \mathscr{L}|U_i \xrightarrow{\simeq} \mathscr{O}|U_i.$$

For each pair $i,\ j \in I$ the holomorphic map

$$\phi_i \circ \phi_j^{-1}$$

defines a holomorphic matrix function

$$g_{ij} : U_i \cap U_j \to \mathbb{C}^*.$$

The cocycle

$$(g_{ij})_{i,j\in I} \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

defines the line bundle L with $\mathscr{L}$ its sheaf of holomorphic sections, q.e.d.

We now show that any divisor $D \in Div(X)$ defines a line bundle

$$p : L \to X$$

such that

$$\mathscr{O}_D \simeq \mathscr{L}.$$

Theorem 10.7 (Line bundle of a divisor). *Let X be a Riemann surface and consider a divisor $D \in Div(X)$. Then for a suitable open covering $\mathscr{U} = (U_i)_{i\in I}$ the divisor has the form*

$$D|U_i = div\ f_i$$

with a cochain of meromorphic functions $f = (f_i)_{i\in I} \in C^0(\mathscr{U}, \mathscr{M})$.

1. The cocycle

$$g = \left(g_{ij} := \frac{f_i}{f_j} \right) \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

defines a line bundle, the line bundle of the divisor *D,*

$$p : L \to X.$$

Denote by $\mathscr{L}$ its sheaf of holomorphic sections.

2. The family

$$(f_i)_{i \in I} \in C^0(\mathscr{U}, \mathscr{M})$$

defines

- *a meromorphic section of L, i.e.*

  $$f = (f_i)_{i \in I} \in H^0(X, \mathscr{M}_L)$$

  - even holomorphic for effective D -

- *and by multiplication*

  $$\mathscr{O}_D \xrightarrow{\cdot (f_i)} \mathscr{L}$$

  a well-defined isomorphism of $\mathscr{O}$-module sheaves.

3. The dual line bundle

$$p^\vee : L^\vee \to X$$

is defined by the cocycle

$$g^\vee := \left(g_{ij}^\vee := \frac{1}{g_{ij}} = \frac{f_j}{f_i} \right)_{i,j \in I} \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

It satisfies

$$\partial D = [g^\vee] \in H^1(X, \mathscr{O}^*)$$

with

$$\partial : H^0(X, \mathscr{D}) \to H^1(X, \mathscr{O}^*)$$

the connecting morphism of the divisor sequence from Proposition 8.6.

Proof. 1. For each pair $i, j \in I$ on $U_{ij} := U_i \cap U_j$

$$D|U_{ij} = div\ (f_i|U_{ij}) = div\ (f_j|U_{ij}).$$

which implies

$$g_{ij} = \frac{f_i}{f_j} \in \mathscr{O}^*(U_{ij})$$

The cocycle relation is obvious.

2. i) By definition for each pair $i, j \in I$

$$f_i = g_{ij} \cdot f_j$$

Hence Proposition 10.5 proves that the family

$$(f_i)_{i \in I} \in C^0(\mathscr{U}, \mathscr{M})$$

defines a meromorphic section of the line bundle. For an effective divisor D each function $f_i,\ i \in I$, is holomorphic by definition.

ii) For any open set $U \subset X$ any meromorphic function

$$f \in \mathscr{O}_D(U),$$

which is not locally-constant equal to zero, satisfies on $U \cap U_i,\ i \in I$,

$$div\ f \geq -D = -div\ f_i$$

or

$$div\ (f \cdot f_i) \geq 0$$

Hence

$$f \cdot f_i \in \mathscr{O}(U \cap U_i).$$

The transition rule

$$f_i = g_{ij} \cdot f_j$$

implies

$$f \cdot f_i = g_{ij} \cdot (f \cdot f_j).$$

According to Proposition 10.5 the family

$$f \cdot f_i$$

represents a holomorphic section from $\mathscr{L}(U)$.

3. The claim follows from Proposition 8.6, q.e.d.

Remark 10.8 (Line bundles and divisors).

1. Note. For a given divisor $D \in H^0(X, \mathscr{D})$ the line bundle

$$p : L \to X$$

defined by the class

$$\partial D \in H^1(X, \mathscr{O}^*)$$

has the sheaf of sections $\mathscr{L}$, which is the dual(!) of $\mathscr{O}_D$, i.e.

$$\mathscr{L} = \mathscr{O}_D^\vee.$$

2. Line bundles of the form $\mathscr{O}_D$ are invertible subsheaves of the sheaf $\mathscr{M}$ of meromorphic functions on X. Theorem 10.23 will show the converse of Theorem 10.7: Any holomorphic line bundle on a compact Riemann surface is the line bundle of a divisor. Moreover, on a non-compact Riemann surface X any line bundle L is even trivial, i.e. it is the line bundle of the zero divisor. For the latter statement see Theorem 10.23.

Definition 10.9 (Canonical line bundle). For a Riemann surface X the line bundle

$$\kappa := O_K \in H^1(X, \mathscr{O}^*)$$

of a canonical divisor $K \in Div(X)$ is named the *canonical line bundle* on X.

The Chern class of a holomorphic line bundle is the connecting morphism of the exponential sequence. This definition does not require any sign convention.

Definition 10.10 (Chern class of a line bundle). Let X be a compact Riemann surface. The connecting morphism ∂ of the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$$

is named the *Chern morphism*

$$c_1 := \partial : H^1(X, \mathscr{O}^*) \to H^2(X, \mathbb{Z}).$$

If a line bundle L is represented by the class $\xi \in H^1(X, \mathscr{O}^*)$ then

$$c_1(L) := \partial\xi \in H^2(X, \mathbb{Z})$$

is named the *Chern class* of L.

Note: Because the Chern class is a group homomorphism we have

$$c_1(L_1 \cdot L_2) = c_1(L_1) + c_1(L_2).$$

The Chern morphism is not restricted to holomorphic line bundles. The Chern morphism is also defined for smooth or continuous complex line bundles: One has to replace in the exponential sequence the sheaves of holomorphic functions by the corresponding sheaves of smooth or continuous functions.

Proposition 10.11 prepares the proof that a global differential form on X, which represents the Chern class of a given line bundle, can be computed by means of an object from differential geometry, a Hermitian metric of the line bundle.

Proposition 10.11 (Hermitian metric on line bundles). *Consider a compact Riemann surface and a holomorphic line bundle*

$$p : L \to X.$$

Assume that the line bundle is represented with respect to the open covering $\mathscr{U} = (\mathscr{U}_\alpha)_{\alpha \in I}$ *by the cocycle*

$$\xi = (\xi_{\alpha\beta}) \in Z^1(\mathscr{U}, \mathscr{O}^*).$$

1. *Then a cochain exists*
$$h = (h_\alpha) \in C^0(\mathscr{U}, \mathscr{E}_{\mathbb{R}_+}^*)$$
with $\mathscr{E}_{\mathbb{R}_+}^*$ *the multiplicative sheaf of positive smooth functions on* X*, such that the coboundary satisfies*
$$\delta h = (|\xi_{\alpha\beta}|^2) \in Z^1(\mathscr{U}, \mathscr{E}_{\mathbb{R}_+}^*), \text{ i.e. } |\xi_{\alpha\beta}|^2 = \frac{h_\beta}{h_\alpha} \text{ for all pairs } \alpha,\ \beta \in I.$$
2. *The cochains from part 1*
$$h = (h_\alpha) \in C^0(\mathscr{U}, \mathscr{E}_{\mathbb{R}_+}^*)$$
correspond bijectively to the smooth Hermitians metrics *of the line bundle: If*
$$p : L \to X$$
has the atlas
$$(\phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{C})_{\alpha \in I}$$
then the Hermitian metric
$$< -, - >: L \times L \to \mathbb{C}$$
is defined on the fibres
$$L_x,\ x \in U_\alpha,$$
by using the linear chart ϕ_α *as the map*
$$L_x \times L_x \to \mathbb{C},\ (z, w) \mapsto < z, w > := u_\alpha \cdot h_\alpha(x) \cdot \overline{v}_\alpha,$$
if
$$\phi_\alpha(z) = (x, u_\alpha) \text{ and } \phi_\alpha(w) = (x, v_\alpha).$$

Proof. 1. *The sheaf $\mathscr{E}_+^*$ is acyclic*: Denote by

$$\mathscr{E}_{\mathbb{R}} \subset \mathscr{E}$$

the subsheaf of smooth real-valued functions on X. By means of a partition of unity one proves

$$H^1(X, \mathscr{E}_{\mathbb{R}}) = 0$$

in an analogous way as Theorem 6.14. The real exponential function defines a sheaf isomorphism

$$exp : \mathscr{E}_{\mathbb{R}} \xrightarrow{\simeq} \mathscr{E}_{\mathbb{R}\,+}^{\,*},\ f \mapsto exp\ f.$$

Hence

$$H^1(X, \mathscr{E}_{\mathbb{R}\,+}^{\,*}) = H^1(X, \mathscr{E}_{\mathbb{R}}) = 0,$$

which finishes the proof.

2. *Independence of the linear charts*: In order to prove that the metric $< -, - >$ derived from the cochain $(h_\alpha) \in C^0(\mathscr{U}, \mathscr{E}_{\mathbb{R}\,+}^{\,*})$ is well-defined we consider a second linear chart

$$\phi_\beta : p^{-1}(U_\beta) \xrightarrow{\simeq} U_\beta \times \mathbb{C}.$$

If $x \in U_\alpha \cap U_\beta$ and

$$(z, w) \in L_x \times L_x \text{ with } \phi_\beta(z) = (x, u_\beta) \text{ and } \phi_\beta(w) = (x, \mathrm{v}_\beta)$$

then

$$u_\alpha = \xi_{\alpha\beta}(x) \cdot u_\beta \text{ and } \mathrm{v}_\alpha = \xi_{\alpha\beta}(x) \cdot \mathrm{v}_\beta.$$

Hence

$$u_\alpha \cdot h_\alpha(x) \cdot \overline{\mathrm{v}}_\alpha = \xi_{\alpha\beta}(x) \cdot u_\beta \cdot \frac{h_\beta(x)}{|\xi_{\alpha\beta}(x)|^2} \cdot \overline{\mathrm{v}}_\beta(x) \cdot \overline{\xi}_{\alpha\beta}(x) = u_\beta \cdot h_\beta(x) \cdot \overline{\mathrm{v}}_\beta,\ q.e.d.$$

Apparently the construction from Proposition 10.11 allows to introduce also on vector bundles a smooth Hermitian metric.

The proof of Proposition 10.13 will make use of the topological result from Lemma 10.12.

For an open covering $\mathscr{U} = (U_i)_{i \in I}$ of a topological space X the *nerve* $N(\mathscr{U})$ of $\mathscr{U}$ is the family of all finite subsets $J \subset I$ with support

$$supp\ J := \bigcap_{i \in J} U_i \neq \emptyset.$$

Lemma 10.12 (Coverings with contractible intersections). *Any open covering of a compact Riemann surface X has a refinement $\mathscr{S} = (S_i)_{i \in I}$ with*

$$\bigcap_{i \in J} S_i$$

contractible for all $J \in N(\mathscr{U})$.

Proof. One uses the existence of a triangulation of X. Let V be its set of vertices. The vertex star S_v of a given vertex v is the union of all singular 2-simplices containing the vertex v. One may assume that each vertex star S_v is contained in an open set of the original covering. Set

$$\mathscr{S} := (S_\mathrm{v}^\circ)_{\mathrm{v} \in V}$$

with S_v° denoting the open kernel. Each 2-simplex of X is contractible to each of its vertices. Hence the intersection of the vertex stars of finitely many vertices is contractible to each of these vertices, q.e.d.

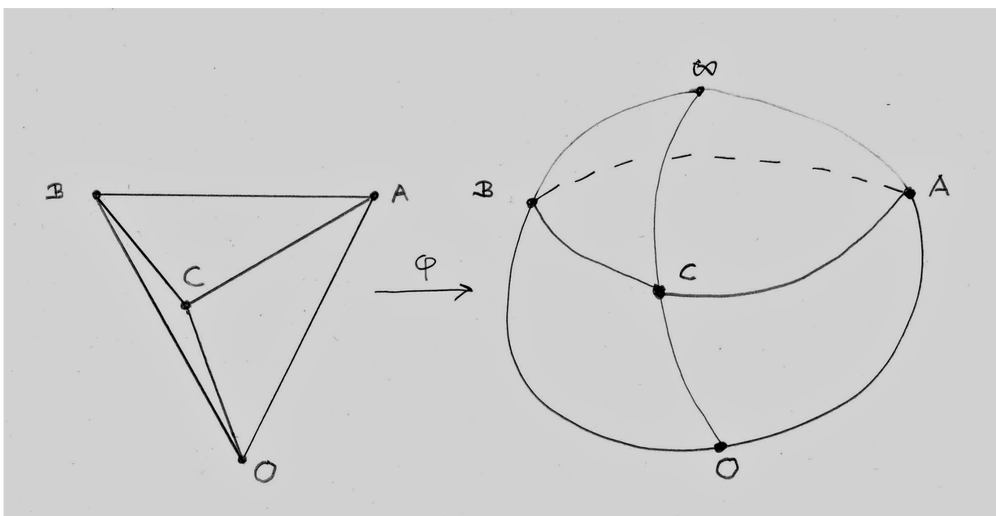


Fig. 10.1 Triangulation of $\mathbb{P}^1$

Figure 10.1 shows a triangulation of the Riemann surface $\mathbb{P}^1$ by singular simplices obtained from the 2-faces of the tetraeder. They are the 2-simplices of the standard simplex in $\mathbb{R}^3$. The triangulation has

- the vertices or 0-simplices O, A, B, C
- the 1-simplices OA, OB, OC, AB, AC, BC
- the 2-simplices OAC, OBC, OAB, ABC.

Typical vertex stars are

$$S_O = OAC, OBC, OAB \text{ and } S_A = OAC, OAB, ABC$$

E.g., the intersection

$$S_O \cap S_A = OAC, OAB$$

is contractible.

The Chern class of a line bundle is an element from $H^2(X,\mathbb{Z})$ and maps via the canonical map

$$j : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}).$$

Like the Chern morphism also the de Rham isomorphism

$$deRham : H^2(X,\mathbb{C}) \to Rh^2(X) = \frac{H^0(X,\mathscr{E}^2)}{im[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]}$$

is independent from any sign convention. The de Rham isomorphism is induced by the de Rham resolution, the proof is analogous to the proof of Theorem 6.15. The composition of both maps allows to represent Chern classes by de Rham classes.

Proposition 10.13 (Mapping Chern classes to de Rham classes). *Consider a compact Riemann surface and a holomorphic line bundle*

$$p : L \to X.$$

Assume that the line bundle is represented with respect to the open covering $\mathscr{U} = (U_\alpha)_{\alpha \in I}$ by the cocycle

$$\xi = (\xi_{\alpha\beta}) \in Z^1(\mathscr{U},\mathscr{O}^*).$$

Any Hermitian metric of L

$$h = (h_\alpha) \in C^0(\mathscr{U},\mathscr{E}_{\mathbb{R}^*_+})$$

defines a global differentialform

$$\zeta := \left(\frac{1}{2\pi i} \cdot d''d' \, log \, h_\alpha\right) \in Z^0(\mathscr{U},\mathscr{E}^{1,1}) = H^0(X,\mathscr{E}^{1,1})$$

with the property: The form ζ is the de Rham representative of the Chern class $c_1(L)$, i.e. the composition of Chern morphism and de Rham morphism

$$\left[H^1(X,\mathscr{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{j} H^2(X,\mathbb{C}) \xrightarrow{deRham} Rh^2(X) = \frac{H^0(X,\mathscr{E}^2)}{im[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]}\right]$$

satisfies

$$[\zeta] = (deRham \circ j \circ c_1)(L) \in Rh^2(X).$$

Proof. The proof has to follow the composition of the de Rham isomorphism with the Chern morphism. Therefore we have to make explicit the two morphisms: The Chern morphism is a connecting morphism and the de Rham isomorphism results as the composition of two connecting morphisms. Hence we have to consider three times the construction "climbing stairs".

1. *Chern morphism*: Passing over to a suitable refinement, Lemma 10.12 assures that we may assume all intersections

$$U_\alpha \cap U_\beta,\ \alpha,\ \beta \in I,$$

simply connnected. Hence on each $U_\alpha \cap U_\beta$ exists a branch of the logarithm. The final result is independent of the choosen branch because it depends only on the derivation of the logarithm. The exponential sequence induces by backward stair climbing

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\ e\ } \mathcal{O}^* \longrightarrow 0$$

$$\begin{array}{ccc} (c_{\alpha\beta\gamma}) \in Z^2(\mathcal{U},\mathbb{Z}) \longrightarrow (c_{\alpha\beta\gamma}) \in Z^2(\mathcal{U},\mathcal{O}) & & \\ \uparrow \delta & & \\ \left(\dfrac{1}{2\pi i}\cdot log\ \xi_{\alpha\beta}\right) \in C^1(\mathcal{U},\mathcal{O}) & \xrightarrow{e} & (\xi_{\alpha\beta}) \in Z^1(\mathcal{U},\mathcal{O}^*) \end{array}$$

the Chern morphism as the connecting morphism

$$c_1 = \partial : H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z}),\ \partial(\xi_{\alpha\beta}) := (c_{\alpha\beta\gamma}) \in Z^2(\mathcal{U},\mathbb{Z})$$

with

$$c_{\alpha\beta\gamma} = \frac{1}{2\pi i}\cdot (log\ \xi_{\beta\gamma} - log\ \xi_{\alpha\gamma} + log\ \xi_{\alpha\beta})$$

2. *De Rham isomorphism*: The de Rham isomorphism is part of the following commutative diagram

$$\begin{array}{ccc} & H^0(X,\mathscr{E}^2) & \\ \swarrow^{\partial_2} & & \searrow \\ H^1(X,\mathscr{F}) & & Rh^2(X) = \dfrac{H^0(X,\mathscr{E}^2)}{im[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]} \\ \searrow_{\partial_1} & & \nearrow_{deRham} \\ & H^2(X,\mathbb{C}) & \end{array}$$

Here ∂_1 and ∂_2 are the connecting morphisms in the splitting of the de Rham resolution

$$0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to 0$$

into the two short exact sequences

$$0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{F} \to 0,\ \mathscr{F} := ker[\mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2],$$

and

$$0 \to \mathscr{F} \to \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \to 0$$

We investigate how to obtain an inverse image

$$\zeta \in H^0(X,\mathscr{E}^2)$$

of a given element

$$[(c_{\alpha\beta\gamma})] \in H^2(X,\mathbb{C})$$

under the composition of the two surjective connecting morphisms

$$H^0(X,\mathscr{E}^2) \xrightarrow{\partial_2} H^1(X,\mathscr{F}) \xrightarrow{\partial_1} H^2(X,\mathbb{C}).$$

- For $\partial_1 : H^1(X,\mathscr{F}) \to H^2(X,\mathbb{C}),\ (f_{\alpha\beta}) \mapsto (c_{\alpha\beta\gamma})$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{E} \xrightarrow{\quad d \quad} \mathscr{F} \longrightarrow 0$$

$$\begin{array}{ll} (c_{\alpha\beta\gamma}) \in Z^2(\mathscr{U},\mathbb{C}) \to (c_{\alpha\beta\gamma}) = \delta\eta \in Z^2(\mathscr{U},\mathscr{E}) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \uparrow \delta & \\ \qquad\qquad\qquad\quad \eta = (\eta_{\alpha\beta}) \in C^1(\mathscr{U},\mathscr{E}) \xrightarrow{d} (f_{\alpha\beta}) \in Z^1(\mathscr{U},\mathscr{F}) & \end{array}$$

- For $\partial_2 : H^0(X,\mathscr{E}^2) \to H^1(X,\mathscr{F}),\ \zeta \mapsto (f_{\alpha\beta})$:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \longrightarrow 0$$

$$\begin{array}{ccc} (f_{\alpha\beta}) \in Z^1(\mathscr{U},\mathscr{F}) \rightarrow (f_{\alpha\beta}) = \delta\tau \in Z^1(\mathscr{U},\mathscr{E}^1) & & \\ \uparrow \delta & & \\ \tau = (\tau_\alpha) \in C^0(\mathscr{U},\mathscr{E}^1) & \xrightarrow{d} & \zeta \in Z^0(\mathscr{U},\mathscr{E}^2) \end{array}$$

3. *Constructing the differential form* $\zeta \in H^0(X,\mathscr{E}^2)$: According to part 2) the task is to find for a Chern cocycle, i.e. the cocycle of the Chern class of a line bundle,

$$(c_{\alpha\beta\gamma}) \in Z^2(\mathscr{U},\mathbb{Z}) \text{ with class } c = [(c_{\alpha\beta\gamma})] \in H^2(X,\mathbb{C})$$

an element $\zeta \in H^0(X,\mathscr{E}^2)$ satisfying

$$deRham(c) = [\zeta] \in Rh^2(X).$$

This task has two steps:

- To find
$$\eta \in C^1(\mathscr{U},\mathscr{E}) \text{ with } \delta\eta = (c_{\alpha\beta\gamma})$$
- and after setting
$$(f_{\alpha\beta}) := d\eta \in Z^1(\mathscr{U}\mathscr{F})$$
to find
$$\tau \in C^0(\mathscr{U},\mathscr{E}^1) \text{ with } \delta\tau = (f_{\alpha\beta})$$

The first step has been solved by part 1) of the proof: The cocycle

$$(c_{\alpha\beta\gamma}) \in Z^2(\mathscr{U},\mathbb{Z})$$

from part 1) with

$$c_{\alpha\beta\gamma} = \frac{1}{2\pi i} \cdot (log\ \xi_{\beta\gamma} - log\ \xi_{\alpha\gamma} + log\ \xi_{\alpha\beta})$$

represents the Chern class of the line bundle. Hence we may set

$$\eta = \left(\eta_{\alpha\beta} := \frac{1}{2\pi i} \cdot log\ \xi_{\alpha\beta}\right) \in C^1(\mathscr{U},\mathscr{E})$$

To solve the second step we define

$$\tau = \left(\tau_\alpha := \frac{1}{2\pi i} \cdot\ d'log\ h_\alpha\right) \in C^0(\mathscr{U},\mathscr{E}^1)$$

We claim:

$$\delta\tau = (f_{\alpha\beta})$$

For the proof recall from Proposition 10.11

$$h_\beta = h_\alpha \cdot |\xi_{\alpha\beta}|^2 = h_\alpha \cdot \xi_{\alpha\beta} \cdot \overline{\xi}_{\alpha\beta}.$$

Hence

$$log\ h_\beta = log\ h_\alpha + log\ \xi_{\alpha\beta} + log\ \overline{\xi}_{\alpha\beta}$$

Because each function $\overline{\xi}_{\alpha\beta}$ is anti-holomorphic, which implies $d'\overline{\xi}_{\alpha\beta} = 0$, we obtain

$$d' log\ h_\beta = d' log\ h_\alpha + d'\ log\ \xi_{\alpha\beta}$$

Hence

$$\tau_\beta = \tau_\alpha + \frac{1}{2\pi i} \cdot d'\ log\ \xi_{\alpha\beta},$$

i.e.

$$f_{\alpha\beta} = \frac{1}{2\pi i} \cdot d'\ log\ \xi_{\alpha\beta} = \tau_\beta - \tau_\alpha$$

which finishes the task.

For all $\alpha \in I$

$$\tau_\alpha \in \mathscr{E}^{1,0}(U_\alpha) \implies d\tau_\alpha = d''\tau_\alpha \in \mathscr{E}^{1,1}(U_\alpha)$$

As a consequence

$$\zeta := d\tau = \left(d\tau_\alpha = \frac{1}{2\pi i} \cdot d''d'\ log\ h_\alpha \right) \in Z^0(\mathscr{U}, \mathscr{E}^2) = H^0(X, \mathscr{E}^2),\ q.e.d.$$

Apparently the form $\zeta \in H^0(X, \mathscr{E}^{1,1})$ constructed in the proof of Proposition 10.13 as an inverse image of the Chern class of a given line bundle is not uniquely determined. De Rham's theorem shows that ζ is determined up to the image

$$im[d : H^0(X, \mathscr{E}^1) \to H^0(X, \mathscr{E}^2)].$$

Remark 10.14 (Curvature form of a connection). Proposition 10.13 represents the Chern class of a holomorphic line bundle on a Riemann surface X

$$p : L \to X$$

by the form

$$\zeta := \left(\frac{1}{2\pi i} \cdot d''d'\ log\ h_\alpha \right) \in Z^0(\mathscr{U}, \mathscr{E}^{1,1}) = H^0(X, \mathscr{E}^{1,1})$$

which derives from a Hermitian metric $h = (h_\alpha)$ of the line bundle.

1. *Connection and curvature in the smooth category*: The form

$$\zeta \in H^0(X, \mathscr{E}^{1,1})$$

is the curvature form of a connection D of L. For a short introduction to connections of vector bundles see Chern [4, §5 and §6]: Consider a smooth manifold X and a smooth complex vector bundle of rank $= k$

$$p : F \to X$$

Denote by $\mathscr{F}$ the sheaf of smooth sections of F. A connection of the vector bundle is a $\mathbb{C}$-linear map

$$D : \mathscr{F} \to \mathscr{E}^1 \otimes_{\mathscr{E}} \mathscr{F}$$

which satisfies the product rule

$$D(f \cdot \sigma) = df \otimes \sigma + f \cdot D\sigma$$

for sections $f \in \Gamma(U, \mathscr{E})$ and $\sigma \in \Gamma(U, \mathscr{F})$ with open $U \subset X$. A *frame* of F on an open set $U \in X$ is a family

$$s = (s_1, ..., s_k)^{\perp}$$

of sections $s_j \in \Gamma(U, \mathscr{F}),\ j = 1, ..., k$ such that for all $x \in U$ the family

$$(s_1(x), ..., s_k(x))$$

is a basis of the fibre F_x. The restriction $D|\Gamma(U, \mathscr{F})$ is determined by the element

$$D(s) = \omega \otimes s$$

with a matrix of 1-forms

$$\omega \in \Gamma\left(U, \mathscr{E}^1 \otimes_{\mathbb{C}} M(k \times k, \mathbb{C})\right)$$

named the *connection matrix* of the frame s. Consider a second frame $s' \in \Gamma(U, \mathscr{F})$ with connection matrix

$$\omega' \in \Gamma\left(U, \mathscr{E}^1 \otimes_{\mathbb{C}} M(k \times k, \mathbb{C})\right)$$

Then

$$s' = g \cdot s$$

with a matrix

$$g \in \Gamma\left(U, \mathscr{E} \otimes_{\mathbb{C}} GL(k, \mathbb{C})\right)$$

and the connection forms transform as

$$\omega' \cdot g = dg + g \cdot \omega \in \Gamma(U, \mathscr{E}^1 \otimes_{\mathbb{C}} \mathbb{C}^n)$$

Taking the exterior derivative of the last equation shows that the corresponding *curvature matrices* of the frame, matrices of 2-forms,

$$\Omega := d\omega - \omega \wedge \omega,\ \Omega' := d\omega' - \omega' \wedge \omega' \in \Gamma(U, \mathscr{E}^2 \otimes_{\mathscr{E}} M(k \times k, \mathbb{C}))$$

transform as $\Omega' \cdot g = g \cdot \Omega$ or

$$\Omega' = g \cdot \Omega \cdot g^{-1}$$

2. *Smooth line bundle*: For a smooth line bundle L on a smooth manifold X, i.e. $k = 1$, the construction from part 1) simplifies due to the Abelian context. The curvature matrix is a single, global form

$$\Omega \in H^0(X, \mathscr{E}^2).$$

If the line bundle L is represented by the cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathscr{U}, \mathscr{E}^*)$ then the linear charts of L satisfy

$$\phi_\alpha = \xi_{\alpha\beta} \cdot \phi_\beta$$

and the corresponding frames

$$s_\alpha := \phi_\alpha^{-1}(-,1) \text{ and } s_\beta := \phi_\beta^{-1}(-,1)$$

transform as

$$s_\beta = \xi_{\alpha\beta} \cdot s_\alpha.$$

A connection on $L|U_\alpha$ is determined by a single connection form

$$\omega_\alpha \in \Gamma(U_\alpha, \mathscr{E}^1),$$

which defines the value of the connection on the frame

$$D(s_\alpha) := \omega_\alpha \cdot s_\alpha$$

3. *Holomorphic line bundle on a Riemann surface* X: A Hermitian metric $h = (h_\alpha)$ on L satisfies

$$h_\beta = |g|^2 h_\alpha \text{ with } g = \xi_{\alpha\beta}.$$

The 1-forms

$$\omega_\alpha := d' log\ h_\alpha \in \Gamma(U_\alpha, \mathscr{E}^{1,0}),\ \alpha \in I,$$

are well-defined due to the splitting induced from the complex structure on X

$$d = d' + d''$$

The local forms define conncections

$$D_\alpha : \mathscr{L}|U_\alpha \to \mathscr{E}^1 \otimes_{\mathscr{E}} \mathscr{L}|U_\alpha,\ D(s_\alpha) := \omega_\alpha \otimes s_\alpha$$

which glue to a global connection

$$D : \mathscr{L} \to \mathscr{E}^1 \otimes_{\mathscr{E}} \mathscr{L}$$

because they transform as

$$\omega_\beta = d' log\ h_\alpha = d' log(h_\alpha \cdot g \cdot \overline{g}) = d' log\ h_\alpha + d' log\ g + d' log \overline{g} =$$

$$\omega_\alpha + d' log\ g = \omega_\alpha + dlog\ g = \omega_\alpha + dg \cdot g^{-1}$$

or

$$\omega_\beta \cdot g = dg + g \cdot \omega_\alpha$$

The last equation is the transformation rule of connection matrices. The connection forms are of type $(1,0)$. Therefore

$$d\omega_\alpha = d'' \omega_\alpha \text{ and } \omega_\alpha \wedge \omega_\alpha = 0$$

and the local curvature forms

$$\Omega | U_\alpha := d\omega_\alpha - \omega_\alpha \wedge \omega_\alpha = d'' \omega_\alpha = d'' d' log\ h_\alpha \in \Gamma(U_\alpha, \mathscr{E}^{1,1})$$

glue to a global curvature form

$$\Omega \in H^0(X, \mathscr{E}^{1,1})$$

Proposition 10.13 shows that the $(1,1)$-form

$$\frac{1}{2\pi i} \cdot \Omega \in H^0(X, \mathscr{E}^{1,1}),$$

which derives from the curvature form of the connection D, is a de Rham representant of the Chern class $c_1(L) \in H^2(X, \mathbb{Z})$.

Any Riemann surface X is an oriented smooth manifold. Hence integration of 2-forms along X is well-defined. Lemma 10.15 defines the integration of the de Rham classes.

Lemma 10.15 (Integration of de Rham classes). *Let X be a compact Riemann surface. Then the integration map*

$$int : Rh^2(X) \to \mathbb{C},\ [\zeta] \mapsto \iint_X \zeta,$$

is well-defined and surjective.

Proof. Well-definedness follows from Stokes' theorem. Integrating a global volume form on the oriented smooth manifold (X, Σ_{smooth}) shows that the map is not zero, hence surjective, q.e.d.

Note. The map from Lemma 10.15 is even an isomorphism.

Lemma 10.15 suggests to interpret Chern classes $c_1(L) \in H^2(X,\mathbb{Z})$ as integers. There are two $\mathbb{Z}$-linear isomorphisms

$$H^2(X,\mathbb{Z}) \to \mathbb{Z}$$

which differ by the factor (-1). Hence the choice of the isomorphism is a matter of convention. Theorem 10.16 and Definition 10.17 show how to calibrate the integer by the degree of the divisor of sections of $\mathscr{L}$.

Theorem 10.16 (Chern class and section of a line bundle). *Let X be a compact Riemann surface and consider a holomorphic line bundle*

$$p : L \to X.$$

Then the divisors of all non-zero meromorphic sections of L have the same degree. The degree can be identified with the Chern class $c_1(L)$ in a canonical way: The composition

$$H^2(X,\mathbb{Z}) \xrightarrow{j} H^2(X,\mathbb{C}) \xrightarrow{deRham} Rh^2(X) \xrightarrow{int} \mathbb{Z}$$

satisfies for all non-zero $s \in H^0(X,\mathscr{M}_L)$

$$(-1) \cdot (int \circ deRham \circ j \circ c_1)(L) = deg\ div(s).$$

Proof. i) *The Hermitian metric (h_α) induced by the section*: Consider a non-zero meromorphic section $s \in H^0(X,\mathscr{M}_L)$. We choose a linear atlas of the line bundle

$$(\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C})_{\alpha \in I}$$

with “matrix” functions

$$\xi_{\alpha\beta} \in \mathscr{O}^*(U_{\alpha\beta}),\ U_{\alpha\beta} := U_\alpha \cap U_\beta,\ \alpha,\beta \in I.$$

Then the section s is represented by a cochain

$$(s_\alpha)_{\alpha \in I} \in C^0(\mathscr{U},\mathscr{M})$$

of meromorphic functions which transform according to

$$s_\alpha = \xi_{\alpha\beta} \cdot s_\beta \text{ on } U_{\alpha\beta}.$$

The divisor

$$D := div\ s \in Div(X)$$

has finite support. It is the sum of multiples of finitely many point divisors

$$D = \sum_{p \in supp\ D} n_p \cdot P$$

We may assume that each $p \in supp\ D$ has an open neighbourhood $V(p)$ with the following properties

- The sets $V(p),\ p \in supp\ D$, are pairwise disjoint,
- for each $p \in supp\ D$ holds
$$V(p) \subset U_{\alpha(p)}$$
for a suitable $\alpha(p) \in I$,
- and for all $\beta \neq \alpha(p)$
$$V(p) \cap U_\beta = \emptyset$$

We carry over the transformation rule from meromorphic functions to smooth functions by modifying the family (s_α): For each $\alpha \in I$ the restricted function

$$s_\alpha | (U_\alpha \setminus supp\ D)$$

is holomorphic and has no zeros. In $(U_\alpha \cap U_\beta) \setminus supp\ D$

$$|s_\alpha|^2 = |\xi_{\alpha\beta}|^2 \cdot |s_\beta|^2$$

or

$$\left|\frac{1}{s_\beta}\right|^2 = |\xi_{\alpha\beta}|^2 \cdot \left|\frac{1}{s_\alpha}\right|^2$$

as a cocycle relation of smooth functions without zeros. For each $\alpha \in I$ a smooth modification of s_α within the set $V(p) \in U_\alpha$ provides a smooth function

$$h_\alpha \in \mathscr{E}^*_+(U_\alpha)$$

satisfying on $U_\alpha \setminus V(p)$

$$h_\alpha = \left|\frac{1}{s_\alpha}\right|^2.$$

As a consequence, on $U_\alpha \cap U_\beta$ holds

$$h_\beta = |\xi_{\alpha\beta}|^2 \cdot h_\alpha$$

Hence the cochain

$$(h_\alpha)_{\alpha \in I} \in C^0(\mathscr{U}, \mathscr{E}^*_+)$$

defines a Hermitian metric on the line bundle according to Proposition 10.11.

ii) *Integrating the de Rham class of the Chern class*: Proposition 10.13 implies:

$$(int \circ deRham \circ j \circ c_1)(L) = \iint_X \zeta$$

with

$$\zeta \in H^0(X, \mathscr{E}^2)$$

satisfying for all $\alpha \in I$

$$\zeta | U_\alpha = \frac{1}{2\pi i} \cdot d''d' \ log \ h_\alpha.$$

Each point in

$$\left(U_\alpha \setminus \bigcup_{p \in supp \ D} V(p) \right)^\circ$$

has an open, simply connected neighbourhood where

$$d''d' log \ h_\alpha = -d''d' log \ s_\alpha - d''d' log \ \bar{s}_\alpha = 0$$

because s_α is holomorphic. Hence the restriction satisfies

$$d''d' log \ h_\alpha | \left(U_\alpha \setminus \bigcup_{p \in supp \ D} V(p) \right) = 0$$

and

$$\iint_X \zeta = \frac{1}{2\pi i} \sum_{p \in supp \ D} \iint_{V(p)} d''d' log \ h_\alpha$$

Because

$$d''d' = dd'$$

Stokes' theorem implies for each $p \in supp \ D$, $\alpha = \alpha(p) \in I$ and positive oriented boundary $\partial V(p)$

$$\iint_{V(p)} d''d' log \ h_\alpha = \iint_{V(p)} dd' log \ h_\alpha = \int_{\partial V(p)} d' log \ h_\alpha =$$

$$= -\int_{\partial V(p)} (d' log \ s_\alpha + d' log \ \bar{s}_\alpha) = -\int_{\partial V(p)} d' log \ s_\alpha = -\int_{\partial V(p)} \frac{d' s_\alpha}{s_\alpha}$$

$$= -2\pi i \cdot res \left(\frac{s'_\alpha}{s_\alpha} \right) = -2\pi i \cdot ord(s_\alpha; \ p)$$

Hence

$$(-1) \cdot (int \circ deRham \circ j \circ c_1)(L) = -int(\zeta) = -\iint_X \zeta = \sum_{p \in supp \ D} ord(s_{\alpha(p)}; \ p) = deg \ div(s), \ q.e.d.$$

Definition 10.17 (Chern number). The *Chern number* of a line bundle

$$p : L \to X$$

on a compact Riemann surface X is the integer

$$c_1^{int}(L) := (-1) \cdot (int \circ deRham \circ j \circ c_1)(L) \in \mathbb{Z}.$$

The *Chern number* of the corresponding invertible sheaf $\mathscr{L}$ is

$$c_1^{int}(\mathscr{L}) := c_1^{int}(L).$$

Note the minus-sign in Definition 10.17 for the Chern number. It results from the minus-sign which is obtained in the formula from Theorem 10.16. Lemma 10.15 shows that the Chern number does not depend on the construction of a particular $\zeta \in H^0(X, \mathscr{E}^2)$ in the proof of Proposition 10.13.

