Joachim Wehler

Problems 01

1. Consider the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Prove that the *Euclidean norm*

$$\|x\| := \sqrt{\sum_{i=1}^{n} |x_i|^2};\ x = (x_1, ..., x_n) \in \mathbb{K}^n$$

is a norm on the vector space $\mathbb{K}^