Example 10.18 (Chern number of the twisted sheaf). On the projective space

$$X := \mathbb{P}^1$$

consider the line bundle

$$p : L \to X$$

with sheaf of holomorphic sections the twisted sheaf

$$\mathscr{L} = \mathscr{O}(1).$$

It's cocycle with respect to the standard covering is defined by

$$\xi_{01} := \frac{z_1}{z_0} \in \mathscr{O}^*(U_{01}),$$

see Example 2.11. A Hermitian metric on the line bundle is given by the family

$$(h_0, h_1) \in C^0(\mathscr{U}, \mathscr{E}_+^*)$$

with

$$h_i := \frac{|z_i|^2}{|z_0|^2 + |z_1|^2} \text{ for } i = 0, 1,$$

because

$$|\xi_{01}|^2 = \frac{h_1}{h_0}.$$

With respect to the coordinate $z = \dfrac{z_1}{z_0}$ on U_0 we have

$$h_0(z) = \frac{1}{1+|z|^2}$$

and

$$d''d'log\ h_0(z) = \frac{-1}{(1+|z|^2)^2}\, d\bar{z} \wedge dz$$

Hence

$$\zeta = \frac{1}{2\pi i} \cdot d''d'log\ h_0(z) = \frac{1}{2\pi i} \cdot \frac{(-1)}{(1+|z|)^2)^2}\, d\bar{z} \wedge dz$$

It implies

$$\iint_{\mathbb{P}^1} \zeta = \frac{1}{2\pi i} \cdot \iint_{\mathbb{C}} \frac{(-1)}{(1+|z|)^2)^2}\, d\bar{z} \wedge dz = \frac{1}{2\pi i} \cdot \iint_{\mathbb{C}} \frac{1}{(1+|z|)^2)^2}\, dz \wedge d\bar{z} =$$

$$= \frac{1}{2\pi i} \cdot \iint_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^2} \cdot (-2i)\ dx \wedge dy = -\frac{1}{\pi} \cdot \int_0^{2\pi} \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr \wedge d\theta =$$

$$= -\frac{1}{\pi} \cdot 2\pi \cdot \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr = -2 \cdot \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr = -2 \cdot \frac{1}{2} = -1$$

because the integrand has the primitive

$$-\frac{1}{2} \cdot \frac{1}{1+r^2}$$

We obtain

$$(-1) \cdot (int \circ deRham \circ j \circ c_1)(L) = -\iint_{\mathbb{P}^1} \zeta = 1,$$

hence according to Definition 10.17

$$c_1^{int}(\mathscr{L}) = 1 \in \mathbb{Z}.$$

The sheaf $\mathscr{O}(1)$ has the holomorphic section

$$s = (s_0, s_1) \in Z^0(\mathscr{U}, \mathscr{O})$$

with

$$s_0 := \frac{z_1}{z_0} \text{ and } s_1 := 1.$$

It satisfies

$$s_1 = \xi_{10} \cdot s_0.$$

Apparently

$$div\ s \in Div(X)$$

is the point divisor of the point $0 \in \mathbb{P}^1$ and

$$deg(div\ s) = 1.$$

Corollary 10.19 (Degree of a divisor and Chern number of its line bundle). *Consider a compact Riemann surface X. Then the following diagram commutes:*

$$\begin{array}{ccccc}
H^0(X,\mathscr{D}) & \xrightarrow{\partial} & H^1(X,\mathscr{O}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\
\downarrow & & & & \downarrow j \\
deg & & & & H^2(X,\mathbb{C}) \\
\downarrow & & & & \downarrow deRham \\
\mathbb{C} & & \xleftarrow{\quad int \quad} & & Rh^2(X)
\end{array}$$

Fig. 10.2 Canonical morphisms around the Chern class

In particular for any divisor $D \in Div(X)$

$$deg\ D = c_1^{int}(\mathscr{O}_D) \in \mathbb{Z}.$$

Proof. Represent a given divisor $D \in H^0(X,\mathscr{D})$ by a cochain

$$(D_i = div\ f_i)_i \in C^0(\mathscr{U},\mathscr{M})$$

with respect to a suitable open covering $\mathscr{U} = (U_i)_{i\in I}$. Theorem 10.7 implies:

- The class $\partial D \in H^1(X,\mathscr{O}^*)$ represents the line bundle $\mathscr{O}_D^\vee$
- The sheaf $\mathscr{O}_D$ has the meromorphic section $(f_i)_i \in C^0(\mathscr{U},\mathscr{M})$.

Theorem 10.16 implies

$$deg\ D = deg(div\ f_i)_i = (-1)\cdot(int \circ deRham \circ j \circ c_1)(\mathscr{O}_D)$$

Because

$$\partial D = \mathscr{O}_D^\vee \text{ and } c_1(\mathscr{O}_D^\vee) = -c_1(\mathscr{O}_D)$$

we obtain

$$deg\ D = (int \circ deRham \circ j \circ c_1 \circ \partial)(D)$$

The claim about the Chern number follows from Theorem 10.7

$$\partial D = \mathscr{O}_D^\vee,\ q.e.d.$$

The commutative diagram from Figure 10.2 relates several canonical morphisms from different categories: The connecting morphism

$$H^0(X,\mathscr{D}) \xrightarrow{\partial} H^1(X,\mathscr{O}^*)$$

induces a map

$$Cl(X) = \frac{H^0(X,\mathscr{D})}{im[H^0(X,\mathscr{M}^*) \to H^0(X,\mathscr{D})]} \longrightarrow H^1(X,\mathscr{O}^*)$$

between objects with a holomorphic structure, while the Chern morphism

$$H^1(X,\mathscr{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z})$$

maps objects with a holomorphic structure to topological invariants. But the Chern morphism is analogously defined for smooth or even continuous line bundles. The morphism, originating from a ring extension,

$$H^2(X,\mathbb{Z}) \xrightarrow{j} H^2(X,\mathbb{C})$$

maps topological invariants. The de Rham morphism

$$H^2(X,\mathbb{C}) \xrightarrow{deRham} Rh^2(X)$$

is a morphism in the smooth category. Integration

$$Rh^2(X) \xrightarrow{int} \mathbb{C}$$

maps the smooth category to numerical invariants. The degree map

$$H^0(X,\mathscr{D}) \xrightarrow{deg} \mathbb{C}$$

induces a map

$$Cl(X) \longrightarrow \mathbb{Z} \subset \mathbb{C}$$

which relates holomorphic entities to numerical invariants.

10.3 The divisor of a line bundle

The aim of the present section is to complete the proof that line bundles and divisors are equivalent concepts on a compact Riemann surface. As a consequence the basic theorems of Riemann-Roch and Serre carry over to line bundles.

Lemma 10.20 (Meromorphic sections of a line bundle). *Let X be a Riemann surface. The following statements are equivalent:*

i) Vanishing

$$H^1(X,\mathscr{M}^*) = 0.$$

ii) Surjectivity of the connecting morphism of the divisor sequence

$$\partial : H^0(X, \mathscr{D}) \to H^1(X, \mathscr{O}^*)$$

iii) Each line bundle on X has a non-zero meromorphic section.

Proof. The divisor sequence on X

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{D} \to 0$$

induces the long exact sequence

$$... \to H^0(X, \mathscr{M}^*) \to H^0(X, \mathscr{D}) \xrightarrow{\partial} H^1(X, \mathscr{O}^*) \to H^1(X, \mathscr{M}^*) \to 0 = H^1(X, \mathscr{D})$$

with the last group vanishing due to Proposition 8.7.

i) $\iff$ ii). Condition i) implies at once the surjectivity of ∂, i.e. condition ii).

For the converse assume the surjectivity of ∂. Then we have the factorization

$$[H^0(X, \mathscr{D}) \xrightarrow{\partial} H^1(X, \mathscr{O}^*) \to H^1(X, \mathscr{M}^*) \to 0] =$$
$$= [H^0(X, \mathscr{D}) \xrightarrow{\partial} H^1(X, \mathscr{O}^*) \to 0 \to H^1(X, \mathscr{M}^*) \to 0]$$

which implies $H^1(X, \mathscr{M}^*) = 0$, i.e. condition i).

ii) $\iff$ iii). Assume condition ii). The surjectivity of ∂ states that any line bundle L has the form

$$L = \partial D$$

with a divisor $D \in Div(X)$. If D is defined with respect to the covering $\mathscr{U} = (U_i)_{i \in I}$ by the cocycle

$$(div\ f_i)_i \in Z^0(\mathscr{U}, \mathscr{D})$$

then the cochain

$$\left(\frac{1}{f_i}\right)_i \in C^0(\mathscr{U}, \mathscr{M}^*)$$

defines a non-zero meromorphic section of the line bundle $L = \partial D$, see Theorem 10.7.

For the converse consider an arbitrary line bundle L represented by a cocycle

$$\xi = (g_{ij})_{ij \in I} \in Z^1(\mathscr{U}, \mathscr{O}^*)$$

and assume that L has a non-zero meromorphic section

$$f \in H^0(X, \mathscr{M}_L).$$

Due to Proposition 10.5 the section provides a non-zero cochain

$$(f_i)_{i\in I} \in C^0(\mathscr{U},\mathscr{M}^*)$$

satisfying

$$f_i = g_{ij} \cdot f_j$$

According to Theorem 10.7 the divisor

$$D := \left(div\ \frac{1}{f_i}\right)_i \in Z^0(\mathscr{U},\mathscr{D})$$

maps to

$$L = \partial D,\ q.e.d.$$

We now show that on any compact Riemann surface the third condition of Lemma 10.20 is satisfied, and hence all conditions are satisfied. The proof of Theorem 10.22 is not trivial, because until now the Riemann-Roch theorem has not been proven for arbitrary invertible sheaves, only for sheaves of the form $\mathscr{O}_D,\ D \in Div(X)$.

Remark 10.21 (Finiteness theorem for line bundles). On a compact Riemann surface X holds for any line bundle $\mathscr{L} \in Pic(X)$

$$dim\ H^0(X,\mathscr{L}) < \infty \text{ and } dim\ H^1(X,\mathscr{L}) < \infty.$$

This result generalizes the Finiteness-Theorem 7.16 for the structure sheaf, i.e. the trivial line bundle. The result can be proved in the same way by introducing Hilbert space topologies and using the compactness of the restriction map, see [16, Kap. IV, § 3, Satz 7].

Theorem 10.22 (Existence of meromorphic sections of line bundles). *Let X be a compact Riemann surface and consider a point $p \in X$. Then each line bundle*

$$pr : L \to X$$

has a non-zero meromorphic section with a pole at most at the point p.

Proof. Let $\mathscr{L}$ be the invertible sheaf of holomorphic sections of the line bundle L. Consider the point divisor $P \in Div(X)$, an effective divisor, corresponding to $p \in X$ and the invertible sheaf

$$\mathscr{L}_1 := \mathscr{O}_P.$$

Due to Corollary 10.19

$$1 = deg\ P = c_1^{int}(\mathscr{L}_1)$$

Any non-zero constant defines a holomorphic non-zero section

$$s \in H^0(X, \mathscr{L}_1).$$

Theorem 10.16 implies

$$deg(div\ s) = c_1^{int}(\mathscr{L}_1) = 1$$

Hence s has exactly one zero at p, and this zero has order $= 1$.

i) *Calculating* $\chi(\mathscr{L} \otimes_{\mathscr{O}} \mathscr{L}_1^{\otimes n})$: The tensor powers of the non-zero section

$$s \in H^0(X, \mathscr{L}_1)$$

define by multiplication an injective sheaf morphism

$$\mathscr{O} \xrightarrow{\cdot s^{\otimes n}} \mathscr{L}_1^{\otimes n}$$

After tensorizing with $\mathscr{L}$ we obtain a short exact sheaf sequence

$$0 \to \mathscr{L} \xrightarrow{\cdot s^{\otimes n}} \mathscr{L} \otimes_{\mathscr{O}} \mathscr{L}_1^{\otimes n} \to \mathscr{H} \to 0$$

The quotient sheaf $\mathscr{H}$ is a *skyscraper sheaf* $\mathscr{H}$ with stalks

$$\mathscr{H}_q = \begin{cases} \mathbb{C}^n & q = p \\ 0 & q \neq p \end{cases}$$

It satisfies

$$H^0(X, \mathscr{H}) \simeq \mathbb{C}^n \text{ and } H^1(X, \mathscr{H}) = 0.$$

The corresponding exact cohomology sequence is

$$0 \to H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) \to \mathbb{C}^n \to H^1(X, \mathscr{L}) \to H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) \to 0$$

The vanishing of the alternate cross sum

$$0 = dim\, H^0(X, \mathscr{L}) - dim\, H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) + n - dim\, H^1(X, \mathscr{L}) + dim\, H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n})$$

implies

$$dim\ H^0(X, \mathscr{L}) - dim\ H^1(X, \mathscr{L}) = dim\ H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - n$$

We have

$$c_1^{int}(\mathscr{L}_1^{\otimes n}) = n \cdot c_1^{int}(\mathscr{L}_1) = n$$

The product rule

$$c_1^{int}(\mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) = c_1^{int}(\mathscr{L}) + c_1^{int}(\mathscr{L}_1^{\otimes n})$$

and subtracting $c_1^{int}(\mathscr{L})$ on both sides of the equation give

$$dim\ H^0(X, \mathscr{L}) - dim\ H^1(X, \mathscr{L}) - c_1^{int}(\mathscr{L}) =$$

$$= dim\, H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - dim\, H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - c_1^{int}(\mathscr{L}_1^{\otimes n}) - c_1^{int}(\mathscr{L})$$

or

$$\chi(\mathscr{L}) - c_1^{int}(\mathscr{L}) = \chi(\mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - c_1^{int}(\mathscr{L} \otimes \mathscr{L}_1^{\otimes n}).$$

Hence for all $n \in \mathbb{N}$

$$const(\mathscr{L}) := \chi(\mathscr{L} \cdot \mathscr{L}_1^{\otimes n}) - c_1^{int}(\mathscr{L} \cdot \mathscr{L}_1^{\otimes n})$$

is a constant independent from n.

ii) *Proving* $H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) \neq 0$ *for suitable* n: Assume on the contrary: For all $n \in \mathbb{N}$

$$H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) = 0.$$

We obtain from part i)

$$const(\mathscr{L}) = \chi(\mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - c_1^{int}(\mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) = -dim\, H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) - c_1^{int}(\mathscr{L} \otimes \mathscr{L}_1^{\otimes n})$$

i.e.

$$-dim\, H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) = c_1^{int}(\mathscr{L}) + n \cdot c_1^{int}(\mathscr{L}_1) + const(\mathscr{L}),$$

a contradiction because

$$dim\, H^1(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n}) \geq 0$$

and

$$c_1^{int}(\mathscr{L}_1) = 1$$

Hence there exists $n_0 \in \mathbb{N}$ and a holomorphic non-zero section

$$\tau \in H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n_0}).$$

iii) *Proving* $H^0(X, \mathscr{M}_L) \neq \{0\}$: Set

$$\rho := \frac{\tau}{s^{\otimes n_0}} \in H^0(X, \mathscr{M}_L)$$

with the holomorphic non-zero sections

$$\tau \in H^0(X, \mathscr{L} \otimes \mathscr{L}_1^{\otimes n_0}) \text{ and } s^{\otimes n_0} \in H^0(X, \mathscr{L}_1^{\otimes n_0})$$

from part ii) and i). Then ρ is a non-zero meromorphic section of $\mathscr{L}$ with a pole at most at the point p, q.e.d.

Theorem 10.22 has a series of important corollaries. The first, Theorem 10.23, is a converse to Theorem 10.7.

Theorem 10.23 (Divisor of a line bundle). *Let X be a compact Riemann surface. For each holomorphic line bundle L on X exists a divisor* $D \in Div(X)$ *satisfying*

$$\mathscr{L} \simeq \mathscr{O}_D.$$

Proof. Due to Theorem 10.22 each line bundle on X has a non-zero meromorphic section. Then Lemma 10.20 implies the surjectivity of the connecting morphism

$$\partial : H^0(X, \mathscr{D}) \to H^1(X, \mathscr{O}^*).$$

Due to Theorem 10.7: If

$$L := -\partial D \in H^1(X, \mathscr{O}^*)$$

then

$$\mathscr{L} \simeq \mathscr{O}_D, \ q.e.d.$$

Theorems 10.7 and 10.23 show: On a compact Riemann surface X divisors and line bundles are in bijective correspondence. More precisely: Let the *Picard group* of X

$$Pic(X) := H^1(X, \mathscr{O}^*)$$

be the multiplicative group of isomorphism classes of line bundles, then the map

$$Cl(X) \to Pic(X), \ [D] \mapsto [\mathscr{O}_D],$$

is an isomorphism between the divisor class group and the Picard group of X. Taking into account Proposition 10.6 the bijective correspondence extends to isomorphism classes of invertible sheaves.

Proposition 10.24 (Examples of Picard groups).

1. *The Picard group of a compact Riemann surface X splits as*

$$Pic(X) = Pic_0(X) \oplus \mathbb{Z}$$

with

$$Pic_0(X) := \{[L] \in Pic(X) : \ c_1(L) = 0\}$$

the subgroup of line bundles with zero Chern class.

2. *The projective space has*

$$Pic_0(\mathbb{P}^1) = \{0\}$$

3. *A torus $T = \mathbb{C}/\Lambda$ has*

$$Pic_0(T) \simeq T$$

Proof. 1. The exponential sequence from Proposition 2.10

$$0 \to \mathbb{Z} \xrightarrow{j} \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0,$$

Theorem 10.23 and the isomorphy

$$H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$$

imply the exact sequence in cohomology

$$0 \to Pic_0(X) \to H^1(X, \mathscr{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \simeq \mathbb{Z} \to 0$$

The sequence splits because the $\mathbb{Z}$-module $\mathbb{Z}$ on the right-hand side is free. Hence

$$Pic(X) \simeq Pic_0(X) \oplus \mathbb{Z}$$

2. For the projective space
$$H^1(\mathbb{P}^1, \mathscr{O}) = 0$$
due to Proposition 6.17. Hence the Chern morphism is also injective and therefore an isomorphism.

3. For a torus T one has
$$Pic_0(T) := ker[H^1(T, \mathscr{O}^*) \xrightarrow{c_1} \mathbb{Z}] = im[H^1(T, \mathscr{O}) \to H^1(T, \mathscr{O}^*)] \simeq$$
$$= \frac{H^1(T, \mathscr{O})}{ker[H^1(T, \mathscr{O}) \to H^1(T, \mathscr{O}^*)]} = \frac{H^1(T, \mathscr{O})}{im[H^1(T, \mathbb{Z}) \to H^1(T, \mathscr{O})]} = \mathbb{C}/\Lambda = T, \ q.e.d.$$

Proposition 10.24 shows that the only line bundles on $\mathbb{P}^1$ are the line bundles $\mathscr{O}(n),\ n \in \mathbb{Z}$, of the twisted sheaves. By Grothendieck's *splitting theorem* any holomorphic vector bundle of rank $k \geq 2$ on $\mathbb{P}^1$ is the direct sum of k line bundles, which are uniquely determined, see [14, Kap. VII, §8, Abschn. 5]. Grothendiecks theorem shows once more the importance of the twisted sheaves.

Remark 10.25 (Jacobi variety). Proposition 10.24 shows that the Picard group of a compact Riemann surface X is already determined by its subgroup $Pic_0(X)$ of line bundles with zero Chern class. The result

$$Pic_0(T) \simeq T$$

for a torus T generalizes to higher degree. Consider a compact Riemann surface X of genus $g = g(X)$. We identify line bundles and divisors on X, in particular

$$Pic_0(X) \simeq Cl_0(X)$$

with
$$Cl_0(X) := Div_0(X)/\mathscr{M}^*(X) \subset Cl(X)$$
the subgroup of divisor classes of divisors D with $deg\ D = 0$. Any choice of a basis $(\omega_1, ..., \omega_g)$ of $H^0(X, \Omega^1)$ defines a g-dimensional torus $Jac(X)$, the *Jacobi manifold* of X: Consider the *period lattice*

$$Per(\omega_1,...,\omega_g) := \left\{ \left(\int_\gamma \omega_1, ..., \int_\gamma \omega_g \right) : \ \gamma \in H_1(X) \right\} \subset \mathbb{C}^g$$

and set

$$Jac(X) := \mathbb{C}^n / Per(\omega_1, ..., \omega_g).$$

There exists a canonical map

$$Jac : Div_0(X) \to Jac(X)$$

defined as follows: For a given divisor $D \in Div_0(X)$ and a 1-chain $c \in C_1(X)$ with boundary

$$\delta c = D.$$

Then

$$Jac(X) := \left[\left(\int_c \omega_1, ..., \int_c \omega_g \right) \right] \in Jac(X).$$

The map *Jac* is well-defined and fits into the exact sequence of Abelian groups

$$0 \to \mathscr{M}^*(X) \to Cl_0(X) \xrightarrow{Jac} Jac(X) \to 0$$

For the details of the construction see [8, §21].

Proposition 10.26 (Vanishing of $H^1(X, \mathscr{M}^*)$). *On a compact Riemann surface X holds*

$$H^1(X, \mathscr{M}^*) = 0.$$

Proof. The proof of the claim follows from Theorem 10.23 and Lemma 10.20, q.e.d.

Theorem 10.27 (Riemann Roch theorem for line bundles). *Consider a compact Riemann surface X. For any line bundle L on X the Euler characteristics satisfies*

$$\chi(\mathscr{L}) := dim\ H^0(X, \mathscr{L}) - dim\ H^1(X, \mathscr{L}) = 1 - g(X) + c_1^{int}(\mathscr{L}).$$

Proof. The claim follows from the Theorems 8.10, 10.23, 10.16 and Corollary 10.19, q.e.d.

Theorem 10.27 shows that in the proof of Theorem 10.22 the number $const(\mathscr{L})$ has the value

$$const(\mathscr{L}) = 1 - g(X)$$

which is independent from $\mathscr{L}$.

Theorem 10.28 (Serre duality for line bundles). *Consider a compact Riemann surface X with dualizing sheaf ω. For any line bundle L on X the residue form*

$$(-,-) : [H^0(X, \mathscr{L}^\vee \otimes_{\mathscr{O}} \omega) \times H^1(X, \mathscr{L}) \to H^1(X, \omega) \xrightarrow{res} \mathbb{C}]$$

is a dual pairing.

Proof. The proof follows from the Theorems 9.10, 10.23 and 10.16, q.e.d.

Chapter 11
Maps to projective spaces

11.1 The projective space $\mathbb{P}^n$

Generalizing the 1-dimensional projective space $\mathbb{P}^1$ from Definition 1.4 we introduce the higher-dimensional complex projective spaces. At this point we leave the domain of Riemann surfaces and presuppose some basic results from the theory of complex-analytic manifolds.

Definition 11.1 (*n*-dimensional projective space). For $n \in \mathbb{N}$ consider the quotient

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$$

with respect to the equivalence relation

$$z = (z_0, ..., z_n) \sim w = (w_0, ..., w_n) \ :<=> \exists \lambda \in \mathbb{C}^* : \ w = \lambda \cdot z \in \mathbb{C}^{n+1} \setminus \{0\}$$

and the canonical projection onto equivalence classes

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n, \ z \mapsto [z].$$

For $z = (z_0, ..., z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ the expression

$$(z_0 : ... : z_n) := \pi(z) \in \mathbb{P}^n$$

is named the *homogeneous coordinate* of $\pi(z)$. The quotient topology is named the Euclidean topology of $\mathbb{P}^n$. It is a countable Hausdorff topology. The space $\mathbb{P}^n$ is named the n-dimensional complex *projective space*. If not stated otherwise we consider $\mathbb{P}^n$ equipped with the Euclidean topology.

Definition 11.2 (Standard atlas of $\mathbb{P}^n$). Consider the n-dimensional projective space $\mathbb{P}^n$. Its *standard atlas* is the family

$$\mathscr{U} := (U_i)_{i=0,...,n}$$

with the open sets

$$U_i := \{(z_0 : ... : z_n) \in \mathbb{P}^n : \; z_i \neq 0\}, \; i = 0, ..., n.$$

For each $i = 0, ..., n$ the i-th *standard chart* is the homeomorphism

$$\phi_i : U_i \to \mathbb{C}^n, \; (z_0 : ... : z_n) \mapsto \left(\frac{z_0}{z_i}, ..., \frac{z_{i-1}}{z_i}, \widehat{\frac{z_i}{z_i}}, \frac{z_{i+1}}{z_i}, ..., \frac{z_n}{z_i} \right), \text{ the “hat” indicates omission of the term.}$$

Remark 11.3 (Projective space and projective-algebraic geometry).

1. *The twisted sheaves*: From the view point of complex analysis of several variables the projective space $\mathbb{P}^n$ is an n-dimensional compact complex manifold: The transformation between two charts ϕ_j and ϕ_i with $i < j$ is the holomorphic map on an open subset of $\mathbb{C}^n$

$$\psi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j),$$

$$\left(w_0, ..., w_{j-1}, \hat{1}, w_{j+1}, ..., w_n\right) \mapsto \left(\frac{w_0}{w_i}, ..., \frac{w_{i-1}}{w_i}, \hat{1}, \frac{w_{i+1}}{w_i}, ..., \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, ..., \frac{w_n}{w_i} \right),$$

see [19, Chap. 9.1].

The definition of line bundles on Riemann surfaces from Definition 10.1 literally carries over to line bundles

$$p : L \to X$$

on complex manifolds X. The group of isomorphy classes of line bundles on X equals the first cohomology group $H^1(X, \mathscr{O}^*)$.

The twisted sheaf $\mathscr{O}(1)$ on $\mathbb{P}^n$ is the sheaf of holomorphic sections of the line bundle

$$p : L \to \mathbb{P}^n$$

defined with respect to the standard charts by the cocycle

$$g = \left(g_{ij} := \frac{z_j}{z_i} \right) \in Z^1(\mathscr{U}, \mathscr{O}^*).$$

Analogously to Example 2.11 the sections of $\mathscr{O}(1)$ are the linear polynomials

$$H^0(\mathbb{P}^n, \mathscr{O}(1)) \subset \mathbb{C}[z_0, ..., z_n].$$

2. *Zariski topology*: Algebraic geometry provides the set $\mathbb{P}^n$ not with the Euclidean topology but with the *Zariski topology*: A subset $A \subset \mathbb{P}^n$ is closed with respect to the Zariski topology iff A is the zero set of a set of homogenous polynomials from $\mathbb{C}[z_0,...,z_n]$. The Zariski topology is coarser than the Euclidean topology.

3. *The functor Proj*: Following Grothendieck's definition $\mathbb{P}^n$ is a functor, see [18, Chap. II.2]. When defined on the category of commutative rings R with unit then

$$\mathbb{P}^n_R := Proj\ R[z_0,...,z_n]$$

is the *projective spectrum* of the graded polynomial ring

$$S := R[z_0,...,z_n]$$

with the ususal grading by the degree of monomials. The projective spectrum is the set of all homogeneous prime ideals

$$\mathfrak{p} \subset S \text{ with } \bigoplus_{d \geq 1} S_d \not\subset \mathfrak{p},$$

provided with the Zariski topology of schemes. Here $S_d \subset S$ denotes the subset of homogeneous polynomials of degree $= d$. The space $\mathbb{P}^n_R$ is the basic example of a projective scheme. In the particular case

$$R := \mathbb{C}$$

we obtain the complex n-dimensional projective scheme $\mathbb{P}^n_{\mathbb{C}}$. Its subspace of closed points forms the set $\mathbb{P}^n$ equipped with the Zariski topology.

11.2 Very ample invertible sheaves and projective embeddings

The first condition, that an invertible sheaf $\mathscr{L}$ on a Riemann surface X has to satisfy in order to define a map into a projective space

$$X \to \mathbb{P}^n$$

is to be base-point-free.

Definition 11.4 (Base-point, globally generated sheaf). Consider a Riemann surface X and an invertible sheaf $\mathscr{L}$ on X.

1. A point $p \in X$ is a *base-point* of $\mathscr{L}$ if for all sections $s \in H^0(X,\mathscr{L})$ the germ at p satisfies
$$s_p \in \mathfrak{m}_p\mathscr{L} \text{ i.e. } s(p) = 0.$$

2. The sheaf is *generated by global sections* or *globally generated* or *base-point-free* if it has no base-points, i.e. if for any $p \in X$ exists a section $s \in H^0(X, \mathscr{L})$ such that the germ $s_p \in \mathscr{L}_p$ generates the stalk $\mathscr{L}_p$ as $\mathscr{O}_p$-module. The latter condition is equivalent to $s(p) \neq 0$.

Remark 11.5 (Inverse image sheaf).

1. *The sheaves f^{-1} and f^**: Any continuous map

$$f : X \to Y$$

between two topological spaces induces a contravariant functor

$$f^{-1} : \underline{Sheaf}_Y \to \underline{Sheaf}_X$$

For a sheaf $\mathscr{F}$ on Y one defines the sheaf $f^{-1}\mathscr{F}$ on X as the sheafification of the presheaf defined by the direct limit

$$(f^{-1}\mathscr{F})(U) := \lim_{\text{open } V \supset f(U)} \mathscr{F}(V), \text{ open } U \subset X.$$

For a point $x \in X$ the stalk satisfies

$$(f^{-1}\mathscr{F})_x \simeq \mathscr{F}_{f(x)}$$

For a holomorphic map between to complex manifolds

$$f : X \to Y$$

one defines the contravariant functor *inverse image*

$$f^* : \mathscr{O}_Y - \underline{mod} \to \mathscr{O}_X - \underline{mod}$$

as

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

2. *Adjointness*: Any holomorphic map $f : X \to Y$ between two complex manifolds induces two functors between the category of module sheaves: The covariant direct image

$$f_* : \mathscr{O}_X - \underline{mod} \to \mathscr{O}_Y - \underline{mod}$$

and the contravariant inverse image

$$f^* : \mathscr{O}_Y - \underline{mod} \to \mathscr{O}_X - \underline{mod}$$

These functors are adjoint : For any pair with an $\mathscr{O}_Y$-module $\mathscr{F}$ and an $\mathscr{O}_X$-module $\mathscr{G}$ there exists a group isomorphism

$$Hom_{\mathscr{O}_X}(f^*\mathscr{F},\mathscr{G}) \xrightarrow{\simeq} Hom_{\mathscr{O}_Y}(\mathscr{F},f_*\mathscr{G})$$

See [18, Chap. II.5].

Proposition 11.6 (Holomorphic maps to projective space). *Consider a compact Riemann surface X and a globally generated invertible sheaf $\mathscr{L}$. After choosing a basis*

$$(s_i)_{i=0,...,n} \in H^0(X,\mathscr{L})$$

the map

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n,\ p \mapsto (s_0(p) : ... : s_n(p)),$$

is well-defined and induces an isomorphism

$$\phi_{\mathscr{L}}^*(\mathscr{O}(1)) \simeq \mathscr{L}.$$

Proof. We set

$$\phi := \phi_{\mathscr{L}}.$$

i) *Definition*: We consider the definition of ϕ in a suitable open neighbourhood U of p. On U we may identify the invertible sheaf $\mathscr{L}$ with the structure sheaf $\mathscr{O}$. Hence sections from $H^0(U,\mathscr{L})$ are holomorphic functions. Because $\mathscr{L}$ is globally generated there is an index $j \in \{0,...,n\}$ with

$$s_j(p) \neq 0.$$

Hence the point

$$(s_0(p) : ... : s_n(p)) \in \mathbb{P}^n$$

is well-defined and independent of the choice of the chart and the isomorphism

$$\mathscr{L}|U \simeq \mathscr{O}|U.$$

Apparently the map ϕ is holomorphic on U.

ii) *Pullback of the twisted line bundle*: For each $i = 0,...,n$ the sets

$$X_i := \phi^{-1}(U_i) = \{x \in X :\ (s_i)_x \notin \mathfrak{m}_x\mathscr{L}_x\}$$

form an open covering $(X_i)_{i=0,...,n}$ of X because $\mathscr{L}$ is globally generated.
For $i = 0,...,n$ the canonical morphisms between the rings of local sections

$$\mathscr{O}(1)(U_i) \to \phi_*(\mathscr{L})(U_i) = \mathscr{L}(X_i) \text{ induced by } z_i|U_i \mapsto s_i|X_i$$

define a canonical sheaf morphism on $\mathbb{P}^n$

$$\mathscr{O}(1) \to \phi_*\mathscr{L}$$

By adjointness the corresponding sheaf morphism on X

$$\phi^*(\mathscr{O}(1)) \to \mathscr{L},$$

is an isomorphism because it is an isomorphism on the stalks of the two line bundles: If for a given $i \in \{0,...,n\}$

$$x \in X_i \subset X \text{ and } y := \phi(x) \in U_i \subset \mathbb{P}^n$$

then

$$(\phi^*\mathscr{O}(1))_x = \mathscr{O}(1)_y \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}_{X,x} \to \mathscr{L}_x,\ (z_i)_y \otimes 1 \mapsto (s_i)_x,\ q.e.d.$$

Remark 11.7 (Holomorphic map into projective space). For a compact Riemann surface X and a globally generated invertible sheaf $\mathscr{L}$ on X the definition of the map

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$

from Proposition 11.6 depends on the choice of a basis of the finite dimensional vector space $H^0(X,\mathscr{L})$. Following Grothendieck an intrinsic definition is obtained by the dual construction: Let

$$\mathbb{P}(H^0(X,\mathscr{L})^\vee)$$

be the projective space of linear functionals on the $n+1$-dimensional vector space $H^0(X,\mathscr{L})$, i.e. the lines through the origin in the dual space $H^0(X,\mathscr{L})^\vee$. The corresponding map is defined as

$$\phi_{\mathscr{L}} : X \to \mathbb{P}(H^0(X,\mathscr{L})^\vee),\ p \mapsto \lambda_p,$$

with

$$\lambda_p : H^0(X,\mathscr{L}) \to \mathscr{L}(p) := \mathscr{L}_p/\mathfrak{m}_p\mathscr{L}_p \simeq \mathbb{C},\ s \mapsto [s_p].$$

Here the value of $s(p) \in \mathbb{C}$ depends on the choice of the isomorphy between $\mathscr{L}$ and the structure sheaf $\mathscr{O}$ in a neighbourhood of p, but the class of λ_p is independent of this choice.

Theorem 11.8 provides a geometric criterion that the map provided by a globally generated invertible sheaf $\mathscr{L}$ is a closed embedding.

Theorem 11.8 (Projective embedding induced by an invertible sheaf). *Consider a compact Riemann surface X and an invertible, globally generated sheaf $\mathscr{L}$ on X. Then the induced map*

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$

is a closed embedding iff $\mathscr{L}$ satisfies both of the following properties:

1. Separating points*: For any two distinct points $p \neq q \in X$ exists a section*

$$s \in H^0(X,\mathscr{L}) \text{ with } s(p) \neq 0 \text{ but } s(q) = 0$$

or vice versa.

2. Separating tangent vectors*: For all $x \in X$ the map*

$$d_{\mathscr{L},x} : \{s \in H^0(X, \mathscr{L}) : \ s_x \in \mathfrak{m}_x \mathscr{L}_x\} \to \mathfrak{m}_x \mathscr{L}_x / \mathfrak{m}_x^2 \mathscr{L}_x, \ s \mapsto [s_x],$$

is surjective.

In Theorem 11.8 note the isomorphy of $\mathscr{O}_x$-modules

$$\mathfrak{m}_x \mathscr{L}_x / \mathfrak{m}_x^2 \mathscr{L}_x \simeq (\mathfrak{m}_x / \mathfrak{m}_x^2) \otimes_{\mathscr{O}_x} \mathscr{L}_x = T_x^1 \otimes_{\mathscr{O}_x} \mathscr{L}_x$$

The map

$$d_{\mathscr{L},x} : \{s \in H^0(X, \mathscr{L}) : \ s_x \in \mathfrak{m}_x \mathscr{L}_x\} \to T_x^1 \otimes_{\mathscr{O}_x} \mathscr{L}_x$$

is induced by the total differential d_x of holomorphic functions: A section

$$s \in H^0(X, \mathscr{L}) \text{ with } s_x \in \mathfrak{m}_x \mathscr{L}_x$$

factorizes in a suitable neighbourhood U of x as

$$s = f \cdot s_1$$

with a holomorphic function $f \in \mathscr{O}(U)$ satisfying $f_x \in \mathfrak{m}_x$ and a holomorphic section $s_1 \in \mathscr{L}(U)$. Then

$$d_{\mathscr{L},x}(s) = d_x f \otimes s_1 \in T_x^1 \otimes_{\mathscr{O}_x} \mathscr{L}_x$$

The map is well-defined: If also

$$s = g \cdot s_2 \text{ and } ord(g;\ x) \geq ord(f;\ x)$$

then

$$g = h \cdot f$$

with $h \in \mathscr{O}(U)$. Hence

$$d_x g \otimes s_2 = d_x(h \cdot f) \otimes s_2 = (h \cdot d_x f) \otimes s_2 = d_x f \otimes h \cdot s_2 = d_x f \otimes s_1$$

Set

$$y := \phi(x) \in \mathbb{P}^n$$

and consider analogously the map

$$d_{\mathscr{O}(1),y} : \{s \in H^0(\mathbb{P}^n, \mathscr{O}(1)) : \ s(y) = 0\} \to \Omega^1_{\mathbb{P}^n,y} \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}(1)_y$$

The proof of Theorem 11.8 will employ the following commutative diagram

The left-hand side of the diagram in Figure 11.1 is induced by the pullback of sections

$$\phi^* : H^0(\mathbb{P}^n, \mathscr{O}(1)) \to H^0(X, \phi^*(\mathscr{O}(1))) = H^0(X, \mathscr{L}), \ s \mapsto \sigma := s \circ \phi.$$

$$\begin{array}{ccc}
\{s \in H^0(\mathbb{P}^n, \mathscr{O}(1)) :\ s(y) = 0\} & \xrightarrow{d_{\mathscr{O}(1),y}} & T^1_{\mathbb{P}^n,y} \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}(1)_y \\
\Big\downarrow \phi^* & & \Big\downarrow \phi^*_{T^1} \\
\{\sigma \in H^0(X, \mathscr{L}) :\ \sigma(x) = 0\} & \xrightarrow{d_{\mathscr{L},x}} & T^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x
\end{array}$$

Fig. 11.1 Separating tangent vectors

which is surjective by definition. Also the upper horizontal map

$$d_{\mathscr{O}(1),y} : \{s \in H^0(\mathbb{P}^n, \mathscr{O}(1)) :\ s(y) = 0\} \to T^1_{\mathbb{P}^n,y} \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}(1)_y$$

is surjective. The right-hand side of the diagram considers germs of rings and modules at the points $x \in X$ and $y \in \mathbb{P}^n$. It is best understood from the view point of commmutative algebra: Consider the rings

$$A := \mathscr{O}_{\mathbb{P}^n,y} \text{ and } B := \mathscr{O}_{X,x}$$

By means of the ring morphism,

$$A \to B,\ f \mapsto f \circ \phi,$$

the pullback of holomorphic functions, B is an A-module. In addition consider the A-modules

$$\Omega_A := \Omega^1_{\mathbb{P}^n,y} \text{ and } F := \mathscr{O}(1)_y$$

as well as the B-modules

$$\Omega_B := \Omega^1_{X,x} \text{ and } F \otimes_A B = \mathscr{L}_x$$

The pullback of differential forms is a B-morphism

$$\Omega_A \otimes_A B \to \Omega_B$$

and the composition

$$\Omega_A \to \Omega_A \otimes_A B \to \Omega_B$$

is an A-morphism. Tensoring by the A-module F gives an A-morphism

$$\Omega_A \otimes_A F \to \Omega_B \otimes_A F = \Omega_B \otimes_B (B \otimes_A F)$$

which is after tensoring with the residue field $k(x)$

$$T^1_{\mathbb{P}^n,y} \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}(1)_y \to T^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$

Hence the right-hand side of the diagram is induced by the base change

$$A \to B$$

and the pullback of differential forms

$$\Omega_A \otimes_A B \to \Omega_B.$$

Proof (Theorem 11.8). Proposition 11.6 implies that the map

$$\phi := \phi_{\mathscr{L}} : X \to \mathbb{P}^n$$

is well-defined and the pullback satisfies

$$\mathscr{L} = \phi^*(\mathscr{O}(1)).$$

i) *Assume ϕ to be a closed embedding*: Then w.l.o.g.

$$\phi : X \hookrightarrow \mathbb{P}^n$$

is the injection. Let

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$$

be the canonical projection. For a given point $p \in \mathbb{P}^n$ the hyperplanes $H \subset \mathbb{C}^{n+1}$ which contain the line

$$L_p := \pi^{-1}(p) \subset \mathbb{C}^{n+1}$$

correspond bijectively to the non-zero sections

$$s_H \in H^0(\mathbb{P}^n, \mathscr{O}(1))$$

with

$$s_H(p) = 0 :$$

For the proof one represents the hyperplane as

$$H = ker\ \lambda$$

with a non-zero linear functional

$$\lambda : \mathbb{C}^{n+1} \to \mathbb{C},\ (x_0, ..., x_n) \mapsto \sum_{j=0}^{n} \lambda_j \cdot x_j$$

Then the section

$$s_H := \sum_{j=0}^{n} \lambda_j \cdot z_j \in H^0(X, \mathscr{L})$$

satisfies $s_H(p) = 0$.

- Consider two distinct points and their homogenenous coordinates

$$p = \pi(u_0) = (z_0 : ... : z_n) \neq q = \pi(u_n) = (w_0 : ... : w_n) \in X$$

By assumption the two vectors

$$u_0 := (z_0, ..., z_n),\ u_{n+1} := (w_0, ..., w_n) \in \mathbb{C}^{n+1}$$

are linearly independent. Hence they extend to a basis

$$(u_i)_{i=0,...,n}$$

of $\mathbb{C}^{n+1}$ and the hyperplane

$$H := span_{\mathbb{C}} \langle u_i :\ i = 0, ..., n-1 \rangle \subset \mathbb{C}^{n+1}$$

contains u_0 but not u_n. The restriction

$$\sigma := s_H|X \in H^0(X, \mathscr{L})$$

satisfies

$$\sigma(p) = 0 \text{ and } \sigma(q) \neq 0.$$

Therefore $\mathscr{L}$ separates points.

- Consider a point $x \in X \subset \mathbb{P}^n$. Then in Figure 11.1 the map

$$\phi^*_{\Omega^1} : \Omega^1_{\mathbb{P}^n,y} \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}(1)_y \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$

is surjective: The map

$$\Omega_A \otimes_A B \to \Omega_B$$

is surjective, and the surjectivity of

$$A \to B$$

implies by tensoring with Ω_A the surjectivity of

$$\Omega_A \to \Omega_A \otimes_A B.$$

Hence the composition

$$\phi^*_{T^1} \circ d_{\mathscr{O}(1),y}$$

is surjective. As a consequence

$$d_{\mathscr{L},x} : \{\sigma \in H^0(X, \mathscr{L}) :\ \sigma(x) = 0\} \to T^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$

is surjective which proves that $\mathscr{L}$ separates tangent vectors at $x \in X$.

ii) *Assume that $\mathscr{L}$ separates points and tangent vectors*:

- Separating points implies that ϕ is injective. The map is continuous and X is compact. Hence the image
$$\phi(X) \subset \mathbb{P}^n$$
is compact and a posteriori closed.

- To show that $\phi : X \to \mathbb{P}^n$ is an immersion we consider the diagram from Figure 11.1. By assumption
$$d_{\mathscr{L},x} : \{\sigma \in H^0(X,\mathscr{L}) : \ \sigma(x) = 0\} \to \Omega^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$
is surjective. Hence the composition
$$d_{\mathscr{L},x} \circ \phi^*$$
is surjective, which implies the surjectivity of
$$\phi^*_{T^1} : T^1_{\mathbb{P}^n,y} \otimes_{\mathscr{O}_{\mathbb{P}^n,y}} \mathscr{O}(1)_y \to T^1_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{L}_x$$
As a consequence the map
$$\Omega_A \otimes_A B \to \Omega_B$$
is surjective which proves that ϕ is an immersion at the point $x \in X$, q.e.d.

Definition 11.9 (Very ample invertible sheaf). A globally generated invertible sheaf $\mathscr{L}$ on a compact Riemann surface X is *very ample* if the induced map
$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$
is an embedding.

A globally generated invertible sheaf $\mathscr{L}$ on X has enough sections to define a holomorphic map
$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n.$$
If $\mathscr{L}$ is very ample then there are enough sections that $\phi_{\mathscr{L}}$ is even an embedding. Due to the compactness of X its image under an embedding is always closed.

Notation 11.10. *For an invertible sheaf $\mathscr{L}$ on a Riemann surface X and a divisor $D \in Div(X)$ we denote by*
$$\mathscr{L}_D := \mathscr{L} \otimes_{\mathscr{O}} \mathscr{O}_D$$
the invertible sheaf of meromorphic sections of $\mathscr{L}$ which are multiples of the divisor $-D$.

Notation 11.10 has been employed already in Definition 8.3 with the sheaf $\mathscr{L} = \Omega^1$.

Consider an invertible sheaf $\mathscr{L}$ on a compact Riemann surface X. Proposition 11.11 states a numerical criterion for the dimension of the vector spaces

$$H^0(X, \mathscr{L}_D),\ D \in Div(X),$$

which assures that the map

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$

is well-defined and a closed embedding. This criterion is very helpful in the applications because the dimension on the vector spaces in question can often be computed by using the theorem of Riemann-Roch in combination with Serre duality.

Proposition 11.11 (Very-ampleness criterion). *Consider a compact Riemann surface X. For an invertible sheaf $\mathscr{L}$ on X are equivalent:*

- *The sheaf $\mathscr{L}$ is very ample.*

- *For the point divisors $P,\ Q \in Div(X)$ of two arbitrary, not necessarily distinct points $p,\ q \in X$ holds*

$$dim\ H^0(X, \mathscr{L}_{-(P+Q)}) = dim\ H^0(X, \mathscr{L}) - 2.$$

Proof. i) *Assume the validity of the dimension formula*: The formula implies for any two point divisors $P,\ Q \in Div(X)$

$$H^0(X, \mathscr{L}_{-(P+Q)}) \subsetneq H^0(X, \mathscr{L}_{-P}) \subsetneq H^0(X, \mathscr{L})$$

and each proper inclusion has codimension $= 1$ because it is defined by one linear equation.

- The equation
$$h^0(X, \mathscr{L}_{-P}) = h^0(X, \mathscr{L}) - 1.$$
states that the kernel of the evaluation
$$H^0(X, \mathscr{L}) \to \mathscr{L}_p/\mathfrak{m}_p\mathscr{L}_p \simeq \mathbb{C},\ s \mapsto [s_p],$$
has codimension $= 1$. Hence p is not a base-point of $\mathscr{L}$. As a consequence, $\mathscr{L}$ is globally generated and
$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$
is well-defined.

- The equation
$$H^0(X, \mathscr{L}_{-(P+Q)}) \subsetneq H^0(X, \mathscr{L}_{-P})$$
implies that for any pair of distinct points $p,\ q \in X$ there exists a section

$$s \in H^0(X, \mathscr{L}_{-P}) \setminus H^0(X, \mathscr{L}_{-(P+Q)})$$

i.e. satisfying

$$s(p) = 0 \text{ but } s(q) \neq 0.$$

Hence the sheaf $\mathscr{L}$ separates points.

- The dimension formula shows for any point divisor $P \in Div(X)$

$$H^0(X, \mathscr{L}_{-2P}) \subsetneq H^0(X, \mathscr{L}_{-P})$$

has codimension $= 1$. Hence there exists a section

$$s \in H^0(X, \mathscr{L}_{-P}) \setminus H^0(X, \mathscr{L}_{-2P})$$

which implies the surjectivity of the composition of the canonical maps

$$H^0(X, \mathscr{L}_{-P}) \to \mathfrak{m}_{X,p}\mathscr{L}_p \to \mathfrak{m}_{X,p}\mathscr{L}_p/\mathfrak{m}_{X,p}^2\mathscr{L}_p$$

due to

$$dim_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2) = 1.$$

Therefore $\mathscr{L}$ separates tangent vectors.

ii) *Assume $\mathscr{L}$ very ample*: Theorem 11.8 implies that $\mathscr{L}$ separates points and tangent vectors. Separating points implies for all point divisors $P \neq Q \in Div(X)$

$$dim\, H^0(X, \mathscr{L}_{-(P+Q)}) = dim\, H^0(X, \mathscr{L}) - 2.$$

Separating tangent vectors implies for each point divisor $P \in Div(X)$

$$dim\, H^0(X, \mathscr{L}_{-2P}) = dim\, H^0(X, \mathscr{L}) - 2,\ q.e.d.$$

As a consequence of the very-ampleness criterion from Proposition 11.11 we now prove the embedding theorem of compact Riemann surfaces. It is one of the main results about compact Riemann surfaces.

Theorem 11.12 (Embedding theorem). *Any compact Riemann surface X has a closed embedding into a projective space $\mathbb{P}^n$.*

Proof. i) *Existence of an embedding*: Let

$$g := g(X)$$

be the genus of X. For an invertible sheaf $\mathscr{L}$ on X and two, not necessarily distinct point divisors $P, Q \in Div(X)$ the Riemann-Roch theorem 10.27 states

$$\chi(\mathscr{L}) = 1 - g + c_1^{int}(\mathscr{L}) \text{ and } \chi(\mathscr{L}_{-(P+Q)}) = 1 - g + c_1^{int}(\mathscr{L}) - 2$$

which implies

$$\chi(\mathscr{L}_{-(P+Q)}) = \chi(\mathscr{L}) - 2.$$

Hence we prove the claim by providing an invertible sheaf $\mathscr{L}$ satisfying

$$h^1(X, \mathscr{L}) = h^1(X, \mathscr{L}_{-(P+Q)}) = 0$$

or by Serre duality, Theorem 10.28,

$$h^0(X, \mathscr{L}^\vee \otimes_{\mathscr{O}} \omega) = h^0(X, \mathscr{L}^\vee_{P+Q} \otimes_{\mathscr{O}} \omega) = 0.$$

Proposition 8.4 states a necessary condition for the vanishing of these dimensions:

$$c_1^{int}(\mathscr{L}^\vee \otimes_{\mathscr{O}} \omega) = -c_1^{int}(\mathscr{L}) + c_1^{int}(\omega) < 0$$

and

$$c_1^{int}(\mathscr{L}^\vee_{P+Q} \otimes_{\mathscr{O}} \omega) = -c_1^{int}(\mathscr{L}) + c_1^{int}(\omega) + 2 < 0.$$

We will use Proposition 9.17

$$c_1^{int}(\omega) = 2(g-1).$$

Hence the claim reduces to the existence of an invertible sheaf $\mathscr{L}$ with

$$-c_1^{int}(\mathscr{L}) + c_1^{int}(\omega) + 2 < 0$$

i.e.

$$2g < c_1^{int}(\mathscr{L})$$

Therefore any sufficiently high multiple of a point divisor on X provides a suitable invertible sheaf $\mathscr{L}$.

ii) *Explicit construction*: In the following we provide an explicit construction of $\mathscr{L}$ depending on g:

- $g = 0$: We choose a point divisor $P \in Div(X)$ and set

  $$\mathscr{L} := \mathscr{O}_P.$$

  We have

  $$h^0(X, \mathscr{L}) = 1 - 0 + 1 = 2$$

  hence

  $$\phi_{\mathscr{L}} : X \to \mathbb{P}^1$$

  is a closed embedding. Because domain and range have the same dimension the map is an isomorphism

  $$\phi_{\mathscr{L}} : X \xrightarrow{\simeq} \mathbb{P}^1.$$

- $g = 1$: We choose a point divisor $P \in Div(X)$ and set
$$\mathscr{L} := \mathscr{O}_{3P}$$
We have
$$h^0(X, \mathscr{L}) = 1 - 1 + 3.$$
Hence
$$\phi_{\mathscr{L}} : X \to \mathbb{P}^2$$
is a closed embedding.

- $g \geq 2$: We choose
$$\mathscr{L} := \omega^{\otimes 3}$$
the sheaf of sections of the *tri-canonical* bundle. Then
$$2g < 6(g-1) = c_1^{int}(\mathscr{L})$$
because $6 < 4g$. We have
$$h^0(X, \mathscr{L}) = 1 - g + 6(g-1) = 5(g-1).$$
Hence
$$\phi_{\mathscr{L}} : X \to \mathbb{P}^{5(g-1)-1}$$
is a closed embedding, q.e.d.

Remark 11.13 (Embedding theorem). Consider a compact Riemann surface X.

1. *Tri-canonical embedding*: Theorem 11.12 shows that for genus $g(X) \geq 2$ the tri-canonical bundle provides a projective embedding of X. For $g = 1$ the canonical bundle is trivial, i.e. $\omega \simeq \mathscr{O}$, hence for each power $\mathscr{L}$ of the canonical bundle the map $\phi_{\mathscr{L}}$ maps X to a point. For $g = 0$ no positive power of the canonical bundle has a holomorphic section.

2. *Fujita conjecture*: In the proof of Theorem 11.12 we showed for an invertible sheaf $\mathscr{G}$ on a compact Riemann surface X of genus g: The estimate
$$c_1^{int}(\mathscr{G}) > 2g$$
is a sufficient condition for $\mathscr{G}$ to be very ample. The *Fujita conjecture* prompts to verify the following very ampleness criterion: For any n-dimensional compact complex manifold X and an invertible sheaf $\mathscr{L}$ on X with $c_1^{int}(\mathscr{L}) \geq 1$
$$m \geq n + 2 \implies \kappa_x \otimes \mathscr{L}^{\otimes m} \text{ very ample.}$$
The Fujita conjecture holds for Riemann surfaces X: If $c_1^{int}(\mathscr{L}) \geq 1$ then

$$m \geq 3 = 2+1 \implies c_1^{int}(\kappa_X \otimes \mathscr{L}^{\otimes m}) = 2 \cdot (g-1) + m \cdot c_1^{int}(\mathscr{L}) \geq 2g+1$$

The Fujita conjecture also holds for compact complex surfaces. But until now (June 2020) it is open for general compact manifolds.

3. *Effective embedding*: The exponent n obtained for the embeddings from Theorem 11.12

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^n$$

is not the smallest possible for a projective embedding of X. One can prove that there always exist closed embeddings into $\mathbb{P}^3$, see [18, Chap IV, Cor. 3.6].

4. *Moduli of complex structures*: The proof of Theorem 11.12 shows that the only compact Riemann surface with genus $g = 0$ is the projective space $X = \mathbb{P}^1$. An analogous statement does not hold for higher genus: The *moduli space* of compact Riemann surfaces of genus $g = 1$ depends on 1 complex parameter, while the moduli spaces of compact Riemann surfaces of genus $g \geq 2$ depends on $3g-3$ complex parameters.

11.3 Tori and elliptic curves

The present section studies in more detail the projective embedding of tori into $\mathbb{P}^2$ from Theorem 11.12. The section presupposes some classical results about elliptic curves. The projective embedding bridges the theory of 1-dimensional complex tori on the side of complex analysis with elliptic curves from algebraic and arithmetic geometry on the other side. The relation between both view points is further explored by the investigation of modular forms, see [40] and the references contained therein.

We recall and expand Remark 1.12.:

Remark 11.14 (Weierstrass function of a torus). Consider a torus

$$T = \mathbb{C}/\Lambda.$$

Attached to T is its *Weierstrass $\wp$-function*, a meromorphic function

$$\wp \in H^0(T, \mathscr{M})$$

with a single pole, located at the origin $0 \in T$ and having order $= 2$. Its derivative $\wp'$ is meromorphic with a single pole, located at $0 \in T$ and having order $= 3$. The function $\wp$ is even, its derivative $\wp'$ is odd. Both functions are related by the differential equation of meromorphic functions

$$\wp'^2 = 4\wp^3 - g_2 \cdot \wp - g_3$$

with the *lattice constants*

$$g_2 := 60 \cdot \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4} \text{ and } g_3 := 140 \cdot \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6},$$

see [40].

On the torus T denote by $Z \in Div(T)$ the point divisor of the point zero $0 \in T$ and define the divisor

$$D := 3Z \in Div(T) \text{ with } deg\ D = 3.$$

Following the proof of Theorem 11.12 the invertible sheaf

$$\mathscr{L} := \mathscr{O}_D$$

defines an embedding

$$\phi_{\mathscr{L}} : T \to \mathbb{P}^2.$$

Its explicit form is obtained from the Weierstrass $\wp$-function of T according to Theorem 11.15.

Theorem 11.15 (Projective embedding of a torus via its Weierstrass $\wp$-function). *Consider a torus*

$$T = \mathbb{C}/\Lambda$$

and its Weierstrass $\wp$-function*. Set*

$$D := 3Z \in Div(T).$$

For the invertible sheaf

$$\mathscr{L} := \mathscr{O}_D$$

the three sections

$$s_0 := 1, s_1 := \wp, s_2 := \wp'$$

are a basis of $H^0(T, \mathscr{L})$. They define the holomorphic embedding

$$\phi_{\mathscr{L}} : T \to \mathbb{P}^2,\ p \mapsto \begin{cases} (1 : \wp(p) : \wp'(p)) & p \neq 0 \\ (0 : 0 : 1) & p = 0 \end{cases}$$

Proof. The function is holomorphic also in a neighbourhood of 0: For a chart

$$z : U \to V$$

of T around 0 we have in $U \setminus \{0\}$

$$\phi_{\mathscr{L}} = (1 : \wp : \wp') = (z^3 : z^3 \cdot \wp : z^3 \cdot \wp')$$

which extends holomorphically into the singularity with value

$$(0:0:1),\ q.e.d.$$

Theorem 11.15 provides a bridge from Riemann surfaces to algebraic curves, or more general from complex analysis to algebraic geometry. Algebraic curves are the zero sets of homogenous polynomials. Hence the image

$$\phi_{\mathscr{L}}(T) \subset \mathbb{P}^2$$

is expected to be the zero set of a well-defined homogeneous polynomial. How to obtain this polynomial?

The affine part

$$\phi_{\mathscr{L}}(T \setminus \{0\}) \subset U_0 \simeq \mathbb{C}^2$$

is contained in the zero set of a polyomial in the two variables

$$x = \wp \text{ and } y = \wp'.$$

Proposition 11.16 (Weierstrass polynomial of a cubic curve).

1. *The affine plane curve*

$$E_{aff} := \{(x,y) \in \mathbb{C}^2 :\ F(x,y) = 0\} \subset \mathbb{C}^2$$

defined by the Weierstrass polynomial

$$F(x,y) := y^2 - (4x^3 - Ax - B) \in \mathbb{C}[x,y] \text{ with constants } A,\ B \in \mathbb{C},$$

is non-singular iff F has a non-zero discriminant

$$\Delta_F := A^3 - 27B^2 \in \mathbb{C}.$$

2. *The projective closure*

$$E := \overline{E}_{aff} \subset \mathbb{P}^2$$

is the projective curve

$$\{(z_0 : z_1 : z_2) \in \mathbb{P}^2 :\ F_{hom}(z_0, z_1, z_2) = 0\}$$

defined by the homogenized *Weierstrass polyomial*

$$F_{hom}(z_0, z_1, z_2) := z_2^2 \cdot z_0 - (4 \cdot z_1^3 - A \cdot z_1 \cdot z_0^2 - B \cdot z_0^3).$$

One has

$$E = E_{aff} \cup \{O\} \text{ with } O = (0:0:1) \in \mathbb{P}^2.$$

3. The point $O \in E$ is non-singular.

Proof. 1. A point $(x_0, y_0) \in \mathbb{C}^2$ is a singular point of E_{aff} iff it satisfies the three equations

$$0 = y_0^2 - (4x_0^3 - Ax_0 - B), \ \frac{\partial F}{\partial y}(x_0, y_0) = 2y = 0, \ \frac{\partial F}{\partial x}(x_0, y_0) = -12x_0^2 + A = 0$$

Introducing the cubic poynomial

$$f(x) := 4x^3 - Ax + B \in \mathbb{C}[x]$$

the condition is equivalent to

$$f(x_0) = 0 \text{ and } f'(x_0) = 0.$$

The latter condition is equivalent to x_0 being a multiple zero of f, i.e. to the vanishing of the discriminant Δ_f of f. One computes

$$\Delta_f = A^3 - 27B^2,$$

see [23, Chap. III, Cor. 3.4].

2. The projective closure of an affine variety - taken in the Zariski topology - is obtained by homogenizing the defining polynomials, see [18, Chap. I, Ex. 2.9]. Because E_{aff} is closed in the Zariski topology it is also closed in the Euclidean topology.

3. To prove non-singularity at the point O we consider the standard coordinate of $\mathbb{P}^2$ around the point O

$$\phi_2 : U_2 \to \mathbb{C}^2, \ (z_0 : z_1 : z_2) \mapsto (u, \mathrm{v}) := \left(\frac{z_0}{z_2}, \ \frac{z_1}{z_2} \right).$$

We have

$$\phi_2(E \cap U_2) = \{(u, \mathrm{v}) \in \mathbb{C}^2 : \ f(u, \mathrm{v}) = 0\}$$

with

$$f(u, \mathrm{v}) := u - (4 \cdot \mathrm{v}^3 - A \cdot u^2 \cdot \mathrm{v} - B \cdot u^3).$$

Then the partial derivatives are

$$\frac{\partial f}{\partial u}(u, \mathrm{v}) = 1 - (2 \cdot A \cdot u \cdot \mathrm{v} - 3 \cdot B \cdot u^2) \text{ and } \frac{\partial f}{\partial \mathrm{v}}(u, \mathrm{v}) = -12 \cdot \mathrm{v}^2 + A \cdot u^2$$

hence

$$\nabla f(0,0) = (1,0) \neq 0, \ q.e.d.$$

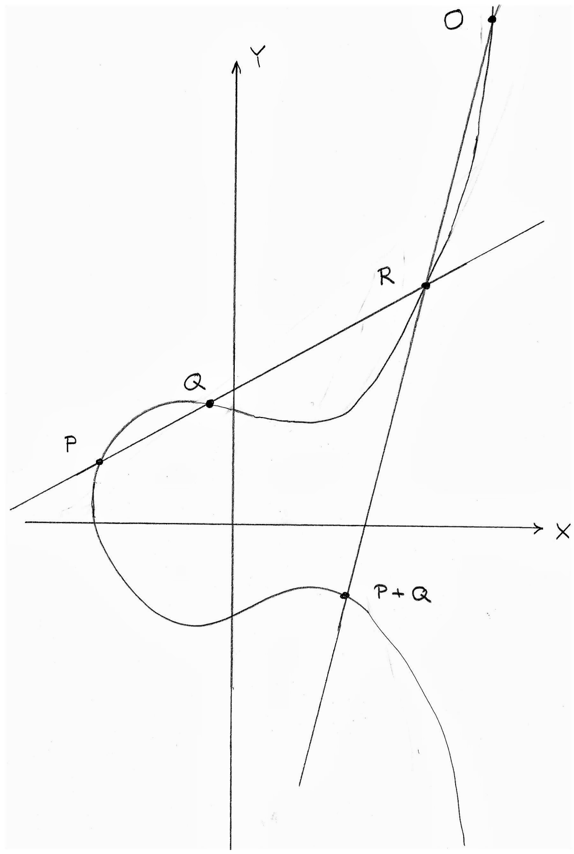


Fig. 11.2 Affine non-singular cubic curve E_{aff} defined by a Weierstrass polynomial

A non-singular cubic has a group structure. Figure 11.2 shows two points $P,\ Q \in E_{aff}$ and the geometric construction of the point $P+Q$. The construction use the fact that the line passing through P and Q intersects the cubic in a third point R and similar the line passing through R and O.

Each complex 1-dimensional torus is biholomorphically equivalent to a torus

$$T = \mathbb{C}/\Lambda$$

defined by a *normalized lattice*

$$\Lambda = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$$

with $\tau \in \mathbb{H}$, the upper halfplane, see [40].

Proposition 11.17 (Discriminant modular form). *The* discriminant *form*

$$\Delta : \mathbb{H} \to \mathbb{C},\ \Delta(\tau) := g_2^3(\tau) - 27 \cdot g_3^2(\tau),$$

is holomorphic and has no zeros.

For the proof of Proposition 11.17 see [40, Chap. 4]. Properties like that are the first fundamental results from the theory of modular forms, see [40].

Corollary 11.18 (Non-singular Weierstrass polynomial). *For any lattice*

$$\Lambda = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau,\ \tau \in \mathbb{H},$$

the plane affine curve

$$E_{aff} := \{(x,y) \in \mathbb{C}^2 :\ P(x,y) = 0\}$$

with Weierstrass polynomial

$$P(x,y) := y^2 - (4x^3 - g_2(\tau) \cdot x - g_3(\tau))$$

is non-singular, see Figure 11.2.

Proof. The claim follows from Proposition 11.16 and Proposition 11.17, q.e.d.

Definition 11.19 (Elliptic curve). A non-singular curve in $X \subset \mathbb{P}^n$ of genus $g(X) = 1$ is an *elliptic curve*.

Corollary 11.20 (Embedding tori as plane elliptic curves). *Consider a torus*

$$T = \mathbb{C}/\Lambda$$

with normalized lattice

$$\Lambda = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \tau,\ \tau \in \mathbb{H},\ \textit{and lattice constants } g_2, g_3 \in \mathbb{C}.$$

The image of the embedding

$$\phi_{\mathscr{L}} : T \hookrightarrow \mathbb{P}^2$$

from Theorem 11.15 is the elliptic curve $E \subset \mathbb{P}^2$ with Weierstrass polynomial

$$F(x,y) = y^2 - (4x^3 - g_2 \cdot x - g_3).$$

Proof. It remains to show the surjectivity of

$$\phi_{\mathscr{L}} : T \hookrightarrow E.$$

Consider a point

$$(x : y : 1) \in E.$$

The $\wp$-function is a non-constant meromorphic function, hence a non-constant holomorphic map

$$\wp : T \to \mathbb{P}^1.$$

The latter is surjective due to Theorem 3.22. Hence there exists $z \in T$ with

$$\wp(z) = x.$$

The function $\wp$ is even, therefore also

$$\wp(-z) = x.$$

From

$$y^2 = 4x^3 - g_2 \cdot x - g_3 = 4\wp(z)^3 - g_2 \cdot \wp(z) - g_3 = \wp'(z)^2$$

follows:

- Either

$$y = \wp'(z) \text{ and } \phi_{\mathscr{L}}(z) = (x : y : 1).$$

- Or

$$y = -\wp'(z) = \wp'(-z) \text{ and } \phi_{\mathscr{L}}(-z) = (x : y : 1), \ q.e.d.$$

Remark 11.21 (Tori, elliptic curves and GAGA).

1. Definition 11.19 introduces elliptic curves as certain 1-dimensional closed submanifolds of $\mathbb{P}^n$. A closed submanifold of $\mathbb{P}^n$ is by definition locally the zero set of holomorphic functions. By a theorem of Chow it can already be defined as the zero set of homogenous polynomials, hence it is a closed subset with respect to the Zariski topology of $\mathbb{P}^n$, notably a non-singular algebraic curve [15, Chap. I, Sect. 3].

2. Replacing power series by polynomials opens up a refined investigation of a projective-algebraic curve E. We distinguish

   - its *field of definition* k which is the smallest subfield $k \subset \mathbb{C}$ containing the coefficients of all homogeneous polynomials from the definition of E,

   - and for all fields

   $$k \subset K \subset \mathbb{C}$$

   the set $E(K)$ of *K-valued points* of E, i.e. of zeros

   $$(z_0 : ... : z_n) \in \mathbb{P}^n$$

   of the defining polynomials with all components $z_j \in K, \ j = 0, ..., n$.

Hence a refined definition considers an elliptic curve as a pair $(E/k, O)$ with

- a non-singular projective-algebraic curve E/k of genus $g = 1$ with k the field of definition

- and a distinguished k-valued point $O \in E$.

3. An elliptic curve $(E/\mathbb{Q}, O)$ is defined by polynomials with rational coefficients. Then the curve can even be defined by polynomials with integer coefficents. Reducing their coefficients modulo a prime $p \in \mathbb{Z}$ provides a series of curves $(E_p)_{p \text{ prime}}$ defined by polynomials with coefficients from the finite fields $\mathbb{F}_p$. The investigation of this family opens up a path from algebraic geometry to arithmetic geometry.

4. The relation between algebraic geometry and complex analytic geometry (en français: *GAGA* = Géométrie Algébrique et Géométrie Analytique) is formalized by a covariant functor
$$an : \underline{Al}_{\mathbb{C}} \to \underline{An}_{\mathbb{C}}$$
from the category $\underline{Al}_{\mathbb{C}}$ of schemes of finite type over $\mathbb{C}$ to the category $\underline{An}_{\mathbb{C}}$ of complex spaces. The functor attaches to a scheme X from $\underline{Al}_{\mathbb{C}}$, provided with the Zariski topology, the complex space X^{an} of the complex points of X, provided with the Euclidean topology. One checks that *an* also maps morphisms f in $\underline{Al}_{\mathbb{C}}$ to morphisms f^{an} in $\underline{An}_{\mathbb{C}}$.

   Of course, a complex manifold, a representative object from the category $\underline{An}_{\mathbb{C}}$, has not necessarily the form X^{an} for a scheme $X \in \underline{Al}_{\mathbb{C}}$. The relation between both categories is closer when one compares projective schemes and compact complex spaces. Here the first important result is Chow's theorem: If a complex space Y is a closed subspace of a projective space then
$$Y = X^{an}$$
for a projective scheme $X \in \underline{Al}_{\mathbb{C}}$. In the other direction one has the obvious result: For any projective scheme X the complex space X^{an} is compact.

   But there are compact complex manifolds Y which do not have the form $Y = X^{an}$ with a projective scheme X. Hence the question: Which additional property assures that a compact manifold Y has the form $Y = X^{an}$? The answer is Kodaira's embedding theorem: One needs the existence of a positive line bundle $\mathscr{L}$ on Y. Then the embedding relies on Kodaira's vanishing theorem for the cohomology of $\mathscr{L}$.

   The study of the functor *an* has been initated by Serre [34] and later generalized by Grothendieck, see also [18, Appendix B] and [28].

Chapter 12
Harmonic theory

The present chapter considers Riemann surfaces from the view point of differential geometry. Section 12.1 and 12.2 take a more general view point than necessary for Riemann surfaces: We consider complex manifold of arbitrary finite dimension $n \in \mathbb{N}^*$ and their underlying higher-dimensional smooth manifolds.

For a compact Riemann surface X both types of cohomology, the topological cohomology groups $H^m(X,\mathbb{C})$ and the holomorphic cohomology groups $H^q(X,\Omega^p)$ are vector spaces of *classes*. Also the elements of the de Rahm groups $Rh^m(X)$ and the Dolbeault groups $Dolb^{q,p}(X)$ are classes. After choosing a metric on X harmonic theory allows to single out from each class a well-defined representative which is a harmonic differential form. As a consequence the whole cohomology of X takes place within the vector spaces of smooth differential forms on X which are respectively d-closed or d''-closed.

The final Chapter 12.3 returns to Riemann surfaces as a low-dimensional example. For Riemann surfaces the de Rahm-Hodge theorem can be obtained without the theory of elliptic differential operators. Instead one uses the finiteness theorem from Chapter 7. By using Dolbeault's theorem we prove in addition the de Rham-Dolbeault-Hodge decomposition theorem, Theorem 12.41. It splits on a compact Riemann surface the *topological* cohomology as a direct sum of the *holomorphic* cohomology. The latter result generalizes to compact complex Kähler manifolds of arbitrary dimension.

12.1 De Rham cohomology with harmonic forms

The underlying smooth structure of a Riemann surface is an oriented Riemann manifold. Section 12.1 makes some first steps to investigate oriented Riemann manifolds of arbitrary dimension by methods from differential geometry and partial differential equations. We introduce the Hodge $*$-operator and define the adjoint differential operator of the exterior derivation. The Laplacian is an elliptic differential operator. The main theorem on elliptic differential operators on compact oriented Riemann manifolds implies the de Rham-Hodge decomposition theorem, see Theorem 12.12, and its Corollary 12.13, a representation of de Rham classes by harmonic forms.

All results hold in the context of real numbers. Hence in this section, we consider for a smooth manifold its real tangent spacec: Partial derivatives from the tangent space multiply by real numbers, and cotangent vectors are linear functionals which take real values.

Remark 12.1 (Fundamentals of Euclidean vector spaces). Consider a finite-dimensional Euclidean vector space $(V, \langle -,- \rangle)$, i.e. an n-dimensional real vector space V provided with a scalar product

$$\langle -,- \rangle : V \times V \to \mathbb{R}.$$

1. The induced map
$$V \to V^*,\ x \mapsto \lambda_x := < x, - >,$$
is an isomorphism. The map becomes an isometry of Euclidean vector spaces when providing the dual space $V^\vee$ with the Euclidean scalar product
$$< \lambda_x, \lambda_y > := < x, y > .$$
2. For each $1 \le k \le m$ the exterior product $\bigwedge^k V$ is an Euclidean space with respect to the induced scalar product
$$< x_1 \wedge ... \wedge x_k, y_1 \wedge ... \wedge y_k > := det(< x_i, y_j >)_{1 \le i,j \le k} \in \mathbb{R}$$
3. For an arbitrary basis $(a_1, ..., a_m)$ of V the matrix
$$g = (g_{ij} := < a_i, a_j >)_{ij}$$
is symmetric and positive definite. For the dual basis $(a^*{}_{,1}, ..., a^*{}_{,m})$ of V^*, the matrix
$$g^* = (< a^*{}_{,i}, a^*{}_{,j} >)_{ij}$$
satifies
$$g^* = g^{-1}$$
4. If $(e_1, ..., e_m)$ is an orthonormal basis of V then also the dual basis
$$(e_1^*, ..., e_m^*)$$
of V^*, is orthonormal. Moreover, for each $1 \le k \le m$ the family
$$(e_{i_1} \wedge ... \wedge e_{i_k})_{1 \le i_1 < ... < i_k \le m}$$
is an orthonormal base of $\bigwedge^k V$.

Definition 12.2 (Volume element of an oriented Euclidean vector space). Consider an m-dimensional real vector space V.

1. An *orientation* of V is an order function on $\bigwedge^m V$ satisfying
   - Each non-zero $\sigma \in \bigwedge^n V$ satisfies either $\sigma > 0$ or $-\sigma > 0$
   - If $\sigma > 0$ and $\tau > 0$ then $\sigma + \tau > 0$
   - If $\sigma > 0$ and $\lambda \in \mathbb{R}^*_+$ then $\lambda \cdot \sigma > 0$
2. An oriented Euclidean m-dimensional vector space $(V, <-,->)$ has a unique element, named its *normalized volume element*,
$$\mu \in \bigwedge^m V$$
satisfying
$$\mu > 0 \text{ and } 1 = \|\mu\| := \sqrt{<\mu,\mu>}$$
Here the scalar product is taken from Remark 12.1, part 2. After choosing a basis $(a_1, ..., a_m)$ of V with
$$a := a_1 \wedge ... \wedge a_m$$
positive, one defines
$$\mu := \frac{a}{\|a\|}$$

Proposition 12.3 (Dual pairing by the volume element). *Consider an n-dimensional oriented Euclidean vector space $(V, \langle -,- \rangle)$ with normalized volume element $\mu \in \bigwedge^m V$.*

1. *The normalized volume element induces an isomorphism*
$$i_\mu : \bigwedge^m V \xrightarrow{\simeq} \mathbb{R},\ \lambda \cdot \mu \mapsto \lambda.$$
2. *For any $1 \leq p \leq m$ the bilinear map*
$$\bigwedge^p V \times \bigwedge^{m-p} V \to \bigwedge^m V \to \mathbb{R},\ (\alpha, \beta) \mapsto i_\mu(\alpha \wedge \beta),$$
is a dual pairing *in the sense of Remark 9.9.*

The $*$-operator of an oriented Euclidean vector space $(V, <-,->)$ from Definition 12.4 combines the duality of the exterior algebra via the volume element and the identification of the vector space and its dual space by the scalar product. In general, the $*$-operator depends on the scalar product.

Definition 12.4 (The $*$-operator for oriented Euclidean vector spaces). On an m-dimensional oriented Euclidean vector space $(V,<-,->)$ with normalized volume element

$$\mu \in \bigwedge^n V$$

the $*$-*operator* is the $\mathbb{R}$-linear morphism

$$* : \bigwedge^p V \longrightarrow \bigwedge^{m-p} V$$

defined as the composition of the two $\mathbb{R}$-linear isomorphisms

$$\bigwedge^p V \longrightarrow \left(\bigwedge^{m-p} V\right)^*, \ \alpha \mapsto i_\mu \circ (\alpha \wedge -),$$

and

$$\left(\bigwedge^{m-p} V\right)^* \longrightarrow \bigwedge^{m-p} V, \ <-, \alpha_{m-p}> \mapsto \alpha_{m-p}.$$

Lemma 12.5 (Properties of the $*$-operator). *Consider an m-dimensional oriented Euclidean vector space* $(V,<-,->)$ *with normalized volume element*

$$\mu \in \bigwedge^n V$$

1. *The $*$-operator is characterized by the property: For all* $p = 1,...,m$ *and* $\alpha, \beta \in \bigwedge^p V$

$$\langle \alpha, \beta \rangle \cdot \mu = \alpha \wedge *\beta \in \bigwedge^m V$$

2. *With respect to an arbitrary positive oriented orthonomal basis of* $(V,<-,->)$

$$(e_1,...,e_m)$$

the $*$-operator is characterized by the formula

$$* : V^p \to V^{m-p}, *(e_{i_1} \wedge ... \wedge e_{i_p}) = sgn\ \sigma \cdot e_{i_{p+1}} \wedge ... \wedge e_{i_m}$$

with

$$\sigma = (i_1,...,i_m)$$

the permutation of the index family $(1,...,m)$.

3. *The iteration of the $*$-star operator*

$$\bigwedge^p V \xrightarrow{*} \bigwedge^{n-p} V \xrightarrow{*} \bigwedge^p V$$

satisfies

$$**=(-1)^{p(m-p)}$$

Proof. ad 3. Due to part 2)

$$*(e_{i_1}\wedge...\wedge e_{i_p})=\varepsilon_1\cdot e_{i_{p+1}}\wedge...\wedge e_{i_m}$$

implies

$$**(e_{i_1}\wedge...\wedge e_{i_p})=\varepsilon_1\cdot\varepsilon_2\cdot e_{i_{p+1}}\wedge...\wedge e_{i_m}$$

with the signs of permutations

$$\varepsilon_1=sgn(i_1,...,i_p,i_{p+1},...,i_m) \text{ and } \varepsilon_2=sgn(i_{p+1},...,i_m,i_1,...,i_p)$$

Hence

$$\varepsilon_2=(-1)^{p(m-p)}\cdot\varepsilon_1$$

which implies

$$**=(-1)^{p(m-p)}\cdot id,\ q.e.d.$$

Definition 12.6 (Riemann manifold). Consider a smooth manifold X with real tangent bundle

$$p:T_{\mathbb{R}}X\to X.$$

1. A *Riemann metric* on the real tangent bundle $T_{\mathbb{R}}X$ is a $\mathbb{R}$-bilinear, symmetric map to the trivial line-bundle

$$g=\langle -,-\rangle : T_{\mathbb{R}}X\times_X T_{\mathbb{R}}X\to X\times\mathbb{R}$$

which induces on each fibre

$$T_p:=(T_{\mathbb{R}}X)_p,\ p\in X,$$

an Euclidean scalar product, i.e. one has for each point $p\in X$ an Euclidean scalar product on the real tangent space

$$g_p:T_p\times T_p\to\mathbb{R},$$

such that its representing symmetric matrix depends smoothly on the base point $p\in X$.

2. A *Riemann manifold* (X,g) is a smooth manifold X with a Riemann metric g on X.

Remark 12.7 (Riemann manifold). Consider a m-dimensional Riemann manifold (X,g). The Riemann metric is a section

$$g\in\Gamma(X,\mathscr{E}^1_{\mathbb{R}}\otimes\mathscr{E}^1_{\mathbb{R}}).$$

1. Consider a smooth chart around of X

$$\phi = (x_1, ..., x_m) : U \to V \subset \mathbb{R}^n.$$

For each point $p \in U$ the tangent vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p, ..., \left.\frac{\partial}{\partial x_m}\right|_p$$

are a basis of the tangent space T_p at p. Its dual basis are the differentials from the real cotangent space

$$dx_1, ..., dx_m \in T_p^*.$$

With respect to the chart the Riemann metric is represented as

$$g(p) = \sum_{i,j=1}^{n} g_{ij}(p)\ dx_i \otimes dx_j \text{ with } g_{ij}(p) = \left\langle \left.\frac{\partial}{\partial x_i}\right|_p, \left.\frac{\partial}{\partial x_j}\right|_p \right\rangle$$

If two sections $\xi, \eta \in \Gamma(U, T_{\mathbb{R}}X)$ are represented as

$$\xi = \sum_{i=1}^{m} \xi_i \cdot \frac{\partial}{\partial x_i}, \text{ and } \eta = \sum_{j=1}^{m} \eta_j \cdot \frac{\partial}{\partial x_j}$$

then

$$g(\xi, \eta) = \sum_{i,j=1}^{m} g_{ij} \cdot \xi_i \cdot \eta_j \in \mathscr{E}_{\mathbb{R}}(U).$$

The length of a tangent vector $\xi \in T_p$ is

$$\|\xi\| := \sqrt{g(\xi, \xi)} = \sqrt{\sum_{i,j=1}^{m} g_{ij}(p) \cdot \xi_i \cdot \xi_j}.$$

2. According to Remark 12.1 the Riemann structure induces scalar products on the real cotangent bundle

$$T_{\mathbb{R}}^*X \text{ and its exterior powers } \bigwedge^p T_{\mathbb{R}}^*X,\ p = 1, ..., m.$$

A smooth manifold X of real dimension m is *orientable* iff it has an atlas of charts such that the functional determinant of the transformation between any two charts is positive. If a smooth manifold is orientable then one of the two orientations is named the *positive orientation*. The manifold with the positive orientation is named a smooth *oriented manifold*. A corresponding atlas defines a *volume form* of the oriented manifold, i.e. a positive differential form of highest degree without zeros. A volume form allows to integrate smooth m-forms with compact support along X.

Definition 12.8 (Euclidean vector spaces of global differential forms using the $*$operator). Consider an oriented Riemann manifold (X,g) of real dimension m.

1. A *volume form* of (X,g) is the form

   $$\mu \in \Gamma(X,\mathscr{E}_{\mathbb{R}}^{m})$$

   which evaluates at a point $x \in X$ as

   $$\mu(x) \in \bigwedge^{m} T_x^*,$$

   the normalized volume element of T_x^* with respect to the orientation of T_x^* induced from the orientation of X and normalized with respect to the metric induced from g, see Definition 12.2.

2. For each $p = 0,...,m$ one defines the $\mathbb{R}$-bilinear map

   $$(-,-) : \Gamma_c(X,\mathscr{E}_{\mathbb{R}}^{p}) \times \Gamma(X,\mathscr{E}_{\mathbb{R}}^{p}) \to \mathbb{R} \text{ with } (\alpha,\beta) := \int_X < \alpha,\beta > \cdot \mu$$

3. For each $p = 0,...,m$ the star-operator on the exterior powers of the cotangent space defines a morphism of $\mathscr{E}_{\mathbb{R}}$-module sheaves

   $$* : \mathscr{E}_{\mathbb{R}}^{p} \longrightarrow \mathscr{E}_{\mathbb{R}}^{m-p}$$

   such that for sections $\alpha,\beta \in \mathscr{E}_{\mathbb{R}}^{p}(U)$, $U \subset X$ open,

   $$< \alpha,\beta > \cdot \mu | U = \alpha \wedge *\beta.$$

Lemma 12.5 implies

$$(\alpha,\beta) = \int_X \alpha \wedge *\beta.$$

For $p = 0,...m$ the bilinear form

$$(-,-) : \Gamma_c(X,\mathscr{E}_{\mathbb{R}}^{p}) \times \Gamma_c(X,\mathscr{E}_{\mathbb{R}}^{p}) \to \mathbb{R}$$

is a scalar product, and $(\Gamma_c(X,\mathscr{E}_{\mathbb{R}}^{p}),(-,-))$ is an Euclidean vector space.

Definition 12.9 (Formal adjoint operator and harmonic forms). Consider an oriented m-dimensional Riemann manifold (X,g). For each $p = 1,...,m$:

1. The *formal adjoint of the exterior derivation*

   $$\delta : \mathscr{E}_{\mathbb{R}}^{p+1} \to \mathscr{E}_{\mathbb{R}}^{p}$$

   is defined as

   $$\delta := (-1)^{mp+1} * d *$$

2. and the *Laplacian*
$$\Delta : \mathscr{E}^p_{\mathbb{R}} \to \mathscr{E}^p_{\mathbb{R}}$$
is defined as
$$\Delta := d^p \circ \delta + \delta \circ d^{p-1}.$$
The kernel of the Laplacian
$$Harm^p(X,\mathbb{R}) := ker[\Delta : \Gamma_c(X,\mathscr{E}^p_{\mathbb{R}}) \to \Gamma_c(X,\mathscr{E}^p_{\mathbb{R}})]$$
is the vector space of *harmonic* p-forms.

Proposition 12.10 (Adjointness). *Consider an oriented m-dimensional Riemann manifold* (X,g). *Denote by* * *("upper star") the operation of taking the adjoint of a linear operator on the Euclidean space*
$$(\Gamma_c(X,\mathscr{E}^p_{\mathbb{R}}),(-,-)),\ p=0,...,m.$$
Then

1. Adjoint operator*:*
$$d^* = \delta$$
2. Selfadjointness*:*
$$\Delta^* = \Delta$$
3. Commutator*:*
$$[\Delta, *] = 0$$

Proof. 1. For arbitrary
$$\alpha \in \Gamma_c(X,\mathscr{E}^p),\ \beta \in \Gamma_c(X,\mathscr{E}^{p+1})$$
set $d = d^p$ and compute
$$(d\alpha,\beta) = \int_X < d\alpha,\beta > \cdot \mu = \int_X d\alpha \wedge *\beta$$
The Leibniz formula
$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^p \alpha \wedge d(*\beta)$$
Stokes' theorem
$$\int_X d(\alpha \wedge *\beta) = 0$$
and Lemma 12.5 imply
$$(d\alpha,\beta) = (-1)^{p+1} \cdot \int_X \alpha \wedge d(*\beta) = (-1)^{p+1} \cdot (-1)^{p(m-p)} \cdot \int_X \alpha \wedge **d(*\beta) =$$

$$= (-1)^{mp+1} \int_X \alpha \wedge *((*d*)\beta) = (-1)^{mp+1} \int_X < \alpha, (*d*)\beta) > \cdot \mu =$$

$$= (-1)^{mp+1} (\alpha, (*d*)(\beta)).$$

2. The selfadjointness of the Laplacian follows from

$$\Delta^* = (d\delta + \delta d)^* = (d \circ \delta)^* + (\delta \circ d)^* = (d \circ d^*)^* + (d^* \circ d)^* =$$

$$= d^{**} \circ d^* + d^* \circ d^{**} = d\delta + \delta d = \Delta$$

3. We have to show that the following square commutes

$$\begin{array}{ccc} \mathscr{E}^p & \xrightarrow{*} & \mathscr{E}^{m-p} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathscr{E}^p & \xrightarrow{*} & \mathscr{E}^{m-p} \end{array}$$

Lemma 12.5 and part 1) imply on one hand

$$\Delta * = (d\delta + \delta d) * = (-1)^{m(m-p)+1} d*d** + (-1)^{m(m-p+1)+1} *d*d* =$$

$$= (-1)^{m(m-p)+1+p(m-p)} d*d + (-1)^{m(m-p+1)+1} *d*d*$$

and on the other hand

$$*\Delta = *(d\delta + \delta d) = (-1)^{mp+1} *d*d* + (-1)^{m(p+1)+1} **d*d =$$

$$(-1)^{mp+1} *d*d* + (-1)^{m(p+1)+1+(m-p)p} d*d$$

Concerning the sign of the exponents one checks

$$sgn(m(m-p)+1+p(m-p)) = sgn(m(p+1)+1+(m-p)p)$$

and

$$sgn(m(m-p+1)+1) = sgn(mp+1), \ q.e.d.$$

Remark 12.11 (Ellipticity of the Laplace operator). Consider an m-dimensional Riemann manifold (X, g). With respect to a smooth chart of X

$$x = (x_1, ..., x_m) : U \to V \subset \mathbb{R}^m$$

the metric has the form

$$g = \sum_{i,j=1}^m g_{ij} \, dx_i \otimes dx_j \in \Gamma(U, \mathscr{E}^1_{\mathbb{R}} \otimes \mathscr{E}^1_{\mathbb{R}})$$

with a matrix function

$$(g_{ij}) \in \Gamma(U, GL(m, \mathbb{R}))$$

Then

$$\Delta = -\sum_{i,j=1}^{m} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + L$$

with

$$(g^{ij}) = g^{-1} \in \Gamma(U, GL(m, \mathbb{R}))$$

the inverse matrix function and L a differential operator of order ≤ 1. The quadratic form defined by the highest order

$$q : \mathbb{R}^m \to \mathbb{R},\ \xi = (\xi_1, ..., \xi_m) \mapsto q(\xi) := -\sum_{i,j=1}^{m} g^{ij} \cdot \xi_i \cdot \xi_j$$

is non-degenerate, i.e.

$$q(\xi) = 0 \iff \xi = 0.$$

Hence the Laplace operator is an *elliptic* differential operator. Notably, when the family $(dx_i)_{i=1,...,m}$ is an orthonormal basis then

$$\Delta = -\sum_{i,j=}^{m} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Theorem 12.12 relies on a deep result from the theory of partial differential equation, which will not be proved in these notes.

Theorem 12.12 (Hodge decomposition). *Consider an m-dimensional compact oriented Riemann manifold (X, g). For each $p = 0, ..., m$ holds*

$$dim\ Harm^p(X, \mathbb{R}) \leq \infty$$

and orthogonal decomposition

$$\Gamma(X, \mathscr{E}^p_{\mathbb{R}}) = Harm^p(X, \mathbb{R}) \overset{\perp}{\oplus} d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) \overset{\perp}{\oplus} \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$

Proof. i) Claim: For all $0 \leq p \leq m$ and $\eta \in \Gamma(X, \mathscr{E}^p_{\mathbb{R}})$

$$\Delta\eta = 0 \iff d\eta = 0 \in \Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}}) \text{ and } \delta\eta = 0 \in \Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) :$$

For the proof note that by definition

$$d\eta = 0 \in \Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}}) \text{ and } \delta\eta = 0 \in \Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) \implies \Delta\eta = 0$$

To prove the converse implication assume

$$\Delta\eta = 0.$$

Then
$$0 = (\Delta\eta, \eta) = (d\delta\eta, \eta) + (\delta d\eta, \eta) = \|\delta\eta\|^2 + \|d\eta\|^2$$
Hence
$$d\eta = 0 \text{ and } \delta\eta = 0.$$
ii) The main theorem of elliptic operators on an m-dimensional compact Riemann manifold X implies for each $j = 0, ..., m$
$$dim\ Harm^j(X, \mathbb{R}) < \infty$$
and
$$\Gamma(X, \mathscr{E}^j_{\mathbb{R}}) = Harm^j(X, \mathbb{R}) \oplus \Delta\Gamma(X, \mathscr{E}^j_{\mathbb{R}}).$$
For a proof of the main theorem see [43, Chap. IV, Theor. 4.12] and [2, 3.10].

iii) Claim: For arbitrary but fixed $p = 0, ..., m$.
$$\Delta\Gamma(X, \mathscr{E}^p_{\mathbb{R}}) = d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) + \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$
The proof will apply the main theorem for the two cases
$$j = p-1,\ p+1.$$
By definition
$$\Delta\Gamma(X, \mathscr{E}^p_{\mathbb{R}}) \subset d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) + \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$
To prove the converse inclusion
$$d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) + \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}}) \subset \Delta\Gamma(X, \mathscr{E}^p_{\mathbb{R}})$$
Consider an arbitrary but fixed element
$$\tau = d\xi + \delta\eta \in d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) \oplus \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$
We apply the main theorem to $\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}})$ and decompose
$$\xi = \Delta\xi_1 + \xi_0 \text{ with } \Delta\xi_0 = 0$$
and to $\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$ and decompose
$$\eta = \Delta\eta_1 + \eta_0 \text{ with } \Delta\eta_0 = 0$$
Due to part i)
$$d\xi_0 = 0 \text{ and } \delta\eta_0 = 0$$
As a consequence
$$\tau = d(\Delta\xi_1 + \xi_0) + \delta(\Delta\eta_1 + \eta_0) = d\Delta\xi_1 + \delta\Delta\eta_1 =$$

$$= d\delta d\xi_1 + dd\delta\xi_1 + \delta\delta d\eta_1 + \delta d\delta\eta_1 = d\delta d\xi_1 + \delta d\delta\eta_1$$

Set

$$\alpha := d\xi_1 + \delta\eta_1 \in \Gamma(X, \mathscr{E}^p_{\mathbb{R}})$$

Then

$$\Delta\alpha = d\delta\alpha + \delta d\alpha = d\delta d\xi_1 + \delta d\delta\eta_1 = \tau$$

iv) As a consequence of part i) and ii)

$$\Gamma(X, \mathscr{E}^p_{\mathbb{R}}) = Harm^p(X, \mathbb{C}) \oplus d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) + \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$

v) The three subspaces

$$Harm^p(X),\ d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}),\ \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}}) \subset \Gamma(X, \mathscr{E}^p_{\mathbb{R}})$$

are mutual orthogonal:

- For each
  $$\alpha \in Harm^p(X) \text{ and } \xi = \delta\eta \in \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$
  we have
  $$(\alpha, \xi) = (\alpha, \delta\eta) = (d\alpha, \eta) = 0$$
  because due to part i)
  $$\Delta\alpha = 0 \implies d\alpha = 0.$$
- For each
  $$\alpha \in Harm^p(X) \text{ and } \xi = d\eta \in d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}})$$
  we have
  $$(\alpha, \xi) = (\alpha, d\eta) = (\delta\alpha, \eta) = 0$$
  because due to part i)
  $$\Delta\alpha = 0 \implies \delta\alpha = 0.$$
- For each
  $$d\eta \in d\Gamma(X, \mathscr{E}^{p-1}_{\mathbb{R}}) \text{ and } \delta\xi \in \delta\Gamma(X, \mathscr{E}^{p+1}_{\mathbb{R}})$$
  we have
  $$(d\eta, \delta\xi) = (dd\eta, \xi) = 0,\ q.e.d.$$

Corollary 12.13 (De Rham-Hodge theorem). *On an m-dimensional compact oriented Riemann manifold (X, g) the Laplacian induces an isomorphism*

$$Rh^p(X, \mathbb{R}) \simeq Harm^p(X, \mathbb{R})$$

between the real de Rham group and the vector space of harmonic forms. In particular,

$$dim\ Rh^p(X, \mathbb{R}) < \infty$$

Proof. We have

$$Rh^p(X,\mathbb{R}) = \frac{ker : [\Gamma(X,\mathscr{E}^p_{\mathbb{R}}) \xrightarrow{d} \Gamma(X,\mathscr{E}^{p+1}_{\mathbb{R}})]}{im : [\Gamma(X,\mathscr{E}^{p-1}_{\mathbb{R}}) \xrightarrow{d} \Gamma(X,\mathscr{E}^p_{\mathbb{R}})]}$$

There is a well-defined canonial map

$$Harm^p(X,\mathbb{R}) \to Rh^p(X,\mathbb{R}),\ \eta \mapsto [\eta]$$

because

$$\Delta\eta = 0 \implies d\eta = 0$$

i) *Surjectivity*: Consider an arbitrary but fixed

$$\eta \in \Gamma(X,\mathscr{E}^p_{\mathbb{R}}) \text{ with } d\eta = 0$$

We have to find a harmonic form

$$\xi \in Harm^p(X,\mathbb{R})$$

satisfying for a suitable $\alpha \in \Gamma(X,\mathscr{E}^{p-1}_{\mathbb{R}})$

$$\eta - \xi = d\alpha$$

Theorem 12.12 provides the decomposition

$$\eta = \eta_0 + d\alpha + \delta\beta$$

with

$$\Delta\eta_0 = 0,\ \alpha \in \Gamma(X,\mathscr{E}^{p-1}_{\mathbb{R}}),\ \beta \in \Gamma(X,\mathscr{E}^{p+1}_{\mathbb{R}}).$$

Then

$$d\eta = 0 \implies d\delta\beta = 0.$$

Apparently

$$\delta\delta\beta = 0.$$

As a consequence

$$\Delta\delta\beta = 0.$$

Hence

$$\delta\beta \in Harm^p(X,\mathbb{R}) \cap \delta\Gamma(X,\mathscr{E}^{p+1}_{\mathbb{R}}) = \{0\}$$

and

$$\eta = \eta_0 + d\alpha \text{ or } \eta - \eta_0 = d\alpha.$$

ii) *Injectivity*: Assume $\eta \in Harm^p(X,\mathbb{R})$ with

$$\eta = d\xi \in d\Gamma(X,\mathscr{E}^{p-1}_{\mathbb{R}})$$

Because
$$Harm^p(X,\mathbb{R}) \cap d\Gamma(X,\mathscr{E}_{\mathbb{R}}^{p-1}) = \{0\}$$
we conclude $\alpha = 0$, q.e.d.

12.2 Harmonic forms on Hermitian manifolds

The present section extends the harmonic theory to Hermitian manifolds, the higher dimensional generalization of Riemann surfaces. Any Hermitian manifold induces in a canonical way an orientation of the underlying Riemann manifold. Thanks to the Hermitian metric the results from the real context of Section 12.1 carry over to the complex context.

On Hermitian manifolds we have an interplay of real and complex structures. Therefore, one has to pay attention and to distinguish carefully whether the object under consideration is

- a real vector space W,
- the *complexification* $W \otimes_{\mathbb{R}} \mathbb{C}$ of a real vector space W,
- a complex vector space V,
- the underlying real vector space V_{real} of a complex vector space V,
- or a *real form* W of a complex vector space V, i.e. V is the complexification of W.

On a complex vector space V we consider $\mathbb{C}$-linear, $\mathbb{C}$-antilinear and $\mathbb{R}$-linear functionals with values in $\mathbb{C}$.

Remark 12.14 (Real linear functionals on a complex vector space). Let V be a complex vector space of complex dimension $= n$.

- $\mathbb{C}$-*antilinear functional*: A $\mathbb{R}$-linear map between complex vector spaces
$$f : U \to W$$
is $\mathbb{C}$-*antilinear* if f satisfies for all $\lambda \in \mathbb{C},\ u \in U$
$$f(\lambda \cdot u) = \overline{\lambda} \cdot f(u).$$

- *Linear and antilinear functionals*: There are two different complex vector spaces of $\mathbb{R}$-linear functionals: The dual space of $\mathbb{C}$-*linear* functionals

$$A^{1,0} := Hom_{\mathbb{C}}(V, \mathbb{C}),$$

a complex vector space, and the vector space of $\mathbb{C}$-*antilinear* functionals

$$A^{0,1} := \overline{Hom_{\mathbb{C}}(V, \mathbb{C})} =$$

$$= \{\phi : V \to \mathbb{C} : \ \phi \ \text{ is } \mathbb{C} - \text{antilinear}\}$$

Also $A^{0,1}$ is a complex vector space under the usual scalar multiplication by $\mu \in \mathbb{C}$

$$(\nu \cdot \phi)(u) := \mu \cdot \phi(u),$$

because for $\phi \in A^{0,1}, \ \lambda, \ \nu \in \mathbb{C}, \ u \in V$,

$$(\nu \cdot \phi)(\lambda \cdot u) = \nu \cdot \phi(\lambda \cdot u) = \nu \cdot (\overline{\lambda} \cdot \phi(u)) = \overline{\lambda} \cdot (\nu \cdot \phi(u)) = \overline{\lambda} \cdot ((\nu \cdot \phi)(u))$$

- *Complex structure on the vector space of* $\mathbb{R}$-*linear functionals*: On the real vector space of $\mathbb{R}$-linear functionals

$$A^1 := Hom_{\mathbb{R}}(V_{\text{real}}, \mathbb{C})$$

the $\mathbb{R}$-linear endomorphism

$$J : A^1 \to A^1, \ \phi \mapsto J\phi \text{ with } (J\phi)(u) := \phi(i \cdot u),$$

satisfies

$$J^2 = -id.$$

The endomorphism has the two complex eigenvalues i and $-i$ with respective eigenspaces $A^{1,0}$ and $A^{0,1}$. Hence

$$A^1 = A^{1,0} \oplus A^{0,1}.$$

Both eigenspaces are complex vector spaces. Hence the real vector space A^1 becomes a complex vector space by defining the complex scalar multiplication

$$i \cdot \phi := J(\phi).$$

We have

$$dim_{\mathbb{C}} \, A^1 = 2 \cdot dim_{\mathbb{C}} \, V.$$

The map J is named a *complex structure* of A^1. For the complex vector space A^1 the above splitting is even a splitting of complex subspaces

$$A^1 = A^{1,0} \oplus A^{0,1}.$$

The *conjugation*

$$A \to A,\ \phi \mapsto \overline{\phi},$$

is a $\mathbb{C}$-antilinear isomorphism with

$$\overline{A^{1,0}} = A^{0,1} \text{and } \overline{A^{0,1}} = A^{1,0}.$$

- *Exterior algebra*: For $j = 0,...,2n$ define

$$A^j := \bigwedge^j A^1.$$

The splitting

$$A^1 \xrightarrow{\simeq} A^{1,0} \oplus A^{0,1}$$

generalizes to the exterior algebra. For $j = 0,...,2n$ it provides a complex structure on A^j.

For $p,q = 0,...,n$ define the complex vector space

$$A^{p,q} := \bigwedge^p A^{1,0} \otimes_{\mathbb{C}} \bigwedge^q A^{0,1}$$

Note the complex scalar multiplication on $A^{p,q}$:

$$\mathbb{C} \times A^{p,q} \to A^{p,q}$$

induced by the multiplication

$$i \cdot (\alpha \otimes \beta) := J(\alpha) \otimes \beta = \alpha \otimes J(\beta) = (i \cdot \alpha) \otimes \beta = \alpha \otimes (-i \cdot \beta)$$

Then for $j = 0,...,2n$

$$A^j \simeq \bigoplus_{p+q=j} A^{p,q}$$

due to the general formula for the exterior algebra of the direct sum of two vector spaces V and W

$$\bigwedge(V \oplus W) \simeq \bigwedge V \otimes \bigwedge W$$

as a graded isomorphism.

Definition 12.15 (Unitary vector space).

1. A *Hermitian form* on a complex vector space V is a map

$$h : V \times V \to \mathbb{C}$$

which is $\mathbb{C}$-linear in the first argument and satisfies for all $u, \mathrm{v} \in V$

$$h(u, \mathrm{v}) = \overline{h(\mathrm{v}, u)}.$$

2. A *unitary vector space* (V,h) is a complex vector space V with a *Hermitian scalar product* h, i.e. h is a Hermitian form which satifies for all non-zero $\mathrm{v} \in V$

$$h(\mathrm{v},\mathrm{v}) > 0.$$

Proposition 12.16 (Hermitian scalar product and induced Euclidean scalar product and alternate form). *Consider a unitary vector space (V,h) with Hermitian scalar product*

$$h : V \times V \to \mathbb{C}.$$

Then:

- *The real part*

$$g := Re\ h : V_{real} \times V_{real} \to \mathbb{R}$$

  is an Euclidean scalar product on V_{real}.

- *The negative of the imaginary part*

$$\omega := -Im\ h : V_{real} \times V_{real} \to \mathbb{R}$$

  is an alternate, real bilinear form on V_{real}, the alternate form associated to *h.*

- *If $(\phi_1,...,\phi_n)$ is a basis of the dual space V^* and*

$$h = \sum_{\alpha,\beta}^{n} h_{\alpha\beta}\ \phi_\alpha \otimes \overline{\phi}_\beta$$

  then

$$\omega = \frac{i}{2} \cdot \sum_{\alpha,\beta}^{n} h_{\alpha\beta}\ \phi_\alpha \wedge \overline{\phi}_\beta$$

Proof. i) *Euclidean scalar product*: We have

$$Re\ h(u,\mathrm{v}) = Re\ \overline{h(u,\mathrm{v})} = Re\ h(\mathrm{v},u)$$

and

$$\omega(u,\mathrm{v}) := -Im\ h(u,\mathrm{v}) = Im\ \overline{h(u,\mathrm{v})} = Im\ h(\mathrm{v},u) = -\omega(\mathrm{v},u).$$

ii) *Alternate form*: Let $(e_1,...,e_n)$ be the basis for V with $(\phi_1,...,\phi_n)$ the dual basis of V^*. If we represent the Hermitian metric as

$$h = \sum_{\alpha,\beta}^{n} h_{\alpha\beta}\ \phi_\alpha \otimes \overline{\phi}_\beta : V \otimes V \to \mathbb{C}$$

then the matrix

$$(h_{\alpha\beta}) \in M(n \times n, \mathbb{C})$$

is Hermitian, i.e. for $\alpha, \beta = 1, ..., n$

$$\overline{h}_{\alpha\beta} = h_{\beta\alpha}.$$

Consider two elements

$$\xi = \sum_{\alpha} \xi_\alpha \cdot e_\alpha,\ \eta = \sum_{\beta} \eta_\beta \cdot e_\beta \in V.$$

On one hand

$$h(\xi, \eta) = \sum_{\alpha,\beta} h_{\alpha\beta} \cdot \xi_\alpha \cdot \overline{\eta}_\beta$$

and

$$\omega(\xi, \eta) = \frac{i}{2} \cdot \sum_{\alpha,\beta} \left(h_{\alpha\beta} \cdot \xi_\alpha \cdot \overline{\eta}_\beta - \overline{h}_{\alpha\beta} \cdot \overline{\xi}_\alpha \cdot \eta_\beta \right)$$

using the formula

$$-Im\, z = z - \overline{z} = \frac{i}{2}(z - \overline{z})$$

for complex numbers $z \in \mathbb{C}$. Because h is Hermitian we obtain

$$\omega(\xi, \eta) = \frac{i}{2} \cdot \sum_{\alpha,\beta} h_{\alpha\beta} \cdot \left(\xi_\alpha \cdot \overline{\eta}_\beta - \overline{\xi}_\beta \cdot \eta_\alpha \right)$$

after changing the indices of the last summand in the bracket. On the other hand

$$(\phi_\alpha \wedge \overline{\phi}_\beta)(\xi, \eta) = \phi_\alpha(\xi)\overline{\phi}_\beta(\eta) - \phi_\alpha(\eta)\overline{\phi}_\beta(\xi) = \xi_\alpha \overline{\eta}_\beta - \eta_\alpha \overline{\xi}_\beta$$

As a consequence

$$\omega(\xi, \eta) = \left(\frac{i}{2} \cdot \sum_{\alpha,\beta} h_{\alpha\beta} \cdot \phi_\alpha \wedge \overline{\phi}_\beta \right)(\xi, \eta),\ q.e.d.$$

Remark 12.17 (Real structure and complexification). Consider an n-dimensional complex vector space V.

1. *Induced Euclidean vector space*: Consider the real dual space dimension $m = 2n$

   $$A^1_{\mathbb{R}} := Hom_{\mathbb{R}}(V_{\text{real}}, \mathbb{R}).$$

   A Hermitian scalar product h on the complex vector space V induces an Euclidean scalar product

   $$g := Re\, h$$

   on the real vector space V_{real}: The unitary vector space (V, h) induces the Euclidean vector space (V_{real}, g). According to Remark 12.1 the Euclidean structure carries over to the dual and to the exterior algebra

$$A^1_{\mathbb{R}} \text{ and } A^j_{\mathbb{R}} := \bigwedge^j A^1_{\mathbb{R}}.$$

2. *Complexification*: The complex vector space from Remark 12.14

$$A^1 = Hom_{\mathbb{R}}(V_{\text{real}}, \mathbb{C})$$

is the *complexification*

$$A^1 \simeq A^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

The complexification carries over to the exterior algebra: For $j = 0, ..., 2n$

$$A^j \simeq A^j_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

3. *Orientation*: Consider a basis

$$\mathscr{B} := (e_1, ..., e_n)$$

of the complex vector space V. The dual basis

$$\mathscr{B}^* := (e_1^*, ..., e_n^*) \text{ of } A^{1,0} = Hom_{\mathbb{C}}(V, \mathbb{C})$$

induces the basis

$$(\mathscr{B}^*)_{\text{real}} = \left(e_1^*, (i \cdot e_1)^*, ..., e_n^*, (i \cdot e_n)^*\right) \text{ of } A^1_{\mathbb{R}} = \left(A^{1,0}\right)_{\text{real}}$$

Here

$$(i \cdot e_k)^* \in A^1_{\mathbb{R}}$$

is the $\mathbb{R}$-linear map

$$V \to \mathbb{R} \text{ with } v \mapsto \begin{cases} 1 & v = i \cdot e_k \\ 0 & v \in \mathscr{B}_{\text{real}} \text{ and } v \neq i \cdot e_k \end{cases}$$

If we provide $A^1_{\mathbb{R}}$ with the orientation such that $(\mathscr{B}^*)_{\text{real}}$ is positively oriented, then the orientation is independent from the choice of the original basis $\mathscr{B}$. Hence the complex vector space V induces a canonical orientation on the real vector spaces V_{real} and $A^1_{\mathbb{R}}$.

4. *Normalized volume element*: Consider a Hermitian scalar product h on V. With respect to the positive orientation of the Euclidean vector space

$$(V_{\text{real}},\ g := Re\ h)$$

and the induced orientation of $A^1_{\mathbb{R}}$ there exists a unique normalized volume element

$$\mu_g \in \bigwedge^{2n} A^1_{\mathbb{R}}$$

see Definition 12.2. Due to the canonical injection

$$A^1_{\mathbb{R}} \hookrightarrow A^1 = A^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C},\ a \mapsto a \otimes 1,$$

we consider

$$\mu_g \in A^{2n}_{\mathbb{R}} \subset A^{2n} = A^{n,n}$$

We now carry over the definition of the $*$-operator from the real case, see Proposition 12.3 and Definition 12.4, to the complex case.

Proposition 12.18 (Dual pairing in unitary vector spaces). *Consider an n-dimensional unitary vector space* (V,h)*, set* $g := Re\ h$ *and recall*

$$A^1 = Hom_{\mathbb{R}}(V_{real}, \mathbb{C}) \text{ and } A^1_{\mathbb{R}} = Hom_{\mathbb{R}}(V_{real}, \mathbb{R}).$$

1. *The normalized volume element*

$$\mu_g \in A^{2n}_{\mathbb{R}} \subset A^{2n} = A^{n,n}$$

from Remark 12.17 induces a $\mathbb{C}$*-linear isomorphism*

$$i_{\mu_g} : A^{n,n} \xrightarrow{\simeq} \mathbb{C},\ \lambda \cdot \mu_g \mapsto \lambda.$$

2. *For each* $0 \le p,q, \le n$ *the* $\mathbb{C}$*-bilinear composition*

$$A^{p,q} \times A^{n-p,n-q} \to A^{n,n} \to \mathbb{C},\ (\alpha, \beta) \mapsto i_{\mu_g}(\alpha \wedge \beta),$$

is a dual pairing.

Definition 12.19 (The $*$-operator for unitary vector spaces). Consider an an n-dimensional unitary vector space (V,h) with normalized volume element

$$\mu_g \in A^{2n}_{\mathbb{R}} \subset A^{n,n},\ g := Re\ h.$$

1. For each $0 \le p,q \le n$ the $*$*-operator*

$$* : A^{p,q} \to A^{n-p,n-q}$$

is defined as the $\mathbb{C}$-antilinear map which is the composition of the $\mathbb{C}$-linear map

$$A^{p,q} \to (A^{n-p,n-q})^*, \alpha_p \wedge \beta_q \mapsto i_{\mu_g}(\alpha_p \wedge \beta_q \wedge -),$$

and the $\mathbb{C}$-antilinear isomorphism

$$(A^{n-p,n-q})^* \to A^{n-p,n-q}$$

induced from

$$< -, \alpha_{n-p} \otimes \beta_{n-q} >_h \mapsto \alpha_{n-p} \otimes \beta_{n-q}$$

2. For each $0 \le j \le 2n$ the $*$-*operator*

$$* : A^j \to A^{2n-j}$$

is defined as the $\mathbb{C}$-antilinear map which is the composition of the $\mathbb{C}$-inear map

$$A^j \to (A^{2n-j})^*, \alpha_j \mapsto i_{\mu_g}(\alpha_j \wedge -),$$

and the $\mathbb{C}$-antilinear isomorphism

$$(A^{2n-j})^* \to A^{2n-j}$$

induced from

$$< -, \alpha_{2n-j} >_h \mapsto \alpha_{2n-j}$$

Note. Definition 12.19 defines the $*$-operator with respect of the Hermitian scalar product h by using the normalized volume element μ_g derived from the Euclidean scalar product g.

Remark 12.20 ($*$-operator on unitary vector spaces).

1. For each $0 \le j \le 2n$ the direct sum of the $*$-operators from Definition 12.19, part 1) is the $*$-operator of part 2)

$$* : A^j = \bigoplus_{p+q=j} A^{p,q} \to \bigoplus_{p+q=j} A^{n-p,n-q} = A^{2n-j}$$

2. As a consequence of Lemma 12.5 and Definition 12.19:

For all $0 \le p, q \le n$ and $\alpha, \beta \in A^{p,q}$

$$< \alpha, \beta >_h \cdot \mu_g = \alpha \wedge *\beta \in A^{n,n}$$

Because $A^1_{\mathbb{R}}$ is a real form of A^1, i.e.

$$Hom_{\mathbb{R}}(V_{\text{real}}, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq Hom_{\mathbb{R}}(V_{\text{real}}, \mathbb{C}),$$

the restriction of the $\mathbb{C}$-antilinear $*$-operator from Definition 12.19

$$* : A^j \to A^{2n-j},\ 0 \le j \le 2n,$$

to the underlying real form is the $\mathbb{R}$-linear $*$-operator

$$*: A^j_{\mathbb{R}} \to A^{2n-j}_{\mathbb{R}}$$

from Definition 12.4.

We now carry over the definition and results from Definition 12.9 to Remark 12.11 from the real context of oriented Riemann manifolds to the complex context of Hermitian manifolds.

Definition 12.21 (Hermitian manifold and underlying Riemann manifold). Consider an n-dimensional complex manifold X.

1. *Hermitian structure*: We denote by

$$TX \to X$$

the *holomorphic tangent bundle* of X. The dual bundle

$$T^*X \to X$$

is the *holomorphic cotangent bundle* and the conjugate

$$\overline{T}^*X \to X$$

is the *anti-holomorphic cotangent bundle*. For a complex chart of X

$$z: U \to V \subset \mathbb{C}^n$$

the restriction

- $TX|U$ is a free $\mathscr{O}(U)$-module with basis the holomorphic tangent vector fields

$$\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}$$

- $T^*X|U$ is a free $\mathscr{O}(U)$-module with basis the holomorphic $(1,0)$-forms

$$dz_1, ..., dz_n$$

- and $\overline{T}^*X|U$ is a free $\mathscr{O}(U)$-module with basis the anti-holomorphic $(0,1)$-forms

$$d\bar{z}_1, ..., d\bar{z}_n$$

- The sheaf of smooth sections of T^*X is denoted $\mathscr{E}^{1,0}$, named the sheaf of smooth $(1,0)$-forms, while the sheaf of smooth sections of $\overline{T}^*$ is denoted $\mathscr{E}^{0,1}$, named the sheaf of smooth $(0,1)$-forms.

- A *Hermitian metric*

$$h: TX \times_X TX \to X \times \mathbb{C}^n$$

on X is a section
$$h \in \mathscr{E}^{1,1}(X)$$
with associated alternate real form
$$\omega := -Im\ h \in \mathscr{E}^{1,1}(X) \cap \mathscr{E}^2_{\mathbb{R}}(X).$$
The pair (X,h) is named a *Hermitian manifold*.

2. *Kähler metric*: A Hermitian metric h is a *Kähler metric* if the associated alternate form is closed:
$$d\omega = 0.$$
Hence any Hermitian metric on a Riemann surface X is a Kähler metric due to the low dimension of X.

3. *Riemann structure*: Denote by X_{smooth} the smooth manifold of dimension $m = 2n$ underlying the complex manifold X. Its real tangent bundle
$$T_{\mathbb{R}}X_{\text{smooth}} \to X_{\text{smooth}}$$
is a real, smooth vector bundle of rank $= m$. The Hermitian matrix h induces a Riemann metric
$$g := Re\ h \text{ on } T_{\mathbb{R}}X_{\text{smooth}}$$
and (X_{smooth}, g) becomes a Riemann manifold.

Next we carry over the results about unitary vector spaces, the induced real volume forms, and the $*$-operator to the tangent and cotangent spaces of a Hermitian manifolds (X,h).

Remark 12.22 (Hermitian manifold and real volume form). Consider an n-dimensional Hermitian manifold (X,h), and denote by
$$g := Re\ h$$
the induced Riemann metric on the real tangent bundle.

1. *Tangent and cotangent spaces*: The linear theory from the first part of the section applies to the fibres of the vector bundles from Definition 12.21 at an arbitrary but fixed point $x \in X$: The fibre
$$V := TX_x$$
is an n-dimensional complex vector space with underlying real vector space
$$V_{\text{real}} = (T_{\mathbb{R}}X_{\text{smooth}})_x$$

of dimension $m = 2n$. Using the notations introduced before we have the real vector spaces

$$A^1_{\mathbb{R}} = Hom_{\mathbb{R}}(V_{\text{real}}, \mathbb{R}) = (T^*_{\mathbb{R}} X_{\text{smooth}})_x,\ A^1 = Hom_{\mathbb{R}}(V_{\text{real}},\ \mathbb{C})$$

and the complex vector spaces

$$A^{1,0} = Hom_{\mathbb{C}}(V, \mathbb{C}) = T^* X_x,\ A^{0,1} = \overline{Hom_{\mathbb{C}}(V, \mathbb{C})} = \overline{T}^* X_x$$

and the induced complex structure on

$$A^1 = A^{1,0} \oplus A^{0,1} = A^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

The Hermitian metric h induces unitary structures on the complex vector spaces

$$A^1,\ A^{1,0},\ A^{0,1}$$

and on their exterior powers. The Euclidean metric g induces an Euclidean structure on the real vector space

$$A^1_{\mathbb{R}}$$

and on its exterior powers.

2. *Orientation, volume form, $*$-operator*: A volume form on the underlying $2n$-dimensional smooth manifold can be obtained as follows. For each complex chart

$$z = (z_1, ..., z_n) : U \to V \subset \mathbb{C}^n$$

and any holomorphic function without zeros $f \in \mathscr{O}^*(U)$ consider the n-form

$$\omega_U := f \cdot dz_1 \wedge ... \wedge dz_n \in \Omega^n(U)$$

Then

$$\omega_U \wedge \overline{\omega}_U = f \cdot dz_1 \wedge ... \wedge dz_n \wedge \overline{(f \cdot dz_1 \wedge ... \wedge dz_n)} =$$

$$f \cdot dz_1 \wedge ... \wedge dz_n \wedge \overline{f} \cdot d\bar{z}_1 \wedge ... \wedge d\bar{z}_n = |f|^2 \cdot dz_1 \wedge ... \wedge dz_n \wedge d\bar{z}_1 \wedge ... \wedge d\bar{z}_n =$$

$$= |f|^2 \cdot (-1)^{n(n-1)/2} (dz_1 \wedge d\bar{z}_1) \wedge ... \wedge (dz_n \wedge d\bar{z}_n) =$$

$$= |f|^2 \cdot (-1)^{n(n-1)/2} (-2i \cdot dx_1 \wedge dy_1) \wedge ... \wedge (-2i \cdot dx_n \wedge dy_n) =$$

$$= |f|^2 \cdot (-1)^{n(n-1)/2} \cdot (-i)^n \cdot 2^n \cdot dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n =$$

$$= |f|^2 \cdot (-1)^{n(n+1)/2} (2i)^n \cdot dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n$$

Set

$$\omega_{\mathbb{R}} := \frac{\omega_U \wedge \overline{\omega}_U}{(-1)^{n(n+1)/2} \cdot (2i)^n} = |f|^2 \cdot dx_1 \wedge dy_1 \wedge ... \wedge dx_n \wedge dy_n$$

and

$$\|\omega_{\mathbb{R}}\| := \sqrt{< \omega_{\mathbb{R}}, \omega_{\mathbb{R}} >_g}.$$

Then

$$\frac{\omega_{\mathbb{R}}}{\|\omega_{\mathbb{R}}\|}$$

is independent from the choice of the chart and the form

$$\mu_g \in \Gamma(X, \mathscr{E}_{\mathbb{R}}^{2n})$$

defined as

$$\mu_g|U := \frac{\omega_{\mathbb{R}}}{\|\omega_{\mathbb{R}}\|}$$

is a volume form of the underlying Riemann manifold (X,g). As a consequence the integration of $2n$-forms along X is well-defined. The volume form evaluates for each $x \in X$ to the normalized volume element

$$\mu_g(x) \in A_{\mathbb{R}}^{2n} = \bigwedge^{2n} A_{\mathbb{R}}^1.$$

3. *The $*$-operator on sheaves of differential forms*: Due to part 2) the $*$-operators on the exterior algebra of the cotangent spaces of the points of X glue to a $\mathbb{C}$-antilinear operator on sheaves

$$* : \mathscr{E}^j \to \mathscr{E}^{2n-j},\ 0 \le j \le 2n,$$

which is compatible with the splitting

$$\mathscr{E}^j = \bigoplus_{p+q=j} \mathscr{E}^{p,q}$$

.

4. *Unitary vector space of global forms*: For each $0 \le p,q \le n$ the map

$$(-,-) : \Gamma_c(X, \mathscr{E}^{p,q}) \times \Gamma_c(X, \mathscr{E}^{p,q}) \to \mathbb{C}$$

defined as

$$(\sigma, \tau) := \int_X < \sigma, \tau >_h \cdot \mu_g = \int_X \sigma \wedge * \tau$$

is a Hermitian scalar product. Hence $(\Gamma_c(X, \mathscr{E}^{p,q}), (-,-))$ is a unitary vector space.

Definition 12.23 (Formal adjoint operators). Consider an oriented n-dimensional Hermitian manifold (X,h).

1. We define

- the *formal adjoint of the d-operator*

$$\delta : \mathscr{E}^{j+1} \to \mathscr{E}^j \text{ as } \delta := (-1) \cdot (* \circ d \circ *)$$

- and the *Laplace operator*

$$\Box : \mathscr{E}^j \to \mathscr{E}^j \text{ as } \Delta := d \circ \delta + \delta \circ d.$$

We define

- the *formal adjoint of the d''-operator*

$$\delta'' : \mathscr{E}^{p,q+1} \to \mathscr{E}^{p,q} \text{ as } \delta'' := (-1) \cdot (* \circ d'' \circ *)$$

- and the *Laplace-Beltrami operator*

$$\Box : \mathscr{E}^{p,q} \to \mathscr{E}^{p,q} \text{ as } \Box := d'' \circ \delta'' + \delta'' \circ d''.$$

The following results 12.24 - 12.28 in the complex context and their proofs are similar to the results 12.10 - 12.13 in the real context and their proofs.

Proposition 12.24 (Adjoint differential operators on $(\Gamma_c(X, \mathscr{E}^{p,q}, (-,-)))$. *Consider an n-dimensional Hermitian manifold (X,h). Denote by * ("upper star") the operation of taking the adjoint of a linear operator on the unitary space*

$$(\Gamma_c(X, \mathscr{E}^{p,q}), (-,-)), \ p,q = 0,...,m.$$

Then

1. Adjoint operator*:*
$$(d'')^* = \delta''$$
2. Selfadjointness*:*
$$\Box^* = \Box$$
3. Commutator*:*
$$[\Box, *] = 0$$

Definition 12.25 (Harmonic (p,q)-forms). On a compact n-dimensional Hermitian manifold X the kernel of the Laplace-Beltrami operators

$$Harm^{p,q}(X) := ker\left[\Box : H^0(X, \mathscr{E}^{p,q}) \to H^0(X, \mathscr{E}^{p,q})\right], \ 0 \le p,q \le n,$$

are the vector spaces of *harmonic* (p,q)-forms.

Remark 12.26 (Ellipticity of the Laplace-Beltrami operator). The Laplace-Beltrami operator $\Box$ on a compact Hermitian manifold is an elliptic operator.

Theorem 12.27 (Hodge decomposition). *Consider an n-dimensional compact Hermitian manifold (X,g). For $0 \le p,q \le n$ holds*

$$dim\ Harm^{p,q}(X) < \infty$$

and the orthogonal decomposition

$$\Gamma(X,\mathscr{E}^{p,q}) = Harm^{p,q}(X) \overset{\perp}{\oplus} d''\Gamma(X,\mathscr{E}^{p,q-1}) \overset{\perp}{\oplus} \delta''\Gamma(X,\mathscr{E}^{p+1,q}).$$

Corollary 12.28 (Dolbeault-Hodge theorem). *On an n-dimensional compact Hermitian manifold (X,h) the Laplace-Beltrami operator induces for all $0 \le p,q, \le n$ an isomorphism*

$$Dolb^{p,q}(X) \simeq Harm^{p,q}(X)$$

between the Dolbeault cohomology group and the vector space of harmonic forms. In particular

$$dim\ Dolb^{p,q}(X) < \infty.$$

Remark 12.29 (Pure Hodge structure). On a compact complex manifold X one has the de Rham-Hodge isomorphism

$$Rh^m(X) \xrightarrow{\simeq} Harm^m(X),$$

see Corollary 12.13. If X has in addition a Kähler metric and one bases the harmonic theory on that Kähler metric then

$$\Delta = 2 \cdot \Box$$

which implies

$$Harm^m(X) = \bigoplus_{p+q=m} Harm^{p,q}(X)$$

In particular, for each pair (p,q) the vector space $Harm^{p+q}(X)$ is a subspace

$$Harm^{p+q}(X) \subset Harm^m(X).$$

One shows, [39, Prop. 6.11]: For each $m \in \mathbb{N}$ the de Rham group splits as

$$Rh^m(X) = \bigoplus_{p+q=m} H^{p,q}(X)$$

with

$$H^{p,q}(X) := \frac{\{\omega \in \mathscr{E}^{p,q}(X) \text{ and } d\omega = 0\}}{\mathscr{E}^{p,q}(X) \cap im[d : \mathscr{E}^{m-1}(X) \to \mathscr{E}^m(X)]} \subset Rh^m(X)$$

the subspace of de Rahm classes represented by closed (p,q)-forms. For each (p,q)

$$H^{p,q}(X) \simeq Harm^{p,q}(X)$$

Here the right-hand side depends on the choice of a Kähler metric. While the left-hand side is independent from any metric. It depends only on the complex structure of X. The following diagram with horizontal injections commutes

$$\begin{array}{ccc} Rh^m(X) & \xrightarrow{\simeq} & Harm^m(X) \\ \uparrow & & \uparrow \\ H^{p,q}(X) & \overset{\simeq}{\dashrightarrow} & Harm^{p,q}(X) \end{array}$$

For each $m \in \mathbb{N}$ the pair

$$(H^m(X,\mathbb{Z}), (H^{p,q}(X))_{p+q=m}) \text{ satisfying } \overline{H^{p,q}(X)} = H^{q,p}(X)$$

is an example of an *integral Hodge structure* of weight m.

12.3 The example of Riemann surfaces

The holomorphic tangent bundle of a Riemann surface X is a complex line bundle. The complex structure on X induces an orientation and a conformal structure on X. They define a $*$-operator on X. Hence the harmonic theory for smooth Riemann manifolds and complex Hermitian manifolds from Sections 12.1 and 12.2 applies to Riemann surfaces. Due to the low dimension of X one gets the following benefits:

- When specializing Theorem 12.12 to the case of real dimension $m = 2$ and Theorem 12.27 to the case of complex dimension $n = 1$ then the Hodge decomposition can be verified by hand - modulo the finitenes theorem. One does not need to invoke the theory of elliptic differential operators.

- On a compact Riemann surface the Laplace operator and the Laplace-Beltrami operator are proportional, see Theorem 12.32. This result can also be checked by explicit computation.

Remark 12.30 ($*$-operator and conformal structure).

1. *Hodge $*$-operator defined by a conformal structure*: Consider an n-dimensional oriented unitary vector space (V,h). The Hodge $*$-operator on the exterior algebra

$$* : A^{p,q} \rightarrow A^{n-p,n-q}$$

   defined in Definition 12.19 is characterized by the equation

$$\alpha \wedge *\beta = < \alpha, \beta >_h \cdot \mu_g$$

with the normalized volume element

$$\mu_g \in \bigwedge^{2n} A^1_{\mathbb{R}}$$

depending on the Euclidean scalar product

$$g := Re\ h.$$

If h_1 and h_2 are two Hermitian forms on V which are *conformally equivalent*, i.e.

$$h_2 = \lambda \cdot h_1$$

with a positive real number $\lambda \in \mathbb{R}^*_+$, then they define the same $*$-operator:

$$\mu_{g_2} = \frac{dx \wedge dy}{\|dx \wedge dy\|_{g_2}} = \frac{dx \wedge dy}{\lambda \cdot \|dx \wedge dy\|_{g_1}} = \frac{1}{\lambda} \cdot \mu_{g_1}$$

implies

$$< \alpha, \beta >_{h_2} \cdot \mu_{g_2} = \lambda \cdot < \alpha, \beta >_{h_1} \cdot \frac{1}{\lambda} \cdot \mu_{g_1} = < \alpha, \beta >_{h_1} \cdot \mu_{g_1}.$$

The computation shows that the $*$-operator does not depend on the Hermitian metric h, but only on the conformal equivalence class of h.

2. *Conformal structure of a Riemann surface*: Consider a Riemann surface X. For each point $p \in X$ one chooses a complex chart around p

$$z = x + i \cdot y : U_1 \to V_1 \subset \mathbb{C}.$$

One chooses an arbitrary Hermitian metric on the complex tangent space T_pX and extends it to a Hermitian metric on

$$A^{1,0} \oplus A^{0,1} = A^1 = Hom_{\mathbb{R}}((T_xX)_{\text{real}}, \mathbb{C})$$

such that

$$A^{1,0} \perp A^{0,1} \text{ and } < dz, dz > = < d\bar{z}, d\bar{z} >$$

From

$$dx \wedge dy = \frac{i}{2} \cdot dz \wedge d\bar{z}$$

one obtains the normalized volume element

$$\mu_{g_1} = \frac{dx \wedge dy}{\|dx \wedge dy\|_{g_1}} = i \cdot \frac{dz \wedge d\bar{z}}{\|dz \wedge d\bar{z}\|_{h_1}}$$

With respect to a second chart

$$w : U_2 \to V_2$$

one obtains the Hermitian scalar product with matrix with respect to $(dw, d\overline{w})$

$$h_2 = \left|\frac{dz}{dw}\right|^2 \cdot h_1$$

Hence both Hermitian metrics on the complex cotangent space are conformally equivalent. Due to part 1) they define the same $*$-operator on the exterior algebra of A^1.

3. *Explicit formulas*: According to part 1) and 2) the $*$-operator is a $\mathbb{C}$-antilinear map, characterized by the equation

$$\alpha \wedge *\beta = <\alpha, \beta>_h \cdot \mu_g$$

The $*$-operator only depends on the conformally equivalence class of a metric. Hence for a given complex chart

$$z : U \to V \subset \mathbb{C}$$

we may choose that Hermitian metric of the 1-dimensional tangent space at a point $p \in U$, which is defined as

$$\left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right\rangle := 1.$$

We extend this Hermitian metric to the 2-dimensional complex vector space A^1 to the metric h, which is defined by the unit matrix

$$H := \mathbb{1} \in GL(2, \mathbb{C})$$

with respect to the complex basis $(dz, d\bar{z})$ of A^1. Then

$$\|dz \wedge d\bar{z}\|_g{}^2 = \|dz \wedge d\bar{z}\|_h{}^2 = det\ H = 1$$

and

$$\mu_g = i \cdot dz \wedge d\bar{z}$$

- *Type* $(0) = (0,0)$:

$$* : A^0 = \mathbb{C} \to A^2,\ *1 = \mu_g,$$

  because

$$1 \cdot *1 = <1, 1>_h \cdot \mu_g = 1 \cdot \mu_g = \mu_g$$

  As a consequence

$$*f = \overline{f} \cdot \mu_g$$

- *Types* $(1,0)$ *and* $(0,1)$:

$$*: A^{1,0} \to A^{0,1},\ *dz := i \cdot d\overline{z}, \text{ and } *: A^{0,1} \to A^{1,0}, *d\overline{z} := -i \cdot dz,$$

because

$$dz \wedge *dz = < dz, dz >_h \cdot \mu_g = 1 \cdot i \cdot dz \wedge d\overline{z} = i \cdot dz \wedge d\overline{z}$$

which implies

$$*dz = i \cdot\ d\overline{z}$$

and similarly

$$d\overline{z} \wedge *d\overline{z} = < d\overline{z}, d\overline{z} >_h \cdot \mu_g = 1 \cdot \mu_g = i \cdot dz \wedge d\overline{z}$$

which implies

$$*d\overline{z} = -i \cdot dz.$$

As a consequence

$$*: A^1 \to A^1$$

satisfies for $\omega_1(x) + \omega_2(x) \in A^{1,0} \oplus A^{0,1} = A^1$

$$*(\omega_1(x) + \omega_2(x)) = i \cdot (\overline{\omega}_1(x) - \overline{\omega}_2(x)).$$

- Type $(2) = (1,1)$:

$$*: A^2 \to A^0,\ *\mu_g := 1,$$

because

$$\mu_g \cdot *\mu_g = < \mu_g, \mu_g >_h \cdot \mu_g = 1 \cdot \mu_g = \mu_g.$$

As a consequence

$$*(f \cdot \mu_g) = \overline{f}$$

4. *The $*$-operator as a sheaf morphism*: For $p = 0, 1, 2$ the $*$-operators on the cotangent spaces glue to $\mathbb{C}$-antilinear $*$-operators of sheaves

$$*: \mathscr{E}^p \to \mathscr{E}^{2-p}$$

Due to part 2) the $*$-operator is independent from the choice of a Hermitian metric on X.

The formulas of Lemma 12.31 will be used in Theorem 12.32 to compute several adjoint operators.

Lemma 12.31 (Relating the $*$-operator and the exterior derivations d' and d''). *On a Riemann surface hold:*

1. Formulas for the d-operator*:*

i)

$$(d \circ *)(\eta_1 + \eta_2) = i \cdot d'\overline{\eta}_1 - i \cdot d''\overline{\eta}_2 \text{ and } (d \circ *)(f \cdot \mu_g) = d\,\overline{f}$$

ii)

$$(* \circ d)f = i \cdot (d''\,\overline{f} - d'\,\overline{f}) \text{ and } (* \circ d)(u\,dz + v\,d\bar{z}) = i \cdot (\overline{\partial}\overline{v} - \partial\overline{u})$$

2. Formulas for the d''-operator*:*

i)

$$(d'' \circ *)(v\,d\bar{z}) = i \cdot \overline{\partial}\,\overline{v}\,dz \wedge d\bar{z} \text{ and } (d'' \circ *)(f \cdot \mu_g) = d''\,\overline{f}$$

ii)

$$(* \circ d')f = i \cdot d''\,\overline{f} \text{ and } (* \circ d'')f = -i \cdot d'\,\overline{f},\ (* \circ d'')(u\,dz) = (-i) \cdot \partial\overline{u}$$

Recall from Definition 12.23 the formal adjoint operators as well as the Laplace operators and the Laplace-Beltrami operators on a Hermitian manifold. On a compact Riemann surface X the Laplace operators, depending on the smooth structure, and the Laplace-Beltrami operators, depending on the complex structure, are proportional. That's a fundamental result. It does not generalize without additional assumptions to higher dimensional compact complex manifolds.

Theorem 12.32 (Relating the Laplace and Laplace-Beltrami operators). *On a compact Riemann surface X the Laplace and Laplace-Beltrami operators satisfy: If*

$$\eta = \eta_1 + \eta_2 \in \Gamma(X, \mathscr{E}^{1,0}) \oplus \Gamma(X, \mathscr{E}^{0,1})$$

then

$$\Delta\eta = 2 \cdot \Box\eta_1 + 2 \cdot \Box\eta_2 \in \Gamma(X, \mathscr{E}^{1,0}) \oplus \Gamma(X, \mathscr{E}^{0,1}),$$

i.e. the following diagram commutes

$$\begin{array}{ccc} H^0(X, \mathscr{E}^1) & \xrightarrow{\ \Delta\ } & H^0(X, \mathscr{E}^1) \\ \simeq \uparrow & & \uparrow \simeq \\ H^0(X, \mathscr{E}^{0,1}) \oplus H^0(X, \mathscr{E}^{1,0}) & \xrightarrow{\ 2\Box \oplus 2\Box\ } & H^0(X, \mathscr{E}^{0,1}) \oplus H^0(X, \mathscr{E}^{1,0}) \end{array}$$

Analogously for smooth functions and smooth 2-forms:

$$\Delta = 2 \cdot \Box : H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E}) \text{ and } \Delta = 2 \cdot \Box : H^0(X, \mathscr{E}^2) \to H^0(X, \mathscr{E}^2).$$

The content of Theorem 12.32 is often expressed in an informal way as the proportionality

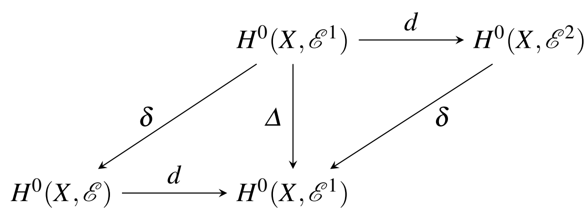
$$\Delta = 2 \cdot \Box.$$

Proof. Those explicit computations in the following, which refer to the choice of a complex chart z, assume w.l.o.g. that the $*$-operator is determined by the Hermitian metric with $(dz, d\bar{z})$ as orthonormal basis of the smooth cotangent space, see Remark 12.30, part 2) and 3).

i) *Computation of* Δ: By definition, the Laplace operator

$$\Delta = \delta \circ d + d \circ \delta : H^0(X, \mathscr{E}^1) \to H^0(X, \mathscr{E}^1)$$

is the sum of the two compositions at the left-hand side and the right-hand side in the diagram below:



By definition

$$\delta \circ d : \mathscr{E}^1 \to \mathscr{E}^1$$

is the composition

$$\delta \circ d = (-1) \cdot (* \circ d \circ * \circ d) = (-1) \cdot (* \circ d) \circ (* \circ d)$$

Successive application of the two formulas Lemma 12.31, part 1.ii):

$$(* \circ d)(u\, dz + \mathrm{v}\, d\bar{z}) = i \cdot (\overline{\partial}\bar{\mathrm{v}} - \partial \bar{u})$$

and

$$\begin{aligned}(* \circ d) \circ (* \circ d)(u\, dz + \mathrm{v}\, d\bar{z}) &= (* \circ d)(i \cdot (\overline{\partial}\bar{\mathrm{v}} - \partial \bar{u})) = \\ i \cdot d''((-i) \cdot (\partial \mathrm{v} - \overline{\partial} u)) &- d'((-i) \cdot (\partial \mathrm{v} - \overline{\partial} u)) = \\ &= \overline{\partial}(\partial \mathrm{v} - \overline{\partial} u)\, d\bar{z} - \partial(\partial \mathrm{v} - \overline{\partial} u)\, dz\end{aligned}$$

hence

$$(\delta \circ d)(u\, dz + \mathrm{v}\, d\bar{z}) = \overline{\partial}(\overline{\partial} u - \partial \mathrm{v})\, d\bar{z} + \partial(\partial \mathrm{v} - \overline{\partial} u)\, dz$$

Analogously, we decompose

$$d \circ \delta = (-1) \cdot (d \circ *) \circ (d \circ *)$$

Successive application of the two formulas Lemma 12.31, part 1.i):

$$(d \circ \delta)(u\, dz + \mathrm{v}\, d\bar{z}) = -\partial(\overline{\partial} u + \partial \mathrm{v})\, dz - \overline{\partial}(\overline{\partial} u + \partial \mathrm{v})\, d\bar{z}$$

Hence

$$\Delta(u\,dz + \mathrm{v}\,d\bar{z}) = -2\cdot\partial\overline{\partial}u\,dz - 2\cdot\overline{\partial}\partial\mathrm{v}\,d\bar{z}$$

ii) *Computation of* $\Box$: Each of the two Laplace-Beltrami operators is the composition in the corresponding diagram below:

$$\begin{array}{ccc} & & H^0(X,\mathscr{E}^{0,1}) \\ & \swarrow{\scriptstyle \delta''} & \downarrow{\scriptstyle \Box} \\ H^0(X,\mathscr{E}) & \xrightarrow{d''} & H^0(X,\mathscr{E}^{0,1}) \end{array} \qquad \begin{array}{ccc} H^0(X,\mathscr{E}^{1,0}) & \xrightarrow{d''} & H^0(X,\mathscr{E}^{2}) \\ {\scriptstyle \Box}\downarrow & \swarrow{\scriptstyle \delta''} & \\ H^0(X,\mathscr{E}^{1,0}) & & \end{array}$$

By definition

$$d''\circ\delta'' = (-1)(d''\circ *)\circ(d''\circ *)$$

Successive application of the two formulas of Lemma 12.31, part 2.i):

$$(d''\circ\delta'')(\mathrm{v}\,d\bar{z}) = (-1)\cdot(d''\circ *)(i\cdot\overline{\partial}\,\overline{v}\,dz\wedge d\bar{z})$$

and

$$(d''\circ\delta'')(\mathrm{v}\,d\bar{z}) = (-1)\cdot(d''\circ *)(\overline{\partial}\,\overline{v}\cdot\mu_g) =$$
$$= -d''\partial\mathrm{v} = -\overline{\partial}\partial\mathrm{v}\,d\bar{z}$$

Analogously one computes the composition

$$\delta''\circ d'' = (-1)\cdot(*\circ d'')\circ(*\circ d'')$$

Successive application of the two formulas of Lemma 12.31, part 2.ii):

$$(\delta''\circ d'')(u\,dz) = (-1)(*\circ d'')((*\circ d'')(u\,dz)) =$$
$$= (-1)*(-i\cdot\partial\overline{u}) = (-1)(-i)\cdot i\cdot d'\partial\overline{u} =$$
$$= -\partial\overline{\partial}u\,dz$$

iii) *Comparing* Δ *and* $\Box$: Comparing the results from part i) and part ii) shows

$$\Box(u\,dz + \mathrm{v}\,d\bar{z}) = -\partial\overline{\partial}u - \overline{\partial}\partial\mathrm{v}$$

and finishes the proof of the theorem for 1-forms.

iv) *Functions and* 2*-forms*: On one hand, by definition

$$\Delta = \delta\circ d : H^0(X,\mathscr{E}) \to H^0(X,\mathscr{E})$$

Successive application of the two formulas of Lemma 12.31, part 1.ii):

$$\delta \circ d = (-1)\dot{(} * \circ\, d) \circ (* \circ d)(f) = (-1) \cdot ((* \circ d)(i \cdot (d''\, \overline{f} - d'\, \overline{f})) =$$

$$= (-1) \cdot (* \circ d)(i \cdot \overline{\partial}\, \overline{f}\, d\overline{z} - i \cdot \partial \overline{f}\, dz) = -(\overline{\partial}\partial f + \partial\overline{\partial} f) =$$

$$= -2 \cdot \overline{\partial}\partial f$$

On the other hand, by definition

$$\Box = \delta'' \circ d'' : H^0(X, \mathscr{E}^{1,0}) \to H^0(X, \mathscr{E}^{1,0})$$

Successive application of the two formulas of Lemma 12.31, part 2.ii):

$$(\delta'' \circ d'')(f) = (-1) \circ (* \circ d'') \circ (* \circ d'')(f) =$$

$$= (-1) \circ (* \circ d'')(-i \cdot d'\, \overline{f}) = (-1) \circ (* \circ d'')(-i \cdot \partial \overline{f})\, dz =$$

$$= -\overline{\partial}\partial f$$

As a consequence,

$$\Delta = 2 \cdot \Box : H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E})$$

Analogously one proves

$$\Delta = 2 \cdot \Box : H^0(X, \mathscr{E}^2) \to H^0(X, \mathscr{E}^2),\ q.e.d.$$

Remark 12.33 (Wirtinger operators, Laplace operator and harmonic functions). Consider a Riemann surface X. The Wirtinger operators from Definition 4.4 with respect to a chart

$$z = x + iy : U \to V$$

satisfy

$$\partial \circ \overline{\partial} = \frac{1}{4} \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

Hence

$$\Delta = -2 \cdot \partial\overline{\partial} = -\frac{1}{2} \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Here the term in brackets is the well-known Laplace operator from real analysis. Hence

$$Harm^0(X) := ker[\Delta : H^0(X, \mathscr{E}) \to H^0(X, \mathscr{E})]$$

is the vector space of complex-valued *harmonic functions* on X.

Definition 12.34 (Harmonic forms). Consider a Riemann surface X. The elements of

$$Harm^m(X) := ker[\Delta : H^0(X, \mathscr{E}^m) \to H^0(X, \mathscr{E}^m)],\ 0 \le m \le 2,$$

are named *harmonic m-forms*. The elements of

$$Harm^{p,q}(X) := ker[\Box : H^0(X, \mathscr{E}^{p,q}) \to H^0(X, \mathscr{E}^{p,q})],\ 0 \le p,q \le 1,$$

are named *harmonic* (p,q)*-forms*.

On a Riemann surface the equation from Theorem 12.32

$$\Delta = 2 \cdot \Box$$

relates the Laplace-operator induced by d and the Laplace-Beltrami operator induced by d''. As a consequence the decomposition of differential forms induces a decomposition of harmonic forms.

Proposition 12.35 (Decomposition of harmonic forms). *On a compact Riemann surface X holds for* $m = 0,1,2$ *the decomposition*

$$Harm^m(X) = \bigoplus_{p+q=m} Harm^{p,q}(X)$$

Proof. The canonical map

$$Harm^m(X) \to \bigoplus_{p+q=m} Harm^{p,q}(X),\ \eta \mapsto \sum_{p+q=m} \eta^{p,q},$$

which splits m-forms into their (p,q) components is well-defined: If $\eta \in H^0(X, \mathscr{E}^m)$ satisfies

$$\Delta\eta = 0$$

then due to Theoem 12.32

$$\Box\eta = 0.$$

As a consequence any $\eta^{p,q} \in H^0(X, \mathscr{E}^{p,q})$ satisfies

$$\Box\eta^{p,q} = 0.$$

Apparently the map is injective. To prove the surjectivity note that any $\eta^{p,q}$ is also an m-form and again due to Theoem 12.32

$$\Box\eta^{p,q} = 0 \implies \Delta\eta^{p,q} = 0,\ q.e.d.$$

Lemma 12.36 (Different characterizations of harmonic forms). *On a Riemann surface X the following properties of a* 1*-form* $\eta \in H^0(X, \mathscr{E}^1)$ *are equivalent:*

1. *The form* η *is harmonic.*

2. The form η satisfies

$$d\eta = 0 \in H^0(X, \mathscr{E}^2) \text{ and } \delta\eta = 0 \in H^0(X, \mathscr{E})$$

3. The form η satisfies

$$d\eta = 0 \in H^0(X, \mathscr{E}^2) \text{ and } d(*\eta) = 0 \in H^0(X, \mathscr{E}^2)$$

4. The form η satisfies

$$d'\eta = 0 \in H^0(X, \mathscr{E}^{1,1}) \text{ and } d''\eta = 0 \in H^0(X, \mathscr{E}^{1,1})$$

5. The form η splits as

$$\eta = \eta_1 + \eta_2 \text{ with } \eta_1 \in H^0(X, \Omega^1) \text{ and } \eta_2 \in H^0(X, \overline{\Omega}^1)$$

6. For each point $x \in X$ exists the germ of a harmonic function

$$f_x \in \mathscr{E}_x$$

with

$$df_x = \eta_x \in \mathscr{E}_x^1$$

Proof. 1) $\iff$ 2) Analogous to the proof of Theorem 12.12. Assume η to harmonic. Then

$$0 = (\Delta\eta, \eta) = ((\delta \circ d + d \circ \delta)\eta, \eta) = ((\delta \circ d)\eta, \eta) + ((d \circ \delta)\eta, \eta) =$$

$$= (d\eta, d\eta) + (\delta\eta, \delta\eta) = \|d\eta\|^2 + \|\delta\eta\|^2$$

Hence

$$d\eta = 0 \text{ and } \delta\eta = 0.$$

The opposite implication follows directly from the representation of Δ.

2) $\iff$ 3) Note that

$$\delta = (-1) * \circ d \circ *$$

and that $*$ is a $\mathbb{C}$-antilinear isomorphism.

3) $\iff$ 4) Assume $d\eta = d(*\eta) = 0$. Split

$$\eta = \eta_1 + \eta_2$$

with

$$\eta_1 \in H^0(X, \mathscr{E}^{1,0}) \text{ and } \eta_2 \in H^0(X, \mathscr{E}^{1,0}).$$

Then

$$d'\eta = d'\eta_2 \text{ and } d''\eta = d''\eta_1$$

On one hand

$$d\eta = d''\eta_1 + d'\eta_2 = 0$$

and on the other hand

$$d(*\eta) = d(i(\overline{\eta}_1 - \overline{\eta}_2)) = i \cdot (d\overline{\eta}_1 - d\overline{\eta}_2) = i \cdot (d'\overline{\eta}_1 - d''\overline{\eta}_2) = 0$$

or

$$d'\overline{\eta}_1 - d''\overline{\eta}_2 = 0$$

Conjugation of the last equation gives

$$d''\eta_1 - d'\eta_2 = 0$$

Summing up:

$$d''\eta = d''\eta_1 = 0 = d'\eta_2 = d'\eta$$

The proof of the opposite direction is obvious.

4) $\iff$ 5) Referring to the splitting

$$\eta = \eta_1 + \eta_2$$

from part 4) we have

$$d''\eta = 0 \iff d''\eta_1 = 0 \iff \eta_1 \in H^0(X, \Omega^1)$$

and

$$d'\eta = 0 \iff d'\eta_2 = 0 \iff d'\overline{\eta}_2 = d''\overline{\eta}_2 = 0 \iff$$

$$\iff \overline{\eta}_2 \in H^0(X, \Omega^1) \iff \overline{\eta}_1 H^0(X, \overline{\Omega}^1)$$

2) $\iff$ 6) The exactness of the De Rahm sequence, see Theorem 5.6, implies the exactness of the sheaf sequence

$$\mathscr{E} \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2$$

Assume

$$d\eta_x = 0 \text{ and } \delta\eta_x = 0$$

Because 2) $\iff$ 1) we know

$$\Delta\eta_x = 0$$

The vanishing

$$d\eta_x = 0$$

implies the existence of a germ

$$f_x \in \mathscr{E}_x \text{ with } df_x = \eta_x$$

And

$$0 = \delta\eta_x = (\delta \circ d)f_x = \Delta f_x$$

implies that f_x is the germ of a harmonic function. For the opposite direction assume

$$\eta_x = df_x$$

with f_x the germ of a harmonic function. In particular

$$d\eta_x = (d \circ d)f_x = 0$$

Moreover

$$0 = \Delta f_x = (\delta \circ d)f_x = \delta(df_x) = \delta\eta_x.$$

Hence

$$d\eta_x = 0 \text{ and } \delta\eta_x = 0$$

Proposition 12.37 (Subspaces of $H^1(X,\mathscr{E})$). *On a compact Riemann surface X hold the following formulas for subspaces of $H^0(X,\mathscr{E}^1)$:*

1.

$$d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E}) = dH^0(X,\mathscr{E}) \oplus \delta H^0(X,\mathscr{E}^2)$$

2.

$$d'H^0(X,\mathscr{E}) = \delta''H^0(X,\mathscr{E}^2)$$

Proof. The proof makes use of some explicit formulas from Lemma 12.31.

1. For the proof consider an element

$$d'f + d''g \in d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E})$$

A formula from Lemma 12.31, part 1.ii) implies

$$df + *dg = d'f + d''f + i(d''\,\overline{g} - d'\,\overline{g}) =$$

$$= d'(f - ig) + d''(\overline{f} + i \cdot \overline{g}) \in d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E})$$

Hence

$$dH^0(X,\mathscr{E}) \oplus *dH^0(X,\mathscr{E}) \subset d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E})$$

Following the last computation in the opposite direction shows

$$d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E}) \subset dH^0(X,\mathscr{E}) \oplus *dH^0(X,\mathscr{E}).$$

Therefore

$$d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E}) = dH^0(X,\mathscr{E}) \oplus *dH^0(X,\mathscr{E})$$

Because

$$* : H^0(X,\mathscr{E}^2) \to H^0(X,\mathscr{E})$$

is a $\mathbb{C}$-antilinear isomorphism we obtain

$$*dH^0(X,\mathscr{E}) = *d*H^0(X,\mathscr{E}^2) = (-1)*d*H^0(X,\mathscr{E}^2) = \delta H^0(X,\mathscr{E}^2)$$

and

$$d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E}) = dH^0(X,\mathscr{E}) \oplus \delta H^0(X,\mathscr{E}^2).$$

2. For $f \in H^0(X,\mathscr{E})$

$$i \cdot d'f = \delta''(f \cdot \mu_g)$$

because a formula from Lemma 12.31, part 2ii) implies

$$\delta''(f \cdot \mu_g) = (-1)\cdot(*\circ d''\circ *)(f\cdot\mu_g) = (-1)\cdot(*\circ d'')(\overline{f}) =$$

$$= (-1)\cdot(-i)\cdot d'f = i\cdot d'f, \ q.e.d.$$

Proposition 12.38 (Unitary vector spaces of global differential forms). *induced from the $*$-operator Consider a compact Riemann surface X.*

1. Hermitian scalar product*: For each $0 \le p,q, \le 1$ the map*

$$(-,-) : H^0(X,\mathscr{E}^{p,q}) \times H^0(X,\mathscr{E}^{p,q}) \to \mathbb{C}$$

defined as

$$(\sigma,\tau) := \int_X \sigma \wedge *\tau$$

is a Hermitian scalar product. Hence $(H^0(X,\mathscr{E}^{p,q}),(-,-))$ is a unitary vector space. Each unitary vector space

$$(H^0(X,\mathscr{E}^m),(-,-)), \ m = 0,1,2,$$

is the orthogonal direct sum

$$H^0(X,\mathscr{E}^m) = \overset{\perp}{\bigoplus}_{p+q=m} H^0(X,\mathscr{E}^{p,q})$$

2. Orthogonality relations*: For each $0 \le p,q \le 1$ the subspaces of $H^0(X,\mathscr{E}^{p,q})$*

$$Harm^{p,q}, \ d''H^0(X,\mathscr{E}^{p,q-1}), \ \delta''H^0(X,\mathscr{E}^{p,q+1})$$

are pairwise orthogonal. For each $m = 0,1,2$, the subspaces of $H^m(X,\mathscr{E})$

$$Harm^m(X), \ dH^0(X,\mathscr{E}^{m-1}), \ \delta H^0(X,\mathscr{E}^{m+1})$$

are pairwise orthogonal.

Note. In Proposition 12.38 the form

$$\sigma \wedge *\tau \in \Gamma(X, \mathscr{E}^2)$$

is independent from the choice of any Hermitian metric h on X. For a given Hermitian metric h it satisfies

$$\sigma \wedge *\tau = <\sigma, \tau>_h \cdot \mu_g$$

with

$$g := Re\ h.$$

Theorem 12.39 (Hodge decomposition of 1-forms). *On a compact Riemann surface X the unitary vector space $H^0(X, \mathscr{E}^1)$ splits as orthogonal direct sum*

$$H^0(X, \mathscr{E}^1) = Harm^1(X) \overset{\perp}{\oplus} dH^0(X, \mathscr{E}) \overset{\perp}{\oplus} \delta H^0(X, \mathscr{E}^2)$$

The vector space of harmonic 1 has finite dimension

$$dim\ Harm^1(X) = 2 \cdot dim\ H^0(X, \Omega^1).$$

Proof. i) *Finite dimension*: Lemma 12.36 implies

$$Harm^1(X) = H^0(X, \Omega^1) \oplus H^0(X, \overline{\Omega}^1)$$

Apparently

$$dim\ H^0(X, \Omega^1) = dim\ H^0(X, \overline{\Omega}^1)$$

Hence

$$dim\ Harm^1(X) = 2 \cdot dim\ H^0(X, \Omega^1) < \infty$$

due to the Finiteness Theorem 7.16.

ii) *Splitting* $(0,1)$*-forms*: We claim

$$H^0(X, \mathscr{E}^{0,1}) = d'' H^0(X, \mathscr{E}) \oplus H^0(X, \overline{\Omega}^1).$$

Dolbeault's Theorem 6.15

$$H^0(X, \Omega^1) = \frac{H^0(X, \mathscr{E}^{0,1})}{im[H^0(X, \mathscr{E}) \xrightarrow{d''} H^0(X, \mathscr{E}^{0,1})]}$$

implies the dimension formula

$$dim\ H^0(X, \overline{\Omega}^1) = dim\ H^0(X, \Omega^1) = dim\ \frac{H^0(X, \mathscr{E}^{0,1})}{d'' H^0(X, \mathscr{E})}$$

Hence

$$dim\, H^0(X,\overline{\Omega}^1)$$

is the codimension of the subspace

$$d''H^0(X,\mathscr{E}) \subset H^0(X,\mathscr{E}^{0,1})$$

The orthogonality relations from Proposition 12.38 imply

$$d''H^0(X,\mathscr{E}) \cap H^0(X,\overline{\Omega}^1) = \{0\}$$

which finishes the proof of the claim.

iii) *Splitting* 1*-forms*: We claim

$$H^0(X,\mathscr{E}^1) = Harm^1(X) \oplus d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E})$$

Part ii) implies by conjugation

$$H^0(X,\mathscr{E}^{1,0}) = d'H^0(X,\mathscr{E}) \oplus H^0(X,\Omega^1).$$

Because

$$\mathscr{E}^1 = \mathscr{E}^{1,0} \oplus \mathscr{E}^{0,1}$$

we obtain

$$H^0(X,\mathscr{E}^1) = d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E}) \oplus H^0(X,\Omega^1) \oplus H^0(X,\overline{\Omega}^1) =$$

$$= d'H^0(X,\mathscr{E}) \oplus d''H^0(X,\mathscr{E}) \oplus Harm^1(X)$$

which finishes the proof of the claim.

The results from part i) - iii), from Proposition 12.37 and the orthogonality results from Proposition 12.38 show

$$H^0(X,\mathscr{E}^1) = Harm^1(X) \overset{\perp}{\oplus} dH^0(X,\mathscr{E}) \overset{\perp}{\oplus} \delta H^0(X,\mathscr{E}^2),\ q.e.d.$$

Theorem 12.40 (De Rham-Hodge theorem). *On a compact Riemann surface X holds*

$$H^1(X,\mathbb{C}) \simeq Rh^1(X) \simeq Harm^1(X).$$

Proof. i) *Splitting the subspace of d-closed* 1*-forms*: We claim

$$ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)] = dH^0(X,\mathscr{E}) \oplus Harm^1(X)$$

The inclusion

$$dH^0(X,\mathscr{E}) \oplus Harm^1(X) \subset ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]$$

holds because

$$d \circ d = 0 \text{ and } ker\ \Delta \subset\ ker\ d,$$

see Lemma 12.36. For the opposite inclusion we have to show, according to Theorem 12.39,

$$ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)] \perp \delta H^0(X,\mathscr{E}^2)$$

Consider

$$\eta \in H^0(X,\mathscr{E}^1) \text{ with } d\eta = 0 \text{ and } \xi \in H^0(X,\mathscr{E}^2)$$

Then

$$(\eta, \delta\xi) = (d\eta, \xi) = 0,$$

which proves the opposite inclusion and finishes the proof of the claim.

ii) *Applying the de Rham theorem*: The de Rahm Theorem 6.15 states

$$H^1(X,\mathbb{C}) \simeq Rh^1(X) = \frac{ker[H^0(X,\mathscr{E}^1) \xrightarrow{d} H^0(X,\mathscr{E}^2)]}{im[H^0(X,\mathscr{E}) \xrightarrow{d} H^1(X,\mathscr{E}^1)]}$$

Appying the result from Part i) implies

$$H^1(X,\mathbb{C}) \simeq Rh^1(X) = \frac{dH^0(X,\mathscr{E}) \oplus Harm^1(X)}{im[H^0(X,\mathscr{E}) \xrightarrow{d} H^1(X,\mathscr{E}^1)]} = Harm^1(X),\ q.e.d.$$

Proposition 12.35 and Theorem 12.41 together provide the Hodge decomposition on a compact Riemann surface. The Hodge decomposition makes manifest the close relation between the topology and the complex structure of a compact Riemann surface.

Theorem 12.41 (De Rham-Dolbeault-Hodge decomposition of cohomology). *For a compact Riemann surface X holds*

$$H^1(X,\mathbb{C}) = \bigoplus_{p+q=1} H^q(X,\Omega^p)$$

with

$$Harm^{0,1} \simeq H^1(X,\mathscr{O}) \text{ and } Harm^{1,0} = H^0(X,\Omega^1)$$

Proof. i) *Splitting* 1*-forms*: Theorem 12.39 proves the decomposition of 1-forms

$$H^0(X,\mathscr{E}^{0,1}) \oplus H^0(X,\mathscr{E}^{1,0}) = H^0(X,\mathscr{E}^1) = Harm^1(X) \oplus dH^0(X,\mathscr{E}) \oplus \delta H^0(X,\mathscr{E}^2)$$

The decomposition splits further as

$$H^0(X,\mathscr{E}^{0,1}) \oplus H^0(X,\mathscr{E}^{1,0}) = Harm^{0,1} \oplus Harm^{1,0} \oplus \delta'' H^0(X,\mathscr{E}^2) \oplus d'' H^0(X,\mathscr{E})$$

because

$$Harm^1(X) = Harm^{0,1}(X) \oplus Harm^{1,0}(X)$$

due to Proposition 12.35 and Proposition 12.37.

Separating $(0,1)$-forms and $(1,0)$-forms shows

$$H^0(X, \mathscr{E}^{0,1}) = Harm^{0,1} \oplus d'' H^0(X, \mathscr{E})$$

and

$$H^0(X, \mathscr{E}^{1,0}) = Harm^{1,0} \oplus \delta'' H^0(X, \mathscr{E}^2)$$

ii) *Harmonic forms and Dolbeault's theorem*:

- $(0,1)$-*forms*: Dolbeault's Theorem 6.15 shows

$$H^1(X, \mathscr{O}) \simeq \frac{H^0(X, \mathscr{E}^{0,1})}{im[H^0(X, \mathscr{E}) \xrightarrow{d''} H^0(X, \mathscr{E}^{0,1})]} = Harm^{0,1}(X)$$

- $(0,1)$-*forms*: Dolbeault's Theorem 6.15 and Lemma 12.36 show

$$Harm^{1,0} = ker[\Box : H^0(X, \mathscr{E}^{1,0}) \to H^0(X, \mathscr{E}^{1,0})] =$$

$$= ker[d'' : H^0(X, \mathscr{E}^{1,0}) \to H^0(X, \mathscr{E}^2)]$$

  and

$$H^0(X, \Omega^1) \simeq ker[H^0(X, \mathscr{E}^{1,0}) \xrightarrow{d''} H^0(X, \mathscr{E}^2)] = Harm^{1,0}(X).$$

  The decomposition from part i)

$$Harm^1(X) = Harm^{0,1}(X) \oplus Harm^{1,0}(X)$$

  and Theorem 12.40 finish the proof, q.e.d.

Remark 12.42 (Hodge decomposition and Serre duality). Consider a compact Riemann surface X.

1. Theorem 12.39 implies

$$dim\, H^1(X, \mathbb{C}) = 2 \cdot dim\, H^0(X, \Omega^1)$$

   and Theorem 12.41 implies

$$H^1(X, \mathbb{C}) \simeq H^0(X, \Omega^1) \oplus H^1(X, \mathscr{O}).$$

   Therefore one obtains the dimension formula from Serre duality

$$dim\, H^0(X, \Omega^1) = dim\, H^1(X, \mathscr{O}).$$

2. The Dolbeault-Hodge decomposition holds on any compact Kähler-manifold, but not on any compact complex manifold of dimension ≥ 2. Each Hermitian metric on the Riemann surface X is a Kähler metric by trivial reason due to the low dimension of X.

3. The Hodge decomposition in the form

$$H^1(X,\mathbb{C}) \simeq H^0(X,\Omega^1) \oplus H^0(X,\overline{\Omega}^1)$$

can also be obtained without any harmonic theory. The following proof is due to Grauert-Remmert [14, Kap. VII, § 7, Abschn. 8]. It relies on Serre duality.

i) *Holomorphic resolution of the sheaf* $\mathbb{C}$: Starting point is the short exact sheaf sequence

$$0 \to \mathbb{C} \to \mathscr{O} \xrightarrow{d''} \Omega^1 \to 0$$

and its complex conjugate

$$0 \to \mathbb{C} \to \overline{\mathscr{O}} \xrightarrow{d'} \overline{\Omega}^1 \to 0$$

The first exact sheaf sequence provides the long exact sequence

$$H^0(X,\mathbb{C}) \to H^0(X,\mathscr{O}) \to H^0(X,\Omega^1) \xrightarrow{\delta} H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}) \to$$

$$H^1(X,\Omega^1) \xrightarrow{\delta} H^2(X,\mathbb{C}) \to H^2(X,\mathscr{O})$$

Here the injection

$$H^0(X,\mathbb{C}) \to H^0(X,\mathscr{O})$$

is an isomorphism because both cohomology groups are isomorphic to $\mathbb{C}$. Due to the low dimension

$$H^2(X,\mathscr{O}) = 0.$$

Corollary 9.15, a consequence of Serre duality, implies

$$H^1(X,\Omega^1) \simeq \mathbb{C}.$$

Becausse X is an oriented 2-dimensional smooth manifold

$$H^2(X,\mathbb{C}) \simeq \mathbb{C}.$$

Therefore

$$\delta : H^1(X,\Omega^1) \to H^2(X,\mathbb{C})$$

is an isomorphism. The long exact sequence reduces to the exact sequence

$$0 \to H^0(X,\Omega^1) \xrightarrow{\delta} H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}) \to 0$$

Lemma 10.12. Analogously, the second short exact sheaf sequence provides the exact sequence

$$0 \to H^0(X, \overline{\Omega}^1) \xrightarrow{\overline{\delta}} H^1(X, \mathbb{C}) \to H^1(X, \overline{\mathscr{O}}) \to 0$$

ii) *No real classes in im* δ: We claim

$$im\ \delta \cap H^1(X, \mathbb{R}) = 0$$

Here the injection $\mathbb{R} \hookrightarrow \mathbb{C}$ induces the injection

$$H^1(X, \mathbb{R}) \hookrightarrow H^1(X, \mathbb{C}).$$

For the proof consider a form $\omega \in H^0(X, \Omega^1)$ with

$$\delta(\omega) \in H^1(X, \mathbb{R})$$

The connecting morphism δ is defined as follows: There exists an open covering

$$\mathscr{U} = (U_\alpha)_{i \in I}$$

of X with all intersections

$$U_{\alpha\beta} := U_\alpha \cap U_\beta,\ \alpha, \beta \in I,$$

connected, see Lemma 10.12, such that for each $\alpha \in I$

$$\omega | U_\alpha = d'' f_\alpha$$

with a cochain

$$(f_\alpha) \in C^0(\mathscr{U}, \mathscr{O})$$

On the connected intersections $U_{\alpha\beta}$ holds

$$d(f_\alpha - f_\beta) = d''(f_\alpha - f_\beta) = 0$$

hence $f_\alpha - f_\beta$ is constant and takes real values. As a consequence the holomorphic functions

$$g_\alpha := exp(2\pi i \cdot f_\alpha)$$

satisfy on the intersections $U_{\alpha\beta}$

$$|g_\alpha / g_\beta| = |exp(2\pi i \cdot (f_\alpha - f_\beta))| = 1$$

or

$$|g_\alpha| = |g_\beta|.$$

The real-valued continuous function

$$|g| = (|g_\alpha|) \in Z^0(\mathscr{U}, \mathscr{C}_\mathbb{R}) = H^0(X, \mathscr{C}_\mathbb{R})$$

assumes its maximum at a point $p \in X$ due to the compactness of X. Let $\alpha \in I$ be an index with $p \in U_\alpha$. By the maximum principle the holomorphic function g_α and a posteriori also f_α are constant, whichs implies $\omega|U_\alpha = 0$. The identity theorem concludes $\omega = 0$.

iii) *Decomposing* $H^1(X,\mathbb{C})$: We claim

$$im\delta \cap im\ \overline{\delta} = 0 \in H^1(X,\mathbb{C})$$

Complex conjugation defines an involution

$$\sigma : H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C})$$

with fixed space the vector space of real cohomology classes $H^1(X,\mathbb{R})$. Apparently σ fixes

$$im\ \delta \cap im\ \overline{\delta}.$$

Hence

$$im\ \delta \cap im\ \overline{\delta} \subset H^1(X,\mathbb{R}), \text{ notably } im\ \delta \cap im\ \overline{\delta} \subset im\ \delta \cap H^1(X,\mathbb{R}) = \{0\}$$

due to part ii). Apparently

$$dim\ H^0(X,\overline{\Omega}^1) = dim\ H^0(X,\Omega^1)$$

and by Serre duality

$$dim\ H^0(X,\Omega^1) = dim\ H^1(X,\mathscr{O})$$

Part ii) and the short exact sequence from part i)

$$0 \to H^0(X,\Omega^1) \xrightarrow{\delta} H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}) \to 0$$

imply

$$dim\ im\ \delta + dim\ im\ \overline{\delta} = dim\ H^1(X,\mathbb{C})$$

As a consequence

$$H^1(X,\mathbb{C}) = \delta(\Omega^1) \oplus \overline{\delta}(\overline{\Omega}^1),\ q.e.d.$$

4. The Hodge decomposition on a Riemann surface is a first example of a Hodge structure. The characteristic of a Hodge structure is the interplay between a lattice $\Lambda \simeq \mathbb{Z}^n$, an arithmetic object, and a complex n-dimensional vector space

$$V \simeq \Lambda_\mathbb{C} := \Lambda \otimes_\mathbb{Z} \mathbb{C}$$

equipped with a complex conjugation $V \to V$.

Definition 12.43 (Betti numbers and Hodge numbers). On a compact Riemann surface X the following complex vector spaces are finite-dimensional. Their dimensions are:

- *Topology*: *Betti numbers* in cohomology

$$b^m := dim\, H^m(X, \mathbb{C})$$

- *Holomorphic structure*: *Hodge numbers*

$$h^{p,q} := dim\, H^q(X, \Omega^p).$$

On a compact Riemann surface X

$$h^{0,0} = b^0 = 1$$

$$h^{0,1} = g(X) \qquad\qquad h^{1,0} = g(X)$$

$$h^{1,1} = b^2 = 1$$

and

$$b^1 = 2 \cdot g(X).$$

Part III
Open Riemann Surfaces

The term *open* Riemann surface is a synonym for *non-compact* Riemann surface. In particular, each domain

$$G \subset \mathbb{C}$$

is an open Riemann surface. Hence the theory of open Riemann surfaces generalizes complex analysis in the affine space $\mathbb{C}$.

Chapter 13
Distributions

The concept of a distribution on an open set

$$X \subset \mathbb{C} \simeq \mathbb{R}^2$$

generalizes the concept of a function defined on X. Distributions are sometimes named *generalized* functions on X. The domain of a distribution is not the set X, but a certain set of functions on X. More precisely, distributions are linear functionals on the vector space of *test functions* defined on X.

Distributions generalize the concept of functions with regard to the following aspect: Any locally integrable function on X defines a distribution by integrating test functions. These distributions are named *regular*. A pleasant property of distributions is their differentiability: Each distribution has partial derivatives of arbitrary order. They are again distributions. IN addition, distributions commute with differentiation and with integration of test functions depending on parameters.

13.1 Definitions and elementary properties

We introduce the topological vetor spaces of test functions and distributions on open subsets of the plane $\mathbb{C}$. For a point $z \in \mathbb{C}$ we denote by $z = x + iy$ its representation with real part x and imaginary part y.

Definition 13.1 (The topological vector space of test functions and distributions). Consider an open set $X \subset \mathbb{C}$.

1. The vector space of *test functions* on X is the complex vector space of smooth functions on X with compact support

$$\mathscr{D}(X) := \{\phi \in \mathscr{E}(X) : supp\ \phi \text{ compact}\}$$

We provide $\mathscr{D}(X)$ with the structure of a topological vector space as follows: A sequence $(\phi_\nu)_{\nu\in\mathbb{N}}$ of test functions from $\mathscr{D}(X)$ is *convergent* towards a function $\phi \in \mathscr{D}(X)$ if the following two properties are satisfied:

- There exists a compact $K \subset X$ such for all $\nu \in \mathbb{N}$

$$supp\ \phi_\nu \subset K$$

- For each multi-index $i = (i_1, i_2) \in \mathbb{N}^2$ holds

$$\lim_{\nu\to\infty} D^i \phi_\nu = D^i \phi$$

  with respect to uniform convergence on K. Here the differential operator is defined for $i = (i_1, i_2) \in \mathbb{N}^2$ as

$$D^i := \frac{\partial^{i_1+i_2}}{\partial x^{i_1} \partial y^{i_2}}$$

  with respect to the coordinates $(x, y) \in X$.

2. The dual vector space

$$\mathscr{D}'(X) := \{T : \mathscr{D}(X) \to \mathbb{C} :\ T \text{ is } \mathbb{C}\text{-linear and continuous}\}$$

is the vector space of *distributions* on U.

Remark 13.2 (Test functions and distributions). Consider an open set $X \subset \mathbb{C}$ and the topological vector space $\mathscr{D}(X)$ of test functions.

1. The topological vector space $\mathscr{D}(X)$ is complete. In general, it is not a Fréchet space; see [31, Theor. 6.5].

2. For a linear map $T : \mathscr{D}(X) \to \mathbb{C}$ the following properties are equivalent:

   i) The map T is continuous

   ii) For every compact subset $K \subset X$ exist an integer $N \in \mathbb{N}$, bounding the order of the derivatives, and a constant $M > 0$, bounding the norm on K, which satisfy: For all $\phi \in \mathscr{D}(X)$ with $supp\ \phi \subset K$

$$T(\phi) \le M \cdot sup\{|D^i\phi(x)| :\ |i| \le N \text{ and } x \in K\}$$

   see [31, Theor. 6.6].

Recall: A Lebesgue measurable function

$$f : X \to \mathbb{C}$$

is *locally integrable* if for each compact set $K \subset X$

$$\iint_K |f(z)| \; dxdy < \infty.$$

Any locally integrable function on X can be considered the kernel of an integral operator on $\mathscr{D}(X)$. The integral operator is a distribution. This construction allows to consider any locally integrable functions as a distribution.

Proposition 13.3 (Regular distributions). *If $L_{loc}(X)$ denotes the vector space of locally integrable functions on the open set $X \subset \mathbb{C}$, then the linear map*

$$reg : L_{loc}(X) \to \mathscr{D}'(X), \; f \mapsto reg_f,$$

with

$$reg_f : \mathscr{D}(X) \to \mathbb{C}, \; \phi \mapsto reg_f[\phi] := \iint_X f \cdot \phi \; dxdy,$$

is well-defined and injective. The distributions from

$$im \; [reg : L_{loc}(X) \to \mathscr{D}'(X)]$$

are named regular distributions.

Proof. Consider a test function $\phi \in \mathscr{D}(X)$ and a compact set $K \subset X$ with $supp \; \phi \subset K$. Then

$$\left| reg_f[\phi] \right| \leq \|\phi\|_K \cdot \iint_K |f| \; dxdy < \infty$$

Consider a convergent sequence $(\phi_\nu)_{\nu \in \mathbb{N}}$ of test functions with

$$\phi = lim_{\nu \to \infty} \; \phi_\nu$$

If $K \subset X$ is a compact set such that for all $\nu \in \mathbb{N}$

$$supp \; \phi_\nu \subset K$$

then

$$\left| reg_f[\phi - \phi_\nu] \right| = \left| \iint_K f \cdot (\phi - \phi_\nu) \; dxdy \right| \leq \|\phi - \phi_\nu\|_K \cdot \iint_K |f| \; dxdy$$

According to Remark 13.2 the linear map reg_f is continuous, hence a distribution, q.e.d.

Definition 13.4 (Differentiation of distributions). Consider an open set $X \subset \mathbb{C}$. For a distribution $T \in \mathscr{D}'(X)$ the *partial derivative of* T of order $i = (i_1, i_2) \in \mathbb{N}^2$ is defined as the distribution

$$D^i T : \mathscr{D}(X) \to \mathbb{C},\ \phi \mapsto D^i T[\phi] := (-1)^{|i|} \cdot T[D^i \phi].$$

In Definition 13.4 continuity of $D^i T$ follows from the fact that convergence in $\mathscr{D}(X)$ implies by definition the uniform convergence of all derivatives on a fixed compact set. Hence $D^i T$ is a distribution.

Definition 13.4 is motivated by the following formula: Consider a smooth function $f \in \mathscr{E}(X)$, a test function $\phi \in \mathscr{D}(X)$ and a multi-index $i \in \mathbb{N}^2$. Then

$$\iint_X D^i f \cdot \phi \ dxdy = (-1)^{|i|} \iint_X f \cdot D^i \phi \ dxdy, \ i.e.$$

$$reg_{D^i f}[\phi] = (-1)^{|i|} \cdot reg_f[D^i \phi].$$

The formula follows by partial integration because the boundary terms vanish due to the compact support of ϕ.

Lemma 13.5 (Interchanging distribution and differentiation of test functions depending on a parameter). *Consider an open subset $X \subset \mathbb{C}$, a compact set $K \subset X$ and an open interval $I \subset \mathbb{R}$. Let*

$$\phi : X \times I \to \mathbb{C},\ (z,t) \mapsto \phi(z,t),$$

be a smooth function with

$$supp\ \phi \subset K \times I.$$

Then for any distribution $T \in \mathscr{D}'(X)$ and any given parameter $t \in I$

$$T_z \left[\frac{\partial \phi(-,t)}{\partial t} \right] = \frac{d}{dt} T_z[\phi(-,t)]$$

Proof. [8, Lemma 24.5]

In Lemma 13.5 and in the following the notation T_z requests to evaluate the distribution T with respect to the first variable z of the test function ϕ.

Lemma 13.6 (Interchanging distribution and integration of test functions depending on a parameter). *Consider open sets $X,\ Y \subset \mathbb{C}$ with compact subsets $K \subset X,\ L \subset Y$, and a smooth function*

$$\phi : X \times Y \to \mathbb{C},\ (z, \zeta) \mapsto \phi(z, \zeta),$$

with

$$supp\ \phi \subset K \times L.$$

Then for any distribution $T \in \mathscr{D}'(X)$

$$T_z \left[\iint_Y \phi(-,\zeta)\ d\xi d\eta \right] = \iint_Y T_z[\phi(-,\zeta)]\ d\xi d\eta,\ \zeta = \xi + i\eta.$$

Proof. [8, Lemma 24.6]

13.2 Weyl's Lemma about harmonic distributions

Proposition 13.7 (Solving the inhomogeneous Laplace equation). *Consider a disk*

$$X := \{z \in \mathbb{C} :\ |z| < R\},\ 0 < R \le \infty.$$

For any $f \in \mathscr{E}(X)$ *exists a function* $\psi \in \mathscr{E}(X)$ *with*

$$\Delta \psi = f.$$

Proof. The Dolbeault Lemma, Theorem 5.2, provides functions

$$\psi_1 \in \mathscr{E}(X) \text{ with } \overline{\partial}\psi_1 = f \text{ and } \psi_2 \in \mathscr{E}(X) \text{ with } \overline{\partial}\psi_2 = \overline{\psi}_1$$

Hence the function

$$\psi := \frac{\overline{\psi}_2}{4}$$

satisfies

$$\Delta \psi = 4 \cdot \partial\overline{\partial}\psi = \partial\overline{\partial}\ \overline{\psi}_2 = \overline{\partial}\partial\overline{\psi}_2 = \overline{\partial}\left(\overline{\overline{\partial}\psi}\right) = \overline{\partial}\psi_1 = f,\ q.e.d.$$

We show that any real-valued harmonic function is the real part of a holomorphic function. Then we derive the mean value formula for harmonic functions from the Cauchy integral theorem.

Theorem 13.8 (The mean value property of harmonic functions). *Consider an open set* $U \subset \mathbb{C}$. *Any harmonic function* $u \in \mathscr{E}(U)$ *satisfies the mean value property, i.e. for all* $z \in U$ *and* $\gamma \subset U$ *a positively oriented circle with radius r around z holds the mean value formula*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\cdot\theta})\ d\theta$$

Proof. i) *Extension to a holomorphic function*: W.l.o.g. we assume $z = 0$. We choose a disk

$$X := \{z \in \mathbb{C} : |z| < R\},\ 0 < R,$$

with $\gamma \subset X \subset U$ and consider the 1-form

$$\omega := u_y\ dx - u_x\ dy \in \mathscr{E}(X)$$

Because u is harmonic the form ω satisfies

$$d\omega = -u_{yy}\ dx \wedge dy - u_{xx}\ dx \wedge dy = -\Delta u\ dx \wedge dy = 0$$

The vanishing

$$H^1(X, \mathbb{C}) = 0$$

provides by Dolbault's Lemma, Theorem 5.2, a function $\mathrm{v} \in \mathscr{E}(X)$ with

$$d\mathrm{v} = \omega$$

i.e.

$$\mathrm{v}_x = u_y \text{ and } \mathrm{v}_y = -u_x$$

The function

$$f := u + i \cdot \mathrm{v} \in \mathscr{E}(X)$$

satisfies the Cauchy-Riemann differential equations. Hence f is holomorphic

$$f \in \mathscr{O}(X)$$

ii) *Cauchy integral formula*: The holomorphic function $f \in \mathscr{O}(X)$ satisfies Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \cdot \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Introducing polar coordinates

$$\zeta = z + r \cdot e^{i\theta} \text{ and } d\zeta = i \cdot re^{i\theta}\ d\theta$$

shows

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + r \cdot e^{i\theta})\ d\theta$$

or

$$u(z) + i \cdot \mathrm{v}(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(z + r \cdot e^{i\theta})\ d\theta + i \cdot \frac{1}{2\pi} \cdot \int_0^{2\pi} \mathrm{v}(z + r \cdot e^{i\theta})\ d\theta,$$

in particular the mean value formula

$$u(z) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(z + r \cdot e^{i\theta})\ d\theta,\ q.e.d.$$

Smoothing converts a continuous function f to a smooth function $sm_\varepsilon(f)$. Hereby one replaces the value of f at each point z by the mean value of f in a disk-neighbourhood of z. The scaling factor $\varepsilon > 0$ gives the diameter of the disk which is employed for averaging. Smoothing is compatible with derivation, i.e. for a smooth function f taking the derivative commutes with smoothing. Smoothing a harmonic function reproduces the function because harmonic functions have the mean-value property a-priori, see Theorem 13.8.

Definition 13.9 (Smoothing continuous functions).

1. A *smoothing function* is a function

$$\rho \in \mathscr{D}(\mathbb{C})$$

with the following properties:

i) *Normalized support*:

$$supp\ \rho \subset \{z \in \mathbb{C} : |z| < 1\}$$

ii) *Rotational symmetry*: For all $z \in \mathbb{C}$ holds

$$\rho(z) = \rho(|z|)$$

iii) *Unit volume*:

$$\iint_{\mathbb{C}} \rho(x + i \cdot y)\ dxdy = 1$$

2. Any smoothing function ρ defines for each open set $U \subset \mathbb{C}$ and for each $\varepsilon > 0$ a corresponding *smoothing map*

$$sm_\varepsilon : \mathscr{C}(U) \to \mathscr{E}(U^\varepsilon)$$

defined as

$$sm_\varepsilon(f)(z) := \iint_{\mathbb{C}} \rho_\varepsilon(z - \zeta) \cdot f(\zeta)\ d\xi d\eta,\ \zeta = \xi + i \cdot \eta.$$

Here

$$U^\varepsilon := \{z \in U :\ B_\varepsilon(z) \subset U\}$$

is the ε-shrinking of U and

$$\rho_\varepsilon(z) := \frac{1}{\varepsilon^2} \cdot \rho\left(\frac{z}{\varepsilon}\right).$$

Note that $sm_\varepsilon(f)$ is indeed smooth because derivation with respect to z and integration with respect to ζ commute. Apparently, near the boundary ∂U one cannot define the smoothing function $sm_\varepsilon(f)$ by the formula above.

Lemma 13.10 (Smoothing and differentiation). *Consider an open set $U \subset \mathbb{C}$, a smooth function $f \in \mathscr{E}(U)$ and $\varepsilon > 0$.*

1. *In U^ε for all $\alpha \in \mathbb{N}^2$*
$$D^\alpha(sm_\varepsilon f) = sm_\varepsilon(D^\alpha f)$$
2. *For harmonic $h \in \mathscr{E}(U)$*
$$sm_\varepsilon(h) = h|U^\varepsilon$$

Proof. 1. For $z \in U^\varepsilon$ we obtain by translation of the integration variable $\zeta = \xi + i \cdot \eta$

$$sm_\varepsilon(f)(z) = \iint_U \rho_\varepsilon(z-\zeta) \cdot f(\zeta)\, d\xi d\eta = \iint_{|\zeta|<\varepsilon} \rho_\varepsilon(\zeta) \cdot f(z+\zeta)\, d\xi d\eta$$

Hence
$$D^\alpha(sm_\varepsilon f)(z) = \iint_{|\zeta|<\varepsilon} \rho_\varepsilon(\zeta) \cdot D^\alpha f(z+\zeta)\, d\xi d\eta =$$
$$= \iint_U \rho_\varepsilon(z-\zeta) \cdot D^\alpha f(\zeta)\, d\xi d\eta = (sm_\varepsilon\, D^\alpha f)(z)$$

2. According to Theorem 13.8 the harmonic function h satisfies the mean-value property: For all $r \in [0, \varepsilon[$
$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} h(z + re^{i\theta})\, d\theta$$
As a consequence

$$sm_\varepsilon(h)(z) = \iint_{|\zeta|<\varepsilon} \rho_\varepsilon(\zeta) \cdot h(z+\zeta)\, d\xi d\eta = \iint_{\substack{0 \le r \le \varepsilon \\ 0 \le \theta \le 2\pi}} \rho_\varepsilon(r) \cdot h(z+re^{i\theta})\, r \cdot dr d\theta =$$
$$= \left(\int_0^r \rho_\varepsilon(r)\, rdr\right) \cdot 2\pi \cdot h(z) = \iint_{\mathbb{C}} \rho_\varepsilon(\xi + i \cdot \eta)\, d\xi d\eta \cdot h(z) = h(z),\ q.e.d.$$

Theorem 13.11 (Weyl's Lemma on the regularity of harmonic distributions). *On an open set $U \subset \mathbb{C}$ each harmonic distribution $T \in \mathscr{D}'(U)$, i.e. satisfying*
$$\Delta T = 0,$$
is regular: There exists a harmonic function $h \in \mathscr{E}(U)$ with
$$reg_h = T,$$
i.e. for all test functions $\phi \in \mathscr{D}(U)$
$$T[\phi] = \iint_U h(z) \cdot \phi(z)\, dxdy.$$

Proof. i) *Definition of the kernel h*: We choose a smoothing function $\rho \in \mathscr{D}(\mathbb{C})$ according to Definition 13.9. For each arbitrary but fixed $z \in U^\varepsilon$ the smooth function of the variable ζ

$$D_\varepsilon(z) \to \mathbb{C},\ \zeta \mapsto \rho_\varepsilon(z-\zeta),$$

has compact support in U and therefore extends by zero to a test function from $\mathscr{D}(\mathbb{C})$. Applying the distribution T and varying $z \in U^\varepsilon$ defines a smooth function of the variable z

$$h : U^\varepsilon \to \mathbb{C},\ h(z) := T_\zeta[\rho_\varepsilon(\zeta - z)].$$

Here the notation T_ζ means to consider the subsequent test function as a function of the variable ζ. The function h is harmonic because by assumption

$$\Delta_z h = \Delta T_\zeta[\rho_\varepsilon(\zeta - z)] = 0$$

ii) *The value of T on the smoothing of test functions*: For a given test function $f \in \mathscr{D}(\mathbb{C})$ with $supp\ f \subset U^\varepsilon$ consider the smooting $sm_\varepsilon(f)$ as a test function of the variable ζ, depending on the paramater z,

$$sm_\varepsilon(f)(\zeta) := \iint_{\mathbb{C}} \rho_\varepsilon(\zeta - z) \cdot f(z)\ dxdy = \iint_{U^\varepsilon} \rho_\varepsilon(\zeta - z) \cdot f(z)\ dxdy$$

Interchanging the distribution with respect to the variable ζ and the integration with respect to the parameter z according to Lemma 13.6 gives

$$T[sm_\varepsilon(f)] = T_\zeta\left[\iint_{U^\varepsilon} \rho_\varepsilon(\zeta - z) \cdot f(z)\ dxdy\right] =$$

$$= \iint_{U^\varepsilon} T_\zeta[\rho_\varepsilon(\zeta - z)] \cdot f(z)\ dxdy = \iint_{U^\varepsilon} h(z) \cdot f(z)\ dxdy$$

Hence the value of T on the smoothing of a test function $sm_\varepsilon(f)$ can be obtained from integration with the kernel h.

iii) *Regularity with respect to the kernel h*: We claim that for any $f \in \mathscr{D}(\mathbb{C})$

$$T[f] = \iint_{U^\varepsilon} h(z) \cdot f(z)\ dxdy.$$

Proposition 13.7 solves the inhomogenenous Laplace equation and provides a smooth function $\psi \in \mathscr{E}(\mathbb{C})$ satisfying

$$\Delta \psi = f.$$

On the complement $\mathbb{C} \setminus supp\ f$ the function ψ is harmonic, hence satisfies

$$\psi = sm_\varepsilon(\psi),$$

see Lemma 13.10. Therefore

$$\phi := \psi - sm_\varepsilon(\psi)$$

has compact support in U. Lemma 13.10, part 2, implies

$$\Delta\phi = \Delta(\psi - sm_\varepsilon(\psi)) = \Delta\psi - sm_\varepsilon(\Delta\psi) = f - sm_\varepsilon(f)$$

The assumption $\Delta T = 0$ and Lemma 13.5 imply

$$0 = \Delta T[\phi] = T[\Delta\phi] = T[f] - T[sm_\varepsilon(f)]$$

hence

$$T[f] = T[sm_\varepsilon(f)] = \iint_{U^\varepsilon} h(z)\cdot f(z)\; dxdy,\; q.e.d.$$

A corollary of Weyl's Lemma states that holomorphic distributions are regular with holomorphic kernel.

Corollary 13.12 (Regularity of holomorphic distributions). *On an open subset $U \subset \mathbb{C}$ each distribution*

$$T \in \mathscr{D}'(U) \textit{ with } \frac{\partial T}{\partial \overline{z}} = 0$$

is regular, i.e. there exists a holomorphic function $f \in \mathscr{O}(U)$ with

$$reg_f = T$$

i.e. for all test function $\phi \in \mathscr{D}(U)$

$$T[\phi] = \iint_U f(z)\cdot\phi(z)\; dxdy.$$

Proof. The assumption

$$\frac{\partial}{\partial \overline{z}} T = 0$$

implies

$$\Delta\ T = 4\cdot\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \overline{z}} T\right) = 0$$

Theorem 13.11 provides a harmonic function $h \in \mathscr{E}(U)$ with

$$reg_h = T.$$

Then

$$\frac{\partial T}{\partial \overline{z}} = 0 \implies \frac{\partial}{\partial \overline{z}} h = 0$$

which shows that h is even holomorphic, q.e.d.

Remark 13.13 (Regularity theorems for solutions of elliptic differential operators). Theorem 13.11 and Corollary 13.12 are examples for the regularity of distributions which satisfy a differential equation with an elliptic operator. The regularity generalizes to arbitrary solution of elliptic differential operators, see [43, Chap. IV, Theor. 4.9].

Chapter 14
Runge approximation

Runge approximation is the decisive means to prove that an open Riemann surface X is a Stein manifold. The present chapter investigates the method of Runge approximation. As application it proves the vanishing

$$H^1(X, \mathscr{O}) = 0$$

Chapter 15 will give the definition of Stein manifolds and complete the proof that X is a Stein manifold.

The principle of approximation is to construct a global solution of a problem

- by finding first local solutions and
- then constructing a global solution bottom-up by extending the modified local solutions.

In general, during the extension step one has to modify the local solutions in order to obtain a convergent global solution. Runge approximation refers to the second step: It determines the domains of the local solution, the Runge domains, which allow to approximate the solution by a solution on a larger domain.

Typical runge domains in the plane $X = \mathbb{C}$ are disks: A holomorphic function f on a disk expands into a Taylor series. Its Taylor polynomials are defined on all of X. They approximate f on any compact subset of the disk with arbitrary precision.

14.1 Prerequisites from functional analysis

Consider a Riemann surface X. We generalize the Fréchet topology on the vector space of holomorphic functions $\mathscr{O}(X)$ from Proposition 7.3 to a Fréchet topology on the vector space $\mathscr{E}(X)$ of smooth functions.

Definition 14.1 (Fréchet space of smooth functions). Consider a Riemann surface X and choose a sequence sequence $(K_i)_{i\in\mathbb{N}}$ of compact sets, each $K_i,\ i\in\mathbb{N}$, contained in a coordinate neighbourhood, and

$$X=\bigcup_{i\in\mathbb{N}}\overset{\circ}{K}_i$$

For each $i\in\mathbb{N}$ and $\nu:=(\nu_1,\nu_2)\in\mathbb{N}^2$ introduce the countable many semi-norms on X

$$p_{i,\nu}:\mathscr{E}(X)\to\mathbb{R}_+,\ p_{i,\nu}(f):=sup\ \{|D^\nu f(x)|:\ x\in K_i\}$$

Taking the finite intersections of the sets

$$V(i,\nu;\varepsilon):=\{f\in\mathscr{E}(U):\ p_{i,\nu}(f)<\varepsilon\},\ i\in\mathbb{N},\ \nu\in\mathbb{N},\ \varepsilon>0,$$

as neighbourhoods of zero defines a topology on $\mathscr{E}(X)$. The topology is independent of the choice of the compact sets and the choice of the charts. It is named the topology of *compact convergence of smooth functions and their derivatives*. The vector space $\mathscr{E}(X)$ becomes a complete topological vector space, hence a Fréchet space.

Note. The canonical injection

$$\mathscr{O}(X)\hookrightarrow\mathscr{E}(X)$$

is an injection of Fréchet spaces. Because Weierstrass' theorem implies that a compact convergent sequence of holomorphic functions has also all derivatives compact convergent.

The injection of the subspace of test functions from Definition 13.1

$$\mathscr{D}(X)\hookrightarrow\mathscr{E}(X)$$

is not continuous, because the topology on $\mathscr{D}(X)$ is coarser than the subspace topology from $\mathscr{E}(X)$: Convergence of a sequence in $\mathscr{D}(X)$ requires a compact subset which contains the support of all functions from the sequence.

Lemma 14.2 (Continuous linear functionals on $\mathscr{E}(X)$ have compact support). *Consider a Riemann surface X. Any continuous linear functional*

$$T:\mathscr{E}(X)\to\mathbb{C}$$

has compact support, i.e. there exists a compact subset $K\subset X$ with

$$T(f)=0$$

for all $f\in\mathscr{E}(U)$ with supp $f\subset X\setminus K$.

Proof. By continunity of T there exists an $\varepsilon > 0$ and neighbourhood of zero V_ε in $\mathscr{E}(X)$ with

$$f \in V_\varepsilon = V(j_1, \nu_1; \varepsilon) \cap ... \cap V(j_n, \nu_n; \varepsilon) \implies |T(f)| < 1$$

Consider the compact set

$$K := K_{j_1} \cup ... \cup K_{j_n} \subset X.$$

For $f \in \mathscr{E}(X)$ with

$$supp\ f \subset X \setminus K$$

and arbitrary $\alpha > 0$ we have

$$|T(\alpha \cdot f)| = 0$$

hence

$$f \in V_\varepsilon \text{ and } \alpha \cdot |T(f)| = |T(\alpha f)| < 1$$

or

$$|T(f)| < 1/\alpha.$$

Because $\alpha > 0$ is arbitrary we obtain

$$T(f) = 0,\ q.e.d.$$

Remark 14.3 (Hahn-Banach theorem).

1. Consider a complex Fréchet space V and a subspace $V_0 \subset V$. Then any continuous linear functional

   $$\lambda : V_0 \to \mathbb{C}$$

   extends to a continuous linear functional

   $$\Lambda : V \to \mathbb{C}.$$

   For a proof see [31, Theor. 3.6].

2. Consider a complex Fréchet space V and two subspaces

   $$V_0 \subset V_1 \subset V.$$

   Then V_0 is dense in V_1 if any continuous linear functional

   $$\lambda : V \to \mathbb{C},$$

   which vanishes on V_0 also vanishes on V_1.

Proposition 14.4 is an application of Weyl's Lemma for distribution-valued solutions of the homogenuous $\overline{\partial}$-equation.

Proposition 14.4 (Integral representation with holomorphic kernel). *Consider a Riemann surface X and an open subset $Y \subset X$ with a continuous linear functional*

$$S : H^0(X, \mathscr{E}^{0,1}) \to \mathbb{C}$$

satisfying

$$S(d''g) = 0$$

for all

$$g \in \mathscr{E}(X) \text{ with supp } g \subset\subset Y.$$

Then there exists a holomorphic form $\sigma \in \Omega^1(Y)$ satisfying for all $\omega \in H^0(X, \mathscr{E}^{0,1})$ with supp $\omega \subset\subset Y$

$$S(\omega) = \iint_Z \sigma \wedge \omega$$

Proof. i) *Construction of local forms σ_U*: Consider a chart of X

$$z : U \to V$$

with $U \subset Y$. Any test function $\phi \in \mathscr{D}(U)$ defines a global $(0,1)$-form $\tilde{\phi} \in H^0(X, \mathscr{E}^{0,1})$ by

$$\tilde{\phi}(x) := \begin{cases} \phi \cdot d\bar{z} & x \in U \\ 0 & x \in X \setminus U \end{cases}$$

and a distribution

$$S_U : \mathscr{D}(U) \to \mathbb{C},\ S_U[\phi] := S(\tilde{\phi}).$$

By assumption for all $g \in \mathscr{D}(U)$

$$S_U[\overline{\partial} g] = 0$$

Corollary 13.12 implies the existence of a unique holomorphic function $h \in \mathscr{O}(U)$ satisfying for all $\phi \in \mathscr{D}(U)$

$$S_U[\tilde{\phi}] = \iint_U h(z) \cdot \phi \ dz \wedge d\bar{z}$$

Define

$$\sigma_U := h\ dz \in \Omega^1(U)$$

Then for all $\omega \in \mathscr{E}^{1,0}(X)$ with *supp* $\omega \subset\subset U$

$$S[\omega] = \iint_U \sigma_U \wedge \omega$$

ii) *Gluing the local forms σ_U*: One checks that the construction from part i) is independent from the choice of the chart. The local forms glue to a global form

$$\sigma \in H^0(Y, \Omega^1)$$

with $\sigma|U = \sigma_U$. For each $\omega \in \mathscr{E}^{0,1}(X)$ with compact support contained in a chart of Y we obtain

$$S[\omega] = \iint_Y \sigma \wedge \omega$$

Any form $\omega \in \mathscr{E}^{0,1}(Y)$ with *supp* $\omega \subset\subset Y$ decomposes by using a partition of unity into a sum

$$\omega = \omega_1 + ... + \omega_n$$

satisfying for $j = 1, ..., n$

$$\omega_j \in \mathscr{E}^{0,1}(Y)$$

and *supp* ω_j a relatively compact subset of the domain of a chart, hence

$$S[\omega_j] = \iint_Y \sigma \wedge \omega_j.$$

As a consequence

$$S[\omega] = \sum_{j=1}^{n} S[\omega_j] = \sum_{j=1}^{n} \iint_Y \sigma \wedge \omega_j = \iint_Y \left(\sigma \wedge \sum_{j=1}^{n} \omega_j\right) = \iint_Y \sigma \wedge \omega, \ q.e.d.$$

14.2 Runge sets

Consider a Riemann surface X and a subset $Y \subset X$. Informally speaking:

Taking the Runge hull of Y means *to plug the relatively compact holes* of Y. It is helpful to conceive the other components of $X \setminus Y$ as the *unbounded components* of the complement of Y.

For an open Riemann surface X the main property of a *Runge domain* Y, i.e. of a domain without relatively compact holes, is Theorem 14.14: Each holomorphic function on Y has a compact approximation by global holomorphic functions on X.

Definition 14.5 (Runge hull and Runge set). Consider a Riemann surface X and a subset $Y \subset X$.

1. The *Runge hull* $h_X(Y)$ of Y with respect to X is the union of Y with all relatively compact connected components of $X \setminus Y$.

2. A *Runge set* $Y \subset X$ is closed with respect to taking the Runge hull, i.e. it satisfies

$$Y = h_X(Y).$$

Recall from Definition 4.18 that the term *relatively compact* refers to the closure taken with respect to X if not stated otherwise.

Example 14.6 (Runge hull).

Consider the open annulus

$$Y := \{z \in \mathbb{C} : \ 1 < |z| < 2\}$$

Then

$$h_{\mathbb{C}}(Y) = \{z \in \mathbb{C} : \ |z| < 2\}$$

while

$$h_{\mathbb{C}^*}(Y) = Y.$$

The hole of Y with respect to $X = \mathbb{C}^*$ is not relatively compact, while the hole of Y with respect to $X = \mathbb{C}$ is relatively compact, even compact.

Lemma 14.7 (Runge hull). *Consider a Riemann surface X and a subset $Y \subset X$.*

1. *For closed Y also $h_X(Y)$ is closed.*
2. *For compact Y also $h_X(Y)$ is compact.*
3. *For a compact subset $K_1 \subset X$ and a compact Runge set $K \subset X$ with*

$$K_1 \subset \overset{\circ}{K}$$

exists an open Runge set $Y \subset X$ with

$$K_1 \subset Y \subset K.$$

4. *Each connected component of an open Runge set is a Runge domain.*

Proof. 1. See [8, Satz 23.5].

2. See [8, Satz 23.5].

3. We shrink the compact Runge set K to a Runge domain Y: We choose for each point $x \in \partial K$ a chart of X around x defined on an open set $U(x) \subset X$ with

$$U(x) \cap K_1 = \emptyset$$

Within each $U(x)$ we choose a compact disk $D(x)$ with center x. By compactness of K finitely many disks

$$D(x_1), ..., D(x_n)$$

provide a covering of ∂K. Set

$$D := D(x_1) \cup \ldots \cup D(x_n)$$

a connected set. Define

$$Y := K \setminus D$$

Then $Y \subset X$ is open and

$$K_1 \subset Y \subset K$$

In order to prove that Y is a Runge domain, let C_Y be a given component of $X \setminus Y$. We employ the equation

$$X \setminus Y = (X \setminus K) \cup D$$

- Either

$$C_Y \cap D \neq \emptyset.$$

  Then $C_Y \cup D$ is connected and

$$C_Y \cup D \subset X \setminus Y,$$

  hence

$$D \subset C_Y$$

  There exists a component C_K of

$$X \setminus K \subset X \setminus Y$$

  with

$$C_K \cap D \neq \emptyset \text{ and a posteriori } C_K \cap C_Y \neq \emptyset$$

  Because C_Y is a component of $X \setminus Y$ then

$$C_Y \subset C_K$$

- Or

$$C_Y \cap D = \emptyset.$$

  Then

$$C_Y \subset (X \setminus K)$$

  which implies

$$C_Y \subset C_K$$

  for a component C_K of $X \setminus K$.

In any case exists a component C_K of $X \setminus K$ with

$$C_Y \subset C_K$$

By assumption C_K is not relatively compact. As a consequence C_Y is not relatively compact.

4. See [8, Satz 23.8], q.e.d.

Lemma 14.8 (Exhaustion by relatively compact domains). *Any open Riemann surface has an exhaustion by relatively compact domains.*

Proof. Choose an arbitrary point $\hat{x} \in X$ and an exhaustion $(Y_i)_{i\in\mathbb{N}}$ of X by open subsets satisfying

$$Y_i \subset\subset Y_{i+1},\ i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$ denote by $\hat{Y}_i$ the connected component of Y_i which contains the point $\hat{x}$. The sequence $(\hat{Y}_i)_{i\in\mathbb{N}}$ is also an exhaustion of X: A given point $x \in X$ can be joined to $\hat{x}$ by a path γ in X. The image of γ is compact, hence contained in Y_i for suitable $i \in \mathbb{N}$ and therefore also contained in $\hat{Y}_i$, q.e.d.

Proposition 14.9 (Existence of a Runge exhaustion). *Any open Riemann surface X has a* Runge exhaustion*, i.e. an exhaustion $(Y_i)_{i\in\mathbb{N}}$ by relatively compact Runge domains*

$$Y_i \subset\subset Y_{i+1},\ i \in \mathbb{N},\ \textit{satisfying } X = \bigcup_{i\in\mathbb{N}} Y_i$$

Proof. It suffices to show that any compact set $K \subset X$ has a relatively compact Runge domain $Y \subset\subset X$

$$K \subset Y \subset\subset X$$

Lemma 14.8 provides a connected compact set K_1 with $K \subset K_1$. Then choose a compact set K_2 with

$$K_1 \subset \mathring{K}_2.$$

Lemma 14.7, part 3) applied to the pair $(K_1, h_X(K_2))$ provides a runge domain $Y_1 \subset X$ satisfying

$$K_1 \subset Y_1 \subset h_X(K_2)$$

Let Y be the connected component of Y_1 which contains K_1. Due to Lemma 14.7, part 4) also Y is a Runge domain, q.e.d.

14.3 Approximation of holomorphic functions

The current section shows the usefullness of Runge domains for the approximative extension of holomorphic functions to larger domains.

Lemma 14.10 (Existence of non-constant holomorphic function). *On an open Riemann surface X for any relatively compact, open subset*

$$Y \subset\subset X$$

exists a holomorphic function

$$f \in H^0(Y, \mathscr{O})$$

which is non-constant on each connected component of Y.

Proof. Lemma 14.8 provides an exhaustion $(Y_i)_{i \in \mathbb{N}}$ of X by relatively compact domains. Because $\overline{Y} \subset X$ is compact, but X is not compact, we have

$$\overline{Y} \subsetneq X$$

Due to the compactenss of $\overline{Y}$ the exhaustion from part i) satisfies for suitable $i_0 \in \mathbb{N}$

$$\overline{Y} \subset Y_{i_0} =: \tilde{Y}$$

The domain $\tilde{Y}$ satisfies

$$Y \subset\subset \tilde{Y} \subset\subset X.$$

The proper inclusion

$$\overline{Y} \subsetneq \tilde{Y}$$

implies the existence of a point

$$a \in \tilde{Y} \setminus Y$$

Proposition 7.18 provides a meromorphic function $F \in \mathscr{M}(\tilde{Y})$ with a single pole at the point a. The restriction

$$f := F|Y$$

satisfies the claim, q.e.d.

For a Riemann surface X Dolbeault's Theorem 6.15 states

$$H^1(X, \mathscr{O}) = \frac{H^0(X, \mathscr{E}^{0,1})}{im[H^0(X, \mathscr{E}) \xrightarrow{d''} H^0(X, \mathscr{E}^{0,1})]}$$

Corollary 14.12 is a first step to conclude for an open Riemann surface X

$$H^1(X, \mathscr{O}) = 0.$$

The proof of Corollary 14.12 relies on the finiteness results for the holomorphic obstructions after restriction along relatively compact pairs, see Chapter 7. The final result will be proved in Theorem 14.16.

Proposition 14.11 (Killing holomorphic obstructions by restriction). *Consider an open Riemann surface X and two open subsets*

$$Y \subset\subset Y' \subset X.$$

Then the restriction of cohomoloy classes vanishes, i.e.

$$im[H^1(Y', \mathscr{O}) \to H^1(Y, \mathscr{O})] = 0.$$

Proof. Because $Y \subset\subset Y'$ we find an open set $\tilde{Y}$ with

$$Y \subset\subset \tilde{Y} \subset\subset Y'$$

The restriction factors

$$H^1(Y',\mathscr{O}) \to H^1(\tilde{Y},\mathscr{O}) \to H^1(Y,\mathscr{O})$$

Hence we may assume $Y' \subset\subset X$.

i) *Finite dimension of the space of obstructions*: Proposition 7.15 implies for the relatively-compact pair $Y \subset\subset Y'$ that the restriction has finite dimension

$$n := dim\ im[H^1(Y',\mathscr{O}) \to H^1(Y,\mathscr{O})] < \infty$$

The assumption that X is an open Riemann surface is not needed for this first result. Choose classes

$$\xi_1,...,\xi_n \in H^1(Y',\mathscr{O})$$

such that their restrictions to Y form a basis of the n-dimensional complex vector space

$$V := im[H^1(Y',\mathscr{O}) \to H^1(Y,\mathscr{O})]$$

ii) *Existence of a non-constant holomorphic function on Y'*: Because X is an open Riemann surface Lemma 14.10 applies and provides a holomorphic function

$$f \in H^0(Y',\mathscr{O})$$

which is not constant on any connected component of Y'. Because the complex vector space $H^1(Y',\mathscr{O})$ is also a module over the ring $H^0(Y',\mathscr{O})$ the elements

$$f \cdot \xi_1,...,f \cdot \xi_n$$

belong to $H^1(Y',\mathscr{O})$ and their restrictians to Y can be represented as

$$f \cdot \xi_\nu = \sum_{\mu=1}^{n} c_{\nu\mu} \cdot \xi_\mu,\ \nu = 1,...,n,$$

with a matrix

$$C := (c_{\mu\nu}) \in M(n \times n,\mathbb{C})$$

The holomorphic function

$$F := det(f \cdot \mathbb{1} - C) \in H^0(Y',\mathscr{O})$$

does not vanish identically on any connected component of Y': Otherwise the equality

$$det(f \cdot \mathbb{1} - C) = 0$$

would represent f on that component as zero of a polynomial from $\mathbb{C}[T]$, in particular f would be constant on that component.

The above system of linear equations in the vector space $H^1(Y,\mathscr{O})$ reads

$$(f\cdot \mathbb{1}-C)\cdot(\xi_1,...,\xi_n)^{\perp}=0$$

Cramer's rule for solving a system of linear equations shows for $\nu=1,...,n$ and the "unknown" ξ_ν

$$det\ (f\cdot \mathbb{1}-C)\cdot\xi_\nu=0,$$

i.e.

$$(F\cdot\xi_\nu)|Y=0.$$

iii) *Killing all obstructions by restriction*: We represent a given class $\zeta\in H^1(Y',\mathscr{O})$ by a cocyle

$$(f_{ij})\in Z^1(\mathscr{U},\mathscr{O})$$

such that each element U_i of the open covering $\mathscr{U}=(U_i)_{i\in I}$ of Y' contains at most one zero of F. Hence for $i\neq j$ the restriction

$$F|(U_i\cap U_j)\in\mathscr{O}^*(U_i\cap U_j)$$

has no zeros. As a consequence, the holomorphic function

$$g_{ij}:=\frac{f_{ij}}{F}$$

defines a cocycle

$$(g_{ij})\in Z^1(\mathscr{U},\mathscr{O})$$

with class

$$\xi:=[(g_{ij})]\in H^1(Y',\mathscr{O}).$$

Then

$$\zeta=[(f_{ij})]=F\cdot[(g_{ij})]=F\cdot\xi\in H^1(Y',\mathscr{O})$$

Due to part ii) the restriction satisfies

$$\zeta|Y=(F\cdot\xi)|Y=0\in H^1(Y,\mathscr{O}),\ q.e.d.$$

Corollary 14.12 (Existence of primitives on relatively compact open subsets). *Consider an open Riemann surface X and two open subsets*

$$Y\subset\subset Y'\subset X.$$

Then for any $\omega\in H^0(Y',\mathscr{E}^{0,1})$ *exists a function* $f\in H^0(Y,\mathscr{E})$ *with*

$$d''f=\omega|Y.$$

Proof. By Dolbeault's Theorem 5.4 for each $x\in X$ the sequence on stalks

$$0\rightarrow\mathscr{O}_x\rightarrow\mathscr{E}_x\xrightarrow{d''}\mathscr{E}_x^{0,1}\rightarrow 0$$

is exact. For a suitable open covering $\mathscr{U} = (U_i)_{i \in I}$ of Y' there exists a cochain

$$(f_i) \in C^0(\mathscr{U}, \mathscr{E})$$

satisfying for each $i \in I$

$$d'' f_i = \omega | U_i$$

One obtains a cocycle

$$\xi := (f_{ij} := f_j - f_i) \in Z^1(\mathscr{U}, \mathscr{O})$$

Proposition 14.11 implies for the restriction

$$[\xi | Y] = 0 \in H^1(Y, \mathscr{O})$$

Hence there exists a cochain

$$(g_i) \in C^0(\mathscr{U} \cap Y, \mathscr{O})$$

satisfying for all $i, j \in I$

$$f_j - f_i = g_j - g_i \text{ or } f_i - g_i = f_j - g_j$$

As a consequence

$$f := (f_i - g_i) \in Z^1(\mathscr{U} \cap Y, \mathscr{E}) = H^0(Y, \mathscr{E})$$

By construction

$$d'' f = \omega | Y, \ q.e.d.$$

Theorem 14.14 proves the fundamental property of a Runge domain $Y \subset X$ in an open Riemann surface X: Any holomorphic function $f \in \mathscr{O}(Y)$ can be approximated by a holomorphic function $F \in \mathscr{O}(X)$ with arbitrary precision on a given compact subset $K \subset Y$. Informally: Holomorphic functions on a Runge domain of an open Riemann surface can be approximated by global holomorphic functions with arbitrary precision on a given compact subset.

Complex analysis in the plane $X = \mathbb{C}$ proves this theorem for disks $Y \subset X$ by using the Taylor polynomials of $f \in \mathscr{O}(Y)$ of sufficiently high order as global functions. The proof of Theorem 14.14 has to show that the role of the Taylor polynomials can be taken over by other global holomorphic functions. The proof will show the existence of such global holomorphic functions, but it will not provide an explicit construction.

Theorem 14.13 contains the main approximation result. Then Theorem 14.14 has only to extend the result by the standard method of a suitable exhaustion to a global approximation.

Theorem 14.13 (Holomorphic approximation on Runge domains). *Consider an open Riemann surface X and a relatively compact Runge domain $Y \subset\subset X$. Then for any open subset*

$$Y \subset Y' \subset\subset X$$

the restriction of Fréchet spaces

$$r : \mathscr{O}(Y') \to \mathscr{O}(Y)$$

has dense image.

Proof. i) *The claim from the view point of functional analysis*: The proof follows from the Hahn-Banach theorem. We apply Remark 14.3, part 2): Consider a continuous linear functional

$$T : \mathscr{O}(Y) \to \mathbb{C}$$

with

$$T|r(\mathscr{O}(Y')) = 0.$$

Then we have to show

$$T = 0.$$

The subsequent part of the proof has to verify a property of holomorphic functions on

$$Y' \subset\subset X$$

from a result about holomorphic functions on the subset

$$Y \subset Y'.$$

To achieve this task one has to extend holomorphic functions on Y to suitable objects on Y'. In general, holomorphic functions do not extend holomorphically from Y to Y'. Therefore we consider holomorphic functions on Y as the primitives of smooth $(0,1)$-forms and extend in a smooth way the coefficients of the forms. Part ii) translates the claim from part i) to a statement about smooth $(0,1)$-forms

$$\omega = d''g \in H^0(X, \mathscr{E}^{0,1})$$

and a claim about the vanishing of $S(\omega) \in \mathbb{C}$ for a suitable linear functional S which derives from T. According to part iii) the functional S will turn out as an integral operator with kernel a holomorphic 1-form

$$\sigma \in \Omega^1(X).$$

Part iv) will derive from the Runge property of Y and the identity theorem of holomorphic functions that σ has compact support. Part v) finally concludes $T = 0$.

ii) *Introducing the linear functional S*: We define a linear functional

$$S : H^0(X, \mathscr{E}^{0,1}) \to \mathbb{C}$$

as follows: For a given $\omega \in H^0(X, \mathscr{E}^{0,1})$ Corollary 14.12 provides a function $f \in H^0(Y', \mathscr{E})$ satisfying

$$d''f = \omega|Y'$$

Then define

$$S(\omega) := T(f|Y)$$

The definition is independent from the choice of f: If also $g \in H^0(Y', \mathscr{E})$ with

$$d''(g) = \omega|Y'$$

then

$$d''(f - g) = 0,$$

hence

$$f - g \in H^0(Y', \mathscr{O})$$

and

$$T((f - g)|Y) = 0.$$

To show that S is continuous we consider the vector space

$$V := \{(\omega, f) \in H^0(X, \mathscr{E}^{0,1}) \times H^0(X, \mathscr{E}) : \ d''f = \omega|Y\}$$

The continuity of d'' implies that the pairs of $(0,1)$-forms and their primitives on Y

$$V \subset H^0(X, \mathscr{E}^{0,1}) \times H^0(X, \mathscr{E})$$

form a closed subspace of a Fréchet space, hence a Fréchet space itself. The following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\ r \circ pr_2\ } & H^0(Y, \mathscr{E}) \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle T} \\ H^0(X, \mathscr{E}^{0,1}) & \xrightarrow{\ S\ } & \mathbb{C} \end{array}$$

The continuity of T, r and pr_2 and the openness of the surjective linear map pr_1, see Remark 7.11, imply that S is continuous.

iii) *Integral representation of S with holomorphic kernel*: Due to Lemma 14.2 the functional T has compact support, i.e. there exists a compact subset $K \subset Y$ with

$$T(f) = 0$$

for all

$$f \in H^0(Y, \mathscr{E}) \text{ with } supp\ f \subset Y \setminus K.$$

By the same Lemma also the functional S has compact support, i.e. there exists a compact set $L \subset X$ with

$$S(\omega) = 0$$

for all

$$\omega \in H^0(X, \mathscr{E}^{0,1}) \text{ with } supp\ \omega \subset X \setminus L.$$

As a consequence for all $g \in \mathscr{E}(X)$ with $supp\ g \subset\subset X \setminus K$

$$T(g) = S(g) = 0.$$

Therefore Proposition 14.4 implies the existence of a holomorphic form

$$\sigma \in \Omega^1(X \setminus K)$$

satisfying for all $\omega \in \mathscr{E}^{0,1}(X)$ with $supp\ \omega \subset\subset X \setminus K$

$$S(\omega) = \iint_{X \setminus K} \sigma \wedge \omega$$

iv) *Vanishing of the kernel due to the Runge condition*: By definition of the Runge hull each component C_K of

$$X \setminus h_X(K)$$

is not relatively compact. Hence C_K is not contained in $K \cup L$, i.e. C_K intersects

$$X \setminus (K \cup L)$$

See Figure 15.2: The unbounded component C_K intersects $X \setminus (K \cup L)$ because it is not contained in the compact set L. While the dashed relatively compact component of $X \setminus K$ is contained in L

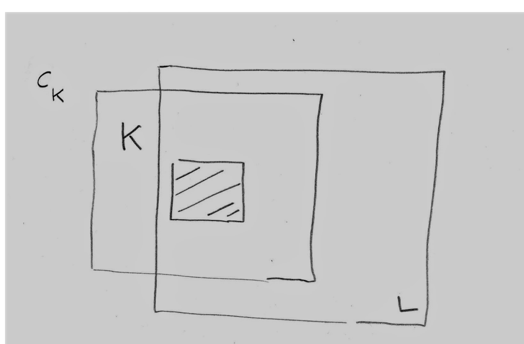


Fig. 14.1 Compact K with dashed relatively compact component of $X \setminus K$

According to the identity theorem

$$\sigma|X \setminus (K \cup L) = 0 \implies \sigma|C_K = 0.$$

As a consequence

$$\sigma|(X \setminus h_X(K)) = 0.$$

We obtain for all $\omega \in H^0(X, \mathscr{E}^{0,1})$ with *supp* $\omega \subset\subset X \setminus h_X(K)$

$$S(\omega) = 0$$

v) *Extending functions from $\mathscr{O}(Y)$ to $\mathscr{E}(X)$ keeping the value of T*: Consider a holomorphic function $f \in \mathscr{O}(Y)$. The compact subset $K \subset Y$ from part iii), which contains the support of f, has a Runge hull

$$h_X(K) \subset h_X(Y) = Y.$$

Hence the restriction $f|K$ has a smooth extension $g \in \mathscr{E}(X)$: In a neighbourhood U of K

$$g|U = f|U \text{ and } supp\ g \subset\subset Y.$$

As a consequence

$$T(f) = T(g|Y)$$

and $g|U$ is holomorphic, hence

$$d''(g|U) = 0$$

or

$$supp\ g \subset X \setminus h_X(K)$$

which implies due to part iii)

$$S(d''(g|U)) = 0$$

The commutative diagram from part ii) implies for all $f \in H^0(Y, \mathscr{O})$

$$T(f) = 0,\ q.e.d.$$

Theorem 14.14 (Runge approximation). *On an open Riemann surface X for any Runge domain $Y \subset X$ the restriction map between Fréchet spaces*

$$\mathscr{O}(X) \to \mathscr{O}(Y),\ f \mapsto f|Y,$$

has dense image.

Proof. One has to show: For given holomorphic function $f \in \mathscr{O}(Y)$, compact $K \subset Y$ and $\varepsilon > 0$ there exists a holomorphic function $F \in \mathscr{O}(X)$ satisfying

$$\|F - f\|_K < \varepsilon.$$

i) *Existence of a Runge exhaustion*: We find an open subset $\tilde{Y} \subset X$ satisfying

$$K \subset \tilde{Y} \subset\subset Y \subset X$$

The restriction $f|K$ factorizes via the restriction

$$r : \mathscr{O}(X) \to \mathscr{O}(\tilde{Y})$$

Therefore we may assume

$$Y = \tilde{Y} \subset\subset X.$$

Proposition 14.9 provides a Runge exhaustion $(Y_i)_{i \in \mathbb{N}}$ of X with $Y_0 = Y$.

ii) *Successive approximation along the Runge exhaustion*: Theorem 14.13 provides a holomorphic function $f_1 \in \mathscr{O}(Y_1)$ with

$$\|f_1 - f\|_K < \frac{1}{2^1} \cdot \varepsilon$$

and by induction on $n \geq 1$ a family of holomorphic functions $f_n \in \mathscr{O}(Y_n),\ n \geq 2$, satisfying

$$\|f_n - f_{n-1}\|_{\overline{Y}_{n-2}} < \frac{1}{2^n} \cdot \varepsilon$$

For arbitrary, but fixed $n \in \mathbb{N}$ the sequence $(f_\nu)_{\nu \geq n}$ is uniformly convergent on Y_n. As a consequence we obtain a global holomorphic function

$$f \in H^0(X, \mathscr{O})$$

satisfying

$$\lim_{\nu \to \infty} (f_\nu | Y_n) = f | Y_n$$

By construction

$$\|F - f\|_K \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \varepsilon = \varepsilon,\ q.e.d.$$

Example 14.15 (Counter example). The pair

$$(X, Y) := (\mathbb{C}, \mathbb{C}^*)$$

shows: In Theorem 14.14 the absence of relatively compact connected components of the complement $X \setminus Y$ is necessary. Here

$$X \setminus Y = \{0\}$$

is compact and the holomorphic function

$$f(z) = \frac{1}{z} \in \mathscr{O}(Y)$$

has no compact approximation by holomorphic functions on X: Assume

$$f = \lim_{n\to\infty} f_n(z),\ f_n \in \mathscr{O}(X) \text{ (Compact convergence)},$$

then Cauchy's integral theorem for integration along the positively oriented unit circle γ implies

$$2\pi i = \int_\gamma \frac{dz}{z} = \lim_{n\to\infty} \int_\gamma f_n(z)\, dz = 0,$$

a contradiction. Hence f has no Runge approximation. Indeed, according to Example 14.6 the domain $\mathbb{C}^*$ is not a Runge domain in $\mathbb{C}$.

Theorem 14.16 (The vanishing theorem on open Riemann surfaces). *On an open Riemann surface X any $(0,1)$-form $\omega \in H^0(X, \mathscr{E}^{0,1})$ has a primitive $f \in H^0(X, \mathscr{E})$, i.e. satisfying*

$$d'' f = \omega.$$

As a consequence

$$H^1(X, \mathscr{O}) = 0.$$

Proof. i) The form $\omega \in H^0(X, \mathscr{E}^{0,1})$ has on any relatively compact open subset $Y \subset\subset X$ a primitive

$$g \in H^0(Y, \mathscr{E}) \text{ satisfying } d''g = \omega|Y$$

due to Corollary 14.12.

ii) Proposition 14.9 provides an exhaustion $(Y_i)_{i\in\mathbb{N}}$ of X by relatively compact Runge domains

$$Y_i \subset\subset Y_{i+1},\ i \in \mathbb{N},\ \text{with } X = \bigcup_{i\in\mathbb{N}} Y_i$$

By induction on $n \in \mathbb{N}$ we construct a sequence of functions $f_n \in \mathscr{E}(Y_n)$ satisfying

$$d'' f_n = \omega|Y_n \text{ and } \|f_{n+1} - f_n\|_{Y_{n-1}} \le \frac{1}{2^n}$$

Induction start: Choose $f_0 \in \mathscr{E}(Y_0)$ according to Corollary 14.12.

Induction step $n \mapsto n+1$: Corollary 14.12 provides a function $g_{n+1} \in \mathscr{E}(Y_{n+1})$ satisfying

$$d'' g_{n+1} = \omega|Y_{n+1}$$

The restriction to Y_n satisfies

$$d''(g_{n+1}|Y_n - f_n) = 0$$

which implies the holomorphy

$$g_{n+1}|Y_n - f_n \in \mathscr{O}(Y_n)$$

The approximation Theorem 14.14 provides a holomorphic function

$$h \in \mathscr{O}(Y_{n+1})$$

with

$$\|(g_{n+1} - f_n) - h\|_{Y_{n-1}} \leq \frac{1}{2^n}.$$

Set

$$f_{n+1} := g_{n+1} - h \in \mathscr{E}(Y_{n+1})$$

Then

$$d'' f_{n+1} = d'' g_{n+1} = \omega|Y_{n+1} \text{ and } \|f_{n+1} - f_n\|_{Y_{n-1}} = \|(g_{n+1} - h) - f_n\|_{Y_{n-1}} \leq \frac{1}{2^n}$$

One checks as ususal the existence of a limit function

$$f = \lim_{n\to\infty} f_n \in \mathscr{E}(X)$$

satisfying

$$d'' f = \omega.$$

iii) By Dolbeault's theorem 6.15

$$H^1(X,\mathscr{O}) \simeq \frac{H^0(X,\mathscr{E}^{0,1})}{im[H^0(X,\mathscr{E}) \xrightarrow{d''} H^0(X,\mathscr{E}^{0,1})]},$$

which proves

$$H^1(X,\mathscr{O}) = 0, \; q.e.d.$$

Remark 14.17 (Leray covering). Consider an arbitrary Riemann surface X. The Vanishing Theorem 14.16 is a far reaching generalization of Theorem 6.16 from a disk in the plane to arbitrary open subsets of X. Theorem 14.16 implies: Any open covering of X is a Leray covering for the structure sheaf $\mathscr{O}$. With respect to a given invertible sheaf $\mathscr{L}$ on X an open covering $\mathscr{U} = (U_i)_{i\in I}$ is a Leray covering if for each $i \in I$

$$\mathscr{L}|U_i \simeq \mathscr{O}|U_i.$$

Chapter 15
Stein manifolds

15.1 Mittag-Leffler problem and Weierstrass problem

The Mittag-Leffler problem and the Weierstrass problem are the two main existence problems from complex analysis for domains in $\mathbb{C}$. We show that both problems are solvable on any open Riemann surface, in particular on any domain $G \subset \mathbb{C}$. While the solution of the Mittag-Leffler problem follows directly from the vanishing theorem, Theorem 14.16, the solution of the Weierstrass problems requires additional approximation results. We derive these results from a theorem from functional analysis about compact operators.

We prove the following theorems for open Riemann surfaces:

- Solution of the Mittag-Leffler problem, Theorem 15.3,
- solution of the Weierstrass problem, Theorem 15.7 and

Definition 9.4 introduced the concept of a Mittag-Leffler distribution of meromorphic differential forms. Analogously we now define the concept of a Mittag-Leffler distribution of meromorphic functions. It formalizes the Mittag-Leffler problem on a Riemann surface X: To find a global meromorphic function on X with given principal parts.

Definition 15.1 (Mittag-Leffler distribution of functions). Consider a Riemann surface X and an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X.

1. A *Mittag-Leffler distribution* of meromophic functions with respect to $\mathscr{U}$ is a cochain of meromorphic functions

$$(f_i)_{i \in I} \in C^0(\mathscr{U}, \mathscr{M})$$

with holomorphic coboundary

$$\delta f \in Z^1(\mathscr{U}, \mathscr{O}),$$

i.e. for all $i \in I$

$$f_j - f_i \in \mathcal{O}(U_i \cap U_j).$$

2. A *solution of the Mittag-Leffler distribution*

$$f = (f_i) \in C^0(\mathscr{U}, \mathscr{M})$$

is a meromorphic function

$$F \in \mathscr{M}(X)$$

with holomorphic

$$F - (f_i) \in C^0(\mathscr{U}, \mathscr{O}),$$

for all $i \in I$

$$F|U_i - f_i \in \mathscr{O}(U_i).$$

A solution of the Mittag-Leffler distribution

$$(f_i) \in C^0(\mathscr{U}, \mathscr{M})$$

from Definition 15.1 glues all local meromorphic functions

$$f_i \in \mathscr{M}(U_i),\ i \in I,$$

to a global meromorphic function F. With respect to any chart of X the function F has the same principal part as the local meromorphic functions $f_i,\ i \in I$. The "glue" are the local holomorphic functions $F|U_i - f_i,\ i \in I$.

Lemma 15.2 (Solvability of Mittag-Leffler distributions). *Consider a Riemann surface X. A Mittag-Leffler distribution of meromorphic functions with respect to an open covering $\mathscr{U} = (U_i)_{i \in I}$ of X*

$$f = (f_i) \in C^0(\mathscr{U}, \mathscr{M})$$

is solvable iff its class vanishes

$$[\delta f] = 0 \in H^1(X, \mathscr{O}).$$

Proof. Apparently

$$f_j - f_i \in B^1(\mathscr{U}, \mathscr{O}) \iff \exists\, (g_i) \in C^0(\mathscr{U}, \mathscr{O}) : f_j - f_i = g_j - g_i$$

i) The existence of (g_i) implies

$$f_i - g_i = f_j - g_j,$$

hence the family

$$(f_i - g_i) \in Z^0(\mathscr{U}, \mathscr{M})$$

defines a global meromorphic function $F = (f_i - g_i) \in \mathscr{M}(X)$ with

$$F|U_i - f_i = -g_i \in \mathscr{O}(U_i).$$

ii) The existence of $F \in \mathscr{M}(X)$ with

$$F|U_i - f_i =: -g_i \in \mathscr{O}()U_i$$

implies

$$f_i - g_i = f_j - g_j \text{ or } (f_j - f_i = g_j - g_i) \in B^1(\mathscr{U}, \mathscr{O}),\ q.e.d.$$

Theorem 15.3 (Solution of the Mittag-Leffler problem). *On an open Riemann surface X any Mittag-Leffler distribution of meromorphic functions is solvable.*

Proof. The Vanishing Theorem 14.16 shows $H^1(X, \mathscr{O}) = 0$, q.e.d.

The Weierstrass problem on a Riemann surface X asks for a global meromorphic function $f \in \mathscr{M}^*(X)$ with a prescribed divisor $D \in Div(X)$. Theorem 15.7 solves the Weierstrass problem. The proof goes along the classical lines for Runge approximation in $\mathbb{C}$, see [30, Kap. 12, § 2, Abschn. 3]. It applies the method of moving poles of a divisor to infinity (deutsch: “Polverschiebung”), see [8, § 26]. We start with the concept of a *weak solution*: It solves D by a smooth but not necessarily meromorphic function.

Definition 15.4 (Weak solution of a divisor). Consider a Riemann surface X and a divisor $D \in Div(X)$. Set

$$X_D := \{x \in X :\ D(x) \geq 0\}.$$

A *weak solution* of D is a smooth function $f \in \mathscr{E}(X_D)$ satisfying: For each point $p \in X$ exists a chart

$$z : U \to V$$

of X around p and a smooth function $\psi \in \mathscr{E}(U)$ with $\psi(p) \neq 0$ such that on $U \cap X_D$

$$f = \psi \cdot z^k,\ k := D(p).$$

Hence a weak solution of a divisor D is a smooth function f, which is defined outside those points where the divisor prescribes a pole, and locally satisfies the divisor. A weak solution f is a solution of the Weierstrass problem if the restriction $f|X_D$ is holomorphic.

Lemma 15.5 (Weak solution of a degree zero divisor). *Consider a Riemann surface X, a path*

$$\gamma : [0,1] \to X$$

and a neighbourhood U of $\gamma([0,1])$ which is relatively compact X. Set

$$a := \gamma(0) \text{ and } b := \gamma(1).$$

Then the difference of point divisors

$$D := B - A \in Div_0(X)$$

has a weak solution f with $f|(X \setminus U) = 1$.

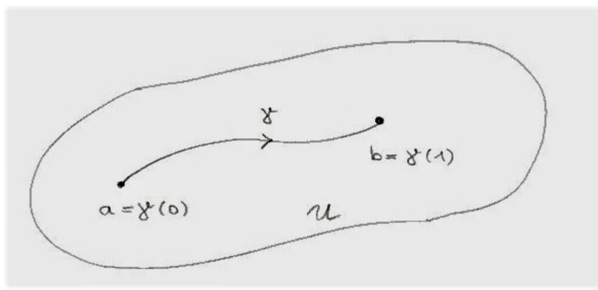


Fig. 15.1 Difference of two point divisors

Proof. i) *γ contained in a coordinate neighbourhood*: Assume that X has a chart

$$z : U \to D_1$$

with $\gamma([0,1]) \subset U$. We identify U with the unit disk D_1. To motivate the following construction note that the meromorphic function on D_1

$$D_1 \setminus \{b\} \to \mathbb{C},\ z \mapsto \frac{z-b}{z-a}$$

solves the divisor D on D_1. We modify the meromorphic function to a weak solution which has constant value $= 1$ near the boundary ∂D and therefore extends to $X \setminus \{a\}$: The function

$$log\ \frac{z-b}{z-a}$$

has a well-defined branch in the annulus

$$\{z \in D_1 :\ r < |z| < 1\}$$

because the additive constants of the logarithm of numerator and denominator cancel. We choose an intermediate radius

$$r < r' < 1$$

and a smooth function $\psi \in \mathscr{E}(U)$ with

$$\psi | D_r = 1 \text{ and } \psi | (D_1 \setminus D_{r'}) = 0.$$

Then the smooth function $f \in \mathscr{E}(U \setminus \{a\})$ with

$$f_0 := \begin{cases} exp\left(\psi \cdot log\ \dfrac{z-b}{z-a}\right) & r < |z| \\ \\ \dfrac{z-b}{z-a} & |z| < r \end{cases}$$

extends by 1 to a weak solution of D with

$$f_0 | (X \setminus U_0) = 1.$$

ii) *General case*: Consider a finite partition of $[0,1]$

$$0 = t_0 < t_1 < ... < t_n = 1$$

and charts of X

$$z_j : U_j \to D_1$$

such that for all $j = 1, ..., n$

$$\gamma([t_{j-1}, t_j]) \subset U_j.$$

For each $j = 1, ..., n$ part i) provides a weak solution f_j of the degree zero divisor

$$D_j := B_j - A_j,\ a_j := \gamma(t_{j-1}),\ b_j := \gamma(t_j)$$

with

$$f_j | (X \setminus U_j) = 1.$$

In the product

$$f := \prod_{j=1}^{n} f_j \in \mathscr{E}(X_D)$$

the singularity of f_{j+1} and the zero of f_j at the point $\gamma(t_j),\ j = 1, ..., n-1$, cancel. As a consequence, $f \in \mathscr{E}(X_D)$ is a weak solution of D which satisfies

$$f | (X \setminus U) = 1,\ U := \bigcup_{j=1}^{n},\ q.e.d.$$

Proposition 15.6 (Construction of a weak solution for a general divisor). *On an open Riemann surface X any divisor $D \in Div(X)$ has a weak solution.*

Proof. Let $(Y_i)_{i\in\mathbb{N}}$ be a Runge exhaustion of X, see Proposition 14.9.

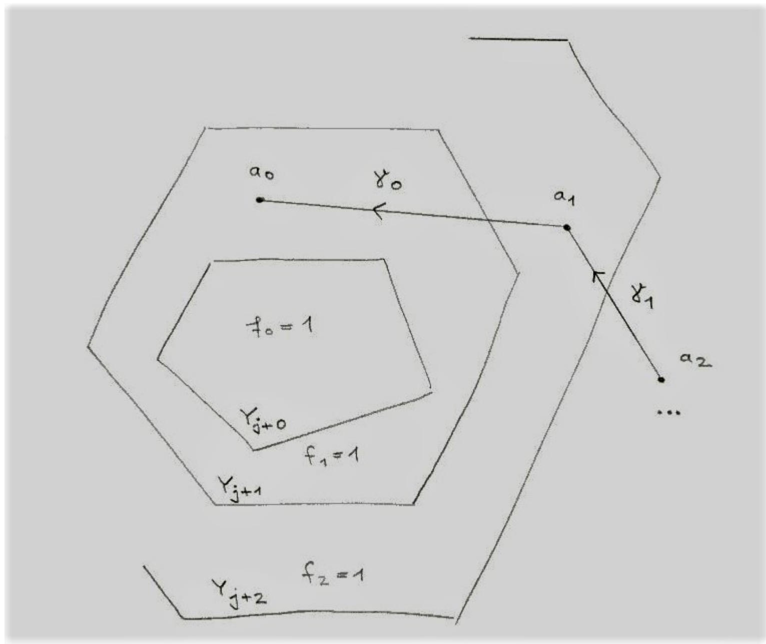


Fig. 15.2 Moving the pole to infinity

i) *Moving the pole of a degree zero divisor to infinity*: Consider an arbitrary but fixed index $j\in\mathbb{N}$ and a point

$$a_0\in X\setminus\overline{Y}_j,\ \text{in particular } a_0\in X\setminus Y_j$$

Because

$$Y_j = h_X(Y_j)$$

is a Runge domain, the point $a_0\in X\setminus Y_j$ belongs to an unbounded connected component C_{Y_j} of $X\setminus Y_j$. The unbound component C_{Y_j} is not contained in the compact set $\overline{Y}_{j+1}$. Hence exists a point

$$a_1\in(C_{Y_j}\setminus\overline{Y}_{j+1})\subset(X\setminus\overline{Y}_{j+1}),\ \text{in particular } a_1\in X\setminus Y_{j+1}$$

We choose a path γ_0 in C_{Y_j} from a_1 to a_0. Consider the corresponding point divisors $A_1,\ A_0\in Div(X)$. Lemma 15.5 provides a weak solution f_0 of the degree zero divisor

$$A_0 - A_1\in Div(X)$$

with
$$f_0|Y_j = 1.$$
Iterating the construction provides

- a sequence of points $(a_\nu)_{\nu\in\mathbb{N}}$ with
$$a_\nu \in X \setminus Y_{j+\nu}$$
- a corresponding sequence of paths γ_ν from $a_{\nu+1}$ to a_ν
- and a sequence of weak solutions f_ν of the degree zero divisors
$$A_\nu - A_{\nu+1} \in Div(X)$$
satisfying
$$f_\nu|Y_{j+\nu} = 1.$$

For given $n \in \mathbb{N}$ the finite product
$$f_0 \cdot \ldots \cdot f_n$$
is a weak solution of the degree zero divisor $A_0 - A_{n+1} \in Div(X)$. The infinite product of smooth functions
$$\lim_{n\to\infty} \left(\prod_{\nu=0}^{n} f_\nu \right)$$
converges towards a smooth function f: For a given point $x \in X$ holds for allmost all $\nu \in \mathbb{N}$
$$x \subset Y_{j+\nu}$$
hence for allmost all $\nu \in \mathbb{N}$
$$f_\nu(x) = 1.$$
The function f is a weak solution of the point divisor A_0 satisfying
$$f|Y_\nu = 1$$

ii) *Constructing a weak solution*: According to the exhaustion of X by Runge domains we split the given divisor D into into its successive building blocks $D_\nu \in Div(X),\ \nu \in \mathbb{N}$,
$$D_\nu(x) := \begin{cases} D(x) & x \in Y_\nu \setminus \overline{Y}_{\nu-1} \\ \\ 0 & x \notin Y_\nu \setminus \overline{Y}_{\nu-1} \end{cases}$$
Here $Y_{-1} := \emptyset$. Then
$$D = \sum_{\nu=0}^{\infty} D_\nu.$$

Each divisor $D_\nu,\ \nu \in \mathbb{N}$, has finite support. Hence it is a finite sum of divisors $B - A \in Div(X)$ which are the difference of two point divisors. Part i) provides a weak solution f_ν of D_ν satisfying

$$f_\nu | Y_{\nu-1} = 1$$

The smooth function

$$f := \prod_{\nu=0}^{\infty} f_\nu$$

is a weak solution of D, q.e.d.

Theorem 15.7 (Solution of the Weierstrass problem). *On an open Riemann surface X any divisor $D \in Div(X)$ is a principal divisor, i.e. D is the divisor of a meromorphic function $f \in \mathscr{M}^*(X)$.*

Proof. With respect to a suitable open covering $\mathscr{U} = (U_i)_{i \in I}$ with simply connected open sets the divisor D is defined by a cochain

$$(f_i) \in C^0(\mathscr{U}, \mathscr{M})$$

satisfying for all $i \in I$

$$div\ f_i = D|U_i.$$

For each $i, j \in I$

$$\frac{f_i}{f_j} \in \mathscr{O}^*(U_i \cap U_j)$$

i) *Existence of a weak solution*: Proposition 15.6 provides a weak solution $\psi \in \mathscr{E}(X_D)$ of D. It satisfies for each $i \in I$

$$\psi | U_i = \psi_i \cdot f_i$$

with a suitable smooth function

$$\psi_i \in \mathscr{E}^*(U_i)$$

which can be assumed as

$$\psi_i = e^{\phi_i} \text{ with } \phi_i \in \mathscr{E}(U_i)$$

because U_i is simply connected.

ii) *Modifying the weak solution to a solution*: For each $i, j \in I$ on $U_i \cap U_j$

$$\frac{f_i}{f_j} = \frac{\psi_j}{\psi_i} = e^{\phi_j - \phi_i} \in \mathscr{O}^*(U_i \cap U_j)$$

The holomorphic functions

$$\phi_{ij} := \phi_j - \phi_i \in \mathscr{O}(U_i \cap U_j)$$

define a cocycle

$$(\phi_{ij}) \in Z^1(\mathscr{U}, \mathscr{O}).$$

The Vanishing Theorem 14.16 provides a cochain

$$(g_i) \in C^0(\mathscr{U}, \mathscr{O})$$

satisfying

$$g_j - g_i = \phi_{ij} = \phi_j - \phi_i$$

Hence we obtain for $i, j \in I$

$$e^{g_j - g_i} = \frac{f_i}{f_j} \text{ i.e. } f_i \cdot e^{g_i} = f_j \cdot e^{g_j}$$

These local function glue to a global meromorphic function $f \in \mathscr{M}(X)$ satisfying for all $i \in I$

$$f|U_i = f_i \cdot e^{g_i}$$

and

$$div\ f|U_i = div\ f_i = D|U_i,\ q.e.d.$$

Proposition 15.8 (Meromorphic functions as quotient of holomorphic functions). *On an open Riemann surface X the field $\mathscr{M}(X)$ of meromorphic functions is the quotient field of the ring $\mathscr{O}(X)$ of holomorphic function, i.e. any meromorphic function $f \in \mathscr{M}(X)$ has the form*

$$f = \frac{g}{h}$$

with two holomorphic functions $g,\ h \in \mathscr{O}(X),\ h \neq 0$.

Proof. If $f \notin \mathscr{O}(X)$ consider the pole divisor

$$D \in Div(X) \text{ of } f.$$

Theorem 15.7 provides a holomorphic function $h \in \mathscr{O}$ with

$$div\ h = -D.$$

The product is holomorphic

$$g := h \cdot f \in \mathscr{O}(X),$$

which proves the claim, q.e.d.

15.2 Triviality of holomorphic line bundles

The present section proves that any line bundle on an open Riemann surface X is holomorphically trivial. The proof relies on the vanishing $H^1(X,\mathscr{O}) = 0$. In addition, the proof makes some preparations and uses further input from functional analysis. We present that input in some detail because it became a standard method in complex analysis of several complex variables. Remark 15.17 shows a second proof which replaces the result from functional analysis by a different argument from algebraic topology in the specific situation.

Lemma 15.9 (Extending cohomology classes to relatively-compact subsets). *Consider a Riemann surface X and a holomorphic line bundle $\mathscr{L}$ on X. Then for any relatively compact subset*

$$Y \subset\subset X$$

and any open subset $Y_0 \subset Y$ the restriction map

$$H^1(Y,\mathscr{L}) \to H^1(Y_0,\mathscr{L})$$

is surjective.

Proof. We choose a finite index set $I = \{1,...,r\}$ and an open covering $\mathscr{U} = (U_i)_{i\in I}$ of Y by open sets $U_i \subset X,\ i \in I$, satisfying for each $i \in I$

$$\mathscr{L}|U_i \simeq \mathscr{O}|U_i$$

For each $k = 0,...,r$ set

$$Y_k := Y_0 \cup \bigcup_{j=1,...,k} U_j$$

Then

$$Y = Y_r$$

We show for each arbitrary but fixed $k \in I$ the surjectivity of the restriction

$$H^1(Y_k,\mathscr{L}) \to H^1(Y_{k-1},\mathscr{L}) :$$

The trick of the proof is to consider the two coverings

$$\mathscr{V} = (V_i := U_i \cap Y_{k-1})_{i=1,...,r} \text{ of } Y_{k-1}$$

and

$$\mathscr{V}' := (V_i')_{i=1,...,r} \text{ of } Y_k,$$

with

$$V_i' := \begin{cases} V_i & i \neq k \\ U_k & i = k \end{cases}$$

For all $i \neq j \in I$ the two coverings have the same intersections

$$V_i' \cap V_j' = V_i \cap V_j$$

because also for $i = k$ and $j \neq k$

$$V_k' \cap V_j' = U_k \cap V_j = (U_k \cap Y_{k-1}) \cap V_k = V_k \cap V_j.$$

Both coverings are Leray coverings for $\mathscr{L}$ due to Remark 14.17. Therefore

$$Z^1(\mathscr{V}', \mathscr{L}) = Z^1(\mathscr{V}, \mathscr{L})$$

implies the surjectivity of

$$H^1(Y_k, \mathscr{L}) = H^1(\mathscr{V}', \mathscr{L}) \to H^1(\mathscr{V}, \mathscr{L}) = H^1(Y_{k-1}, \mathscr{L}), \ q.e.d.$$

In Chapter 7 we proved that the cohomology of the structure sheaf $\mathscr{O}$ becomes finite-dimensional under restriction to a relatively compact subset. The cochain groups under consideration were infinite-dimensional complex vector spaces in general. Therefore we had to provide them with the structure of topological vector spaces. Chapter 7 identified suitable Hilbert spaces of holomorphic functions and cochains. A main input from functional analysis was the existence of orthogonal complements of closed subspaces of Hilbert spaces. Remark 10.21 states that the proof generalizes to holomorphic line bundles $\mathscr{L}$.

The present section gives another proof for the finiteness result by using a deep theorem about compact operators between Fréchet spaces. We topologize the cochain groups of $\mathscr{L}$ by a Fréchet topology which generalizes the topology of compact convergence of holomorphic functions. The existence of complementary vector spaces does no longer hold true in the category of Fréchet spaces. Instead we have to apply a stronger result from functional analysis: Schwartz' theorem on compact operators. This result turns out important also for many finiteness results in the theory of several complex variables. Remark 15.10 gives a short introduction.

Remark 15.10 (Fréchet topology of compact convergence for holomorphic sections).

1. *Fréchet topology*: Consider a Riemann surface X and an invertible sheaf $\mathscr{L}$. We choose a countable covering
$$\mathscr{U} = (U_i)_{i \in I}$$
of X by open sets such that each $U_i,\ i \in I$, has the following properties:
   - There exists a chart of X
   $$z_i : U_i \to V_i \subset \mathbb{C}$$
   - There exists a trivialization of the invertible sheaf
   $$\mathscr{L}|U_i \simeq \mathscr{O}|U_i$$

For each $i \in I$ the topology of compact convergence on the vector space $\mathscr{O}(V_i)$ provides the vector space $\mathscr{L}(U_i)$ with a Fréchet topology. For each $q \in \mathbb{N}$ the product topology for the countable family of Fréchet spaces

$$C^q(\mathscr{U},\mathscr{L}) := \prod_{(i_0,\ldots,i_q)} \mathscr{L}(U_{i_0,\ldots,i_q})$$

is a Fréchet space. The linear coboundary operator

$$\delta^q : C^q(\mathscr{U},\mathscr{L}) \to C^{q+1}(\mathscr{U},\mathscr{L})$$

is continuous. Hence the closed subspace of cocycles

$$Z^q(\mathscr{U},\mathscr{L}) \subset C^q(\mathscr{U},\mathscr{L})$$

is a Fréchet space.

2. *Laurent Schwartz' theorem about compact linear maps*: Let V, W be complex Fréchet spaces. A linear map

$$\phi : V \to W$$

is *compact* (before: *completely continuous*) if a suitable neighbourhood of zero $V_0 \subset V$ is mapped to a relatively compact subset

$$\phi(V_0) \subset\subset W.$$

Schwartz' theorem states: If

$$f : V \to W$$

is a linear continuous surjective map and

$$g : V \to W$$

a linear compact map then

$$im[f - g : V \to W] \subset W$$

is a closed subspace with

$$codim\ im[f - g : V \to W] < \infty$$

For a proof see [32, Cor. de Theor. 2]. Note that any compact linear map is continuous. Schwartz theorem can be rephrased: The image of any compact perturbation of a surjective linear continuous map between Fréchet spaces has finite codimension.

Proposition 15.11 (Finiteness of the cohomology on relatively compact subsets). *Consider a Riemann surface X and a holomorphic line bundle $\mathscr{L}$ on X. Then for* *any open, relatively compact subset*

$$Y \subset\subset X$$

holds

$$dim\, H^1(Y, \mathscr{L}) < \infty$$

Proof. We choose a relatively compact subset

$$Y \subset\subset Y' \subset\subset X.$$

There exist a finite index set $I = \{1, ..., r\}$ and open coverings

$$\mathscr{V} = (V_i)_{i \in I} \text{ of } Y \text{ and } \mathscr{U} = (U_i)_{i \in I} \text{ of } Y'$$

which satisfy for each $i \in I$

$$V_i \subset\subset U_i \text{ and } \mathscr{L}|U_i \simeq \mathscr{O}|U_i.$$

i) *Restriction as a compact linear map*: The restriction

$$\rho : Z^1(\mathscr{U}, \mathscr{L}) \to Z^1(\mathscr{V}, \mathscr{L})$$

is a compact linear map: For the proof one applies Montel's theorem to the restrictions

$$\mathscr{O}(U_i) \to \mathscr{O}(V_i),\ i \in I.$$

Hence also the linear map

$$g : C^0(\mathscr{V}, \mathscr{L}) \times Z^1(\mathscr{U}, \mathscr{L}) \to Z^1(\mathscr{V}, \mathscr{L}),\ (\eta, \xi) \mapsto \rho(\xi),$$

is compact.

ii) *Leray coverings with respect to $\mathscr{L}$*: The coverings $\mathscr{V}$ and $\mathscr{U}$ are Leray coverings for $\mathscr{L}$ due to Remark 14.17. Therefore Lemma 15.9 shows the surjectivity of the restriction

$$H^1(\mathscr{U}, \mathscr{L}) = H^1(Y', \mathscr{L}) \to H^1(Y, \mathscr{L}) = H^1(\mathscr{V}, \mathscr{L}),\ [\xi] \mapsto [\rho(\xi)].$$

As a consequence the linear map

$$f : C^0(\mathscr{V}, \mathscr{L}) \times Z^1(\mathscr{U}, \mathscr{L}) \to Z^1(\mathscr{V}, \mathscr{L}),\ (\eta, \xi) \mapsto \delta\eta + \rho(\xi),$$

is surjective.

iii) *Finite codimension due to Schwartz' theorem*: Schwartz' theorem, see Remark 15.10, implies: The image of the operator

$$f - g : C^0(\mathscr{V},\mathscr{L}) \times Z^1(\mathscr{U},\mathscr{L}) \to Z^1(\mathscr{V},\mathscr{L}),\ (\eta,\xi) \mapsto \delta\eta,$$

has finite codimension and a posteriori also the image of the coboundary map

$$\delta : C^0(\mathscr{V},\mathscr{L}) \to Z^1(\mathscr{V},\mathscr{L})$$

As a consequence

$$H^1(Y,\mathscr{L}) = H^1(\mathscr{V},\mathscr{L}) = \frac{Z^1(\mathscr{V},\mathscr{L})}{im[\delta : C^0(\mathscr{V},\mathscr{L}) \to Z^1(\mathscr{V},\mathscr{L})]}$$

has finite dimension, q.e.d.

Proposition 15.12 is a further example of the principle that finiteness of the holomorphic cohomology implies the existenc of a meromorphic object with suitable properties. Here the finiteness of the cohomology with values in a line bundles implies the existence of a non-zero meromorphic section. For a similar example recall Proposition 7.18.

Proposition 15.12 (Triviality of holomorphic line bundles on relative compact subsets). *Consider an open Riemann surface X and an open, relatively compact subset*

$$Y \subset\subset X.$$

Then for any line bundle

$$p : L \to Y$$

the invertible sheaf $\mathscr{L}$ on Y is isomorphic to the structure sheaf

$$\mathscr{L} \simeq \mathscr{O}_Y$$

Proof. i) *Existence of a non-zero meromorphic section*: Proposition 15.11 implies

$$dim\, H^1(Y,\mathscr{L}) =: k < \infty$$

Hence the present step is completely analoguous to the proof of Proposition 7.18 about the existence on non-constant meromorphic functions: Choose a point $p \in Y$ and consider a chart of Y around p

$$z : U_0 \to D_R(0)$$

such that

$$\mathscr{L}|U_0 \simeq \mathscr{O}|U_0.$$

Setting

$$U_1 := Y \setminus U_0$$

defines an open covering

$$\mathscr{U} := (U_0, U_1)$$

of Y. Due to Lemma 6.4 the canonical map

$$H^1(\mathscr{U},\mathscr{L}) \to H^1(Y,\mathscr{L})$$

is injective. On

$$U_0 \cap U_1 = U_0^*$$

the holomorphic functions

$$1/z_j \in \mathscr{O}^*(U_0^*),\ j = 1,...,k+1,$$

define under the isomorphism

$$\mathscr{L}|U_0 \simeq \mathscr{O}|U_0, \text{ in particular} \mathscr{L}|U_0^* \simeq \mathscr{O}|U_0^*,$$

the $k+1$ cocycles

$$\zeta_j \in Z^1(\mathscr{U},\mathscr{L})$$

Their classes in $H^1(\mathscr{U},\mathscr{L})$ are linearly dependent: There exist complex numbers

$$c_1,...,c_{k+1} \in \mathbb{C},$$

not all zero, and a cochain

$$\eta = (s_0,s_1) \in C^0(\mathscr{U},\mathscr{L})$$

such that

$$\sum_{j=1}^{k+1} c_j \cdot \zeta_j = \delta\eta$$

As a consequence on U_0^* holds

$$\sum_{j=1}^{k+1} c_j \cdot \zeta_j = s_1 - s_0 \in \mathscr{L}(U_0 \cap U_1)$$

The cocycle

$$s := \left(s_0 + \sum_{j=1}^{k+1} c_j \cdot \zeta_j,\ s_1\right) \in Z^0(\mathscr{U},\mathscr{M} \otimes_{\mathscr{O}} \mathscr{L}) = H^0(Y,\mathscr{M} \otimes_{\mathscr{O}} \mathscr{L})$$

is a meromorphic section in $\mathscr{L}$ with a single pole at $p \in Y$.

ii) *Existence of a holomorphic section without zeros*: Part i) provides a non-zero meromorphic section

$$s \in H^0(Y,\mathscr{M} \otimes_{\mathscr{O}} \mathscr{L}).$$

The Weierstrass Theorem 15.7 provides a meromorphic function $f \in \mathscr{M}^*(Y)$ with

$$div\ f = -div\ s$$

We obtain a holomorphic section

$$f \cdot s \in H^0(Y, \mathscr{L})$$

without zeros. By multiplication it defines an isomorphism

$$\mathscr{O}_Y \xrightarrow{\simeq} \mathscr{L}, g \mapsto (f \cdot s) \cdot g, \ q.e.d.$$

Theorem 15.13 (Holomorphic line bundles on open Riemann surfaces are trivial). *If X is an open Riemann surface then*

$$H^1(X, \mathscr{O}^*) = 0$$

The proof makes a Runge approximation to modify the trivialization from Proposition 15.12, obtained on relatively compact domains, to a global trivialization.

Proof. Consider the invertible sheaf $\mathscr{L}$ of a given line bundles on X. We choose a Runge exhaustion $(Y_\nu)_{\nu \in \mathbb{N}}$ of X, see Proposition 14.9. Proposition 15.12 implies that each restriction $\mathscr{L}|Y_\nu,\ \nu \in \mathbb{N}$, is isomorphic to the structure sheaf. Hence for any $\nu \in \mathbb{N}$ sections of $\mathscr{L}|Y_\nu$ are holomorphic functions. Theorem 14.13 allows to approximate each section of $\mathscr{L}$ on Y_ν by a section on $Y_{\nu+1}$ with arbitrary precision on $Y_{\nu-1}$. We assume $Y_0 \neq \emptyset$ and choose a point $p \in Y_0$ and a section

$$s_0 \in \mathscr{L}(Y_0)$$

with $s_0(p) \neq 0$. Runge approximation provides a sequence of sections

$$s_\nu \in \mathscr{L}(Y_\nu),\ \nu \in \mathbb{N},$$

such that for each arbitrary but fixed $\nu_0 \in \mathbb{N}$ the sequence

$$(s_{\nu_0+\nu})_{\nu \in \mathbb{N}}$$

is compact convergent on Y_{ν_0}. The limit

$$s := \lim_{\nu \to \infty} s_\nu \in \mathscr{L}(X)$$

is holomorphic and non-zero. Theorem 15.7 provides a meromorphic function $f \in \mathscr{M}^*(X)$ with

$$div\ f = -div\ s$$

Then

$$f \cdot s \in H^0(X, \mathscr{L})$$

is a holomorphic section without zeros and defines by multiplication a sheaf isomorphism

$$\mathscr{O} \simeq \mathscr{L}, \ q.e.d.$$

Corollary 15.14 (Second Betti number of an open Riemann surface). *For an open Riemann surface X*

$$H^2(X,\mathbb{Z}) = 0$$

and $H_1(X)$ is torsion free.

Proof. i) *Vanishing of $H^2(X,\mathbb{Z})$*: The vanishing

$$H^1(X,\mathscr{O}^*) = 0$$

and the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{ex} \mathscr{O}^* \to 0$$

provide the following exact cohomology sequence

$$0 = H^1(X,\mathscr{O}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}) = 0.$$

ii) $H_1(X)$ *torsion free*: The universal coefficient theorem compares homology and cohomology by the split exact sequence

$$0 \to Ext^1_{\mathbb{Z}}(H_{p-1}(X),\mathbb{Z}) \to H^p(X,\mathbb{Z}) \to Hom_{\mathbb{Z}}(H_p(X),\mathbb{Z}) \to 0$$

Hence the vanishing $H^2(X,\mathbb{Z}) = 0$ from part i) implies

$$Ext^1_{\mathbb{Z}}(H_1(X),\mathbb{Z}) = 0$$

Assume for an indirect argument the existence of a torsion element

$$0 \neq \alpha \in H_1(X) \text{ with } n \cdot \alpha = 0 \text{ for a suitable } n \geq 2$$

The $\mathbb{Z}$-linear morphism

$$j : \mathbb{Z}/n\mathbb{Z} \to H_1(X), \ 1 \mapsto \alpha,$$

is injective. The long exact $Ext^\bullet_{\mathbb{Z}}$-sequence has the segment

$$Ext^1_{\mathbb{Z}}(H_1(X),\mathbb{Z}) \to Ext^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \to Ext^2_{\mathbb{Z}}(coker\ j,\mathbb{Z}) = 0$$

Here the last group vanishes because both arguments are Abelian groups, see [42, Lemma 3.3.1]. As a consequence, the first morphism is surjective, hence

$$Ext^1_{\mathbb{Z}}(H_1(X),\mathbb{Z}) \neq 0,$$

a contradiction. As a consequence $H_1(X)$ does not have any torsion elements, q.e.d.

Corollary 15.15 is a companion to Proposition 10.26.

Corollary 15.15 (Vanishing of $H^1(X,\mathscr{M}^*)$). *On an open Riemann surface X*

$$H^1(X,\mathscr{M}^*)=0$$

Proof. The divisor sequence provides the exact sequence

$$0=H^1(X,\mathscr{O}^*)\rightarrow H^1(X,\mathscr{M}^*)\rightarrow H^1(X,\mathscr{D})=0,\ q.e.d.$$

Proposition 15.16 (De Rham group with holomorphic forms). *Consider an open Riemann surface X. Then the holomorphic de Rham sequence*

$$0\rightarrow\mathbb{C}\rightarrow\mathscr{O}\xrightarrow{d}\Omega^1\rightarrow 0$$

is an exact sequence of sheaves and the holomorphic de Rham group

$$Rh^1_{\mathscr{O}}(X):=\frac{\Omega^1(X)}{im[d:\mathscr{O}(X)\rightarrow\Omega^1(X)]}$$

and

$$H^1(X,\mathbb{C})\simeq Rh^1_{\mathscr{O}}(X).$$

Proof. Exactness of the de Rahm sequence with holomorphic forms follows similarly to the proof of Theorem 5.6. For any open subset $U\subset X$ the line bundle $\Omega^1|U$ is holomorphically trivial according to Theorem 15.13. Hence any open covering of X is a Leray covering for Ω^1. Leray's theorem 6.8 implies

$$H^1(X,\mathbb{C})=Rh^1_{\mathscr{O}}(X),\ q.e.d.$$

Remark 15.17 (Algebraic Topology and the vanishing $H^1(X,\mathscr{O}^*)=0$).

1. *CW-complex*: Any non-compact, smooth n-dimensional manifold X has the homotopy type of an $(n-1)$-dimensional CW-complex $K(X)$, see Whitehead's theorem [45, Lem. 2.1] and also [26, Theor. 0.1]. As a consequence

$$H_n(X)=0 \text{ and } H_{n-1}(X) \text{ is free,}$$

because

$$H_{n-1}(K(X))=Z_{n-1}(K(X))=ker\ [C_{n-1}(K(X))\xrightarrow{\delta}C_{n-2}(K(X))]$$

and because a subgroup of a free $\mathbb{Z}$-module is free itself. The universal coefficient theorem compares homology and cohomology by the split exact sequence

$$0 \to Ext^1_{\mathbb{Z}}(H_{p-1}(X),\mathbb{Z}) \to H^p(X,\mathbb{Z}) \to Hom_{\mathbb{Z}}(H_p(X),\mathbb{Z}) \to 0$$

see [20, Sect. 3.1]. For $p = n$ the theorem implies

$$H^n(X,\mathbb{Z}) = Ext^1_{\mathbb{Z}}(H_{n-1}(X),\mathbb{Z}) \oplus Hom_{\mathbb{Z}}(H_n(X),\mathbb{Z}) = 0$$

2. *Application to open Riemann surfaces*: Let X be an open Riemann surface. Due to the previous part

$$H^2(X,\mathbb{Z}) = 0$$

The exponential sequence on X implies the exact sequence

$$0 = H^1(X,\mathscr{O}) \to H^1(X,\mathscr{O}^*) \to 0 = H^2(X,\mathbb{Z})$$

and therefore

$$H^1(X,\mathscr{O}^*) = 0$$

The divisor sequence implies the exact sequence

$$H^0(X,\mathscr{M}^*) \to H^0(X,\mathscr{D}) \to 0 = H^1(X,\mathscr{O}^*)$$

and therefore the surjectivity of the first morphism: Each divisor on X is a principal divisor.

As a consequence, Theorem 15.13 and a posteriori also Theorem 15.7, Corollary 15.14 and Corollary 15.15 follow from the vanishing

$$H^1(X,\mathscr{O}) = 0,$$

see Theorem 14.16, by purely topological arguments.

15.3 Open Riemann surfaces and Stein manifolds

Stein manifolds have been introduced in [38]. They arise from the study of *domains of holomorphy* in complex affine spaces $\mathbb{C}^n$.

For $n = 1$ any domain $G \subset \mathbb{C}$ is a domain of holomorphy: There exists a holomorphic function $f \in \mathscr{O}(G)$ such that for any given point $p \in \partial G$ of the boundary f does not extend to a holomorphic function in any open neighbourhoood of p. The result does not carry over to higher dimensions $n \geq 2$. The punctured space

$$X := \mathbb{C}^2 \setminus \{0\}$$

is no domain of holomorphy: Any holomorphic function on X extends holomorphically to $\mathbb{C}^2$. Stein identifies in [38, p. 212] three properties of a complex manifold which characterize domains of holomorphy $G \subset \mathbb{C}^n$. He calls these three properties *R-convexity*.

Stein's three conditions on the existence of global holomorphic functions with distinguished properties have been been made the requirement for a *Stein manifold*, see [29, Chap. 3.6]:

Definition 15.18 (Stein manifold). A Stein manifold X is a paracompact, complex manifold with the following three properties:

1. *Holomorphically separable*: For any two points $x_1 \neq x_2$ on X exists a holomorphic function $f \in \mathscr{O}(X)$ with
$$f(x_1) \neq f(x_2)$$
2. *Holomorphically regular*: For any point $x \in X$ the cotangent space T_x^*X is spanned by the differentials of the functions f$\in \mathscr{O}(X)$.
3. *Holomorphically convex*: For any discrete sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of pairwise distinct points $x_\nu \in X$ exists a holomorphic function $f \in \mathscr{O}(X)$ with
$$\lim_{\nu \to \infty} |f(x_\nu)| = \infty.$$

Remark 15.19 (Stein domains). Any domain $G \subset \mathbb{C}^n$ has a countable topology.

1. Apparently any domain
$$G \subset \mathbb{C}^n$$
satisfies conditions 1) and 2) of Definition 15.18. Condition 3) is equivalent to G being a domain of holomorphy. The original definition of holomorphic convexity refers to a different concept, namely to the holomorphically convex hull of relatively compact sets $K \subset G$, see [13, Kap. II, Satz 6.2] and [10].
2. For any domain $G \subset \mathbb{C}$ the validity of condition 3) follows from the Weierstrass product theorem for G, see Theorem 15.20. As a consequence, any domain $G \subset \mathbb{C}$ is a Stein manifold.

We now show more generally that any open Riemann surface is a Stein manifold.

Theorem 15.20 (Holomorphic functions attaining prescribed values). *Let X be an open Riemann surface. For any sequence $(a_\nu)_{\nu \in \mathbb{N}}$ of pairwise distinct points $a_\nu \in X,\ \nu \in \mathbb{N}$, without accumulation point and for any sequence $(c_\nu)_{\nu \in \mathbb{N}}$ of complex numbers exists a holomorphic function*
$$f \in \mathscr{O}(X) \text{ with } f(a_\nu) = c_\nu \text{ for all } \nu \in \mathbb{N}.$$

The particular case that all prescribed values c_ν are zero, i.e. that one prescribes a sequence of zeros, is the exactly the content of the Weierstrass problem. The general case prescribes arbitrary values c_ν. It can be reduced to the Mittag-Leffler problem when encoding the values as the residues of locally defined meromorphic functions with poles of first order at the points of the sequence. The solution of the Mittag-Leffler problem then provides a global meromorphic function. The proof of Theorem 15.20 combines a solution of the Weierstrass problem with a solution of the Mittag-Leffler problem.

Proof. i) *Particular case* $c_\nu = 0$: Consider the divisor

$$D := \sum_{\nu \in \mathbb{N}} A_\nu \in Div(X),\ A_\nu \in Div(X) \text{ point divisor of } a_\nu \in X.$$

Theorem 15.7 provides a holomorphic function $h \in \mathscr{O}(X)$ with

$$div\ h = D.$$

ii) *Encoding the values* c_ν *as "residues"*: Because

$$supp\ D \subset X$$

is closed, the open sets

$$U_i := (X \setminus supp\ D) \cup \{a_i\},\ i \in \mathbb{N},$$

form a covering

$$\mathscr{U} := (U_i)_{i \in \mathbb{N}}$$

of X. The cochain

$$\left(g_i := \frac{c_i}{h}\right) \in C^0(\mathscr{U}, \mathscr{M})$$

is a Mittag-Leffler distribution of meromorphic functions with respect to $\mathscr{U}$ because for $i \neq j$

$$U_i \cap U_j \cap supp\ D = \emptyset.$$

Theorem 15.3 provides a solution

$$g \in \mathscr{M}(X)$$

of the Mittag-Leffler distribution, i.e. satisfying for all $i \in \mathbb{N}$

$$g - g_i \in \mathscr{O}(U_i).$$

The function

$$f := g \cdot h \in \mathscr{M}(X)$$

satisfies on U_i

$$f = g \cdot h = g_i \cdot h + (g - g_i) \cdot h = c_i + (g - g_i) \cdot h$$

Because

$$g - g_i \in \mathscr{O}(U_i) \text{ and } h(a_i) = 0.$$

the function f is even holomorphic

$$f \in \mathscr{O}(X).$$

Apparently, it satisfies for all $i \in \mathbb{N}$

$$f(a_i) = c_i, \ q.e.d.$$

Theorem 15.21 (Open Riemann surfaces are Stein manifolds). *Any open Riemann surface is a Stein manifold.*

Proof. A countable Hausdorff space is paracompact, and for a Riemann surface countability follows from the other properties of the definition, see Proposition 4.19 and Remark 4.20.

The conditions 1) and 3) from Definition 15.18 follow from Theorem 15.20, and condition 2) follows from Theorem 15.7, q.e.d.

Corollary 15.22 (Leray covering with two elements). *Any Riemann surface X has an open covering $\mathscr{U} = (U_0, U_1)$ with two subsets $U_i \subset X, \ j = 0, 1,$ which are Stein manifolds.*

Proof. The claim is obvious if X is an open Riemann surface. If X is compact then choose an arbitrary point $p \in X$ and an arbitrary open neighbourhood U_0 of p in X. Set

$$U_1 := X \setminus \{p\}.$$

Theorem 15.21 implies that $\mathscr{U}$ is a covering by Stein manifolds, q.e.d.

Hence for any invertible sheaf on a Riemann surface one always has a Leray covering with only two open sets, see Remark 14.17, Theorem 15.13 and Corollary 15.22.

Remark 15.23 (Stein manifolds).

1. In the years after introducing the concept of a Stein manifold X the original definition has been modified and replaced by equivalent conditions about coherent sheaves on X. The definition has also been translated to an equivalent characterization of the algebra of holomorphic functions on X. The latter shows the analogy to Grothendieck's definition of affine spaces as the spectrum of commutative rings. In addition the concept of being Stein has been generalized to complex spaces.

The following properties of a complex manifold X are equivalent:

- *Stein manifold*: The manifold X is a Stein manifold.
- *Exactness of the functor Γ*: For any short exact sequence of coherent $\mathscr{O}$-module sheaves
$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$
the induced sequence of global sections is exact
$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$
- *Vanishing theorem*: For any coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}$
$$H^1(X, \mathscr{I}) = 0$$
- *Spectrum of $\Gamma(X, \mathscr{O})$*: The canonical evaluation map
$$X \to Spec\ \Gamma(X, \mathscr{O}),\ x \mapsto \lambda_x \text{ with } \lambda_x(f) := f(x),$$
is a homeomorphism.

Coherent $\mathscr{O}$-module sheaves on a complex manifold X generalize the sheaf of holomorphic sections of line bundles or complex vector bundles: An $\mathscr{O}$-module sheaf $\mathscr{F}$ is *coherent* if it satisfies both of the following conditions:

i) *$\mathscr{F}$ is finite*: Each $x \in X$ has a neighbourhood U and finitely many sections
$$f_1, ..., f_k \in \mathscr{O}(U)$$
whose germs at each $y \in U$ generate the stalk $\mathscr{F}_y$ as $\mathscr{O}_y$-module

ii) *$\mathscr{F}$ has finite relation sheaves*: For each open $Y \subset X$ and for each finite set
$$f_1, ..., f_k \in \mathscr{O}(Y)$$
the *sheaf of relations*
$$\mathscr{R}(f_1, ... f_k)$$
on Y is finite. The sheaf of relations is defined as
$$\mathscr{R}(f_1, ... f_k)(V) := \{(\phi_1, ..., \phi_k) \in \mathscr{O}(V)^k : \sum_{j=1}^{k} \phi_j \cdot (f_j | V) = 0\},\ V \subset Y \text{ open}.$$

The most prominent example of a coherent sheaf is the structure sheaf $\mathscr{O}$, the most prominent counter example is the sheaf $\mathscr{M}$ of meromorphic functions.

The ring $\Gamma(X, \mathscr{O})$, provided with the Fréchet topology, is a topological $\mathbb{C}$-algebra. Its spectrum

$$Spec\ \Gamma(X,\mathscr{O})$$

is the set of all continuous $\mathbb{C}$-algebra morphisms

$$\Gamma(X,\mathscr{O}) \to \mathbb{C}.$$

The set $Spec\ \Gamma(X,\mathscr{O})$ becomes a topological space when provided with the coarsest topology such that for each $f \in \Gamma(X,\mathscr{O})$ the complex valued function

$$\hat{f} : Spec\ \Gamma(X,\mathscr{O}) \to \mathbb{C},\ x \mapsto f(x),$$

is continuous, see [1, Anhang zu Kap. VI, Satz 7].

2. *Main results*: The two main theorems on a Stein manifold X deal with the cohomology of coherent $\mathscr{O}$-module sheaves $\mathscr{F}$ on X:

   - *Theorem A*: For each $x \in X$ the stalk $\mathscr{F}_x$ is globally generated, i.e. the germs at x of all sections $f \in \mathscr{F}(X)$ generate the stalk $\mathscr{F}_x$ as $\mathscr{O}_x$-module.

   - *Theorem B*: For each $q \geq 1$

   $$H^q(X,\mathscr{F}) = 0.$$

   These theorems allow to solve the *Cousin-I problem* and the *Cousin-II problem*, which are analogous to the Mittag-Leffler problem and the Weierstrass problem of the 1-dimensional case.

3. *Holomorphic line bundles*: The exponential sequence on a Stein manifold X provides the following exact sequence

   $$0 = H^1(X,\mathscr{O}) \to H^1(X,\mathscr{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to H^2(X,\mathscr{O}) = 0$$

   Hence the group of holomorphic line bundles equals via the Chern morphism the group $H^2(X,\mathbb{Z})$ which is a topological invariant. On a Stein manifold it reduces to a topological question whether a given holomorphic line bundle is holomorphically trivial.

   For a Stein manifold X with $dim\ X \geq 2$ in general $H^2(X,\mathbb{Z}) \neq 0$, e.g.

   $$H^2(\mathbb{C}^* \times \mathbb{C}^*,\mathbb{Z}) \simeq H^2(S^1 \times S^1,\mathbb{Z}) = \mathbb{Z}.$$

4. *Embedding theorem*: Any n-dimensional Stein manifold X has a closed embedding

   $$f : X \to \mathbb{C}^{2n+1}.$$

   In particular, any open Riemann surface embeds as a closed submanifold into $\mathbb{C}^3$. Because a closed submanifold of a Stein manifold is a Stein manifold itself, the embedding theorem is an equivalence: A manifold is a Stein manifold iff it embedds as a closed submanifold into an affine space $\mathbb{C}^k$.

5. *Homology*: The result about the homology of an open Riemann surface from Remark 15.17 generalizes to an n-dimensional Stein manifold X:

$$H_n(X) \text{ is free and } H_q(X) = 0 \text{ for } q > n.$$

For these and further, even stronger results on Stein manifolds see [17], [7], [14], [22, Part Three], [9] and [1, Anhang Kapitel VI].

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Global meromorphic functions as quotient of holomorphic functions (Prop. 15.8)

Holomorphic line bundles are trivial (Theor. 15.13)

Open Riemann surfaces are Stein manifolds (Theor. 15.21)

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The main reference for these notes is the textbook [8].

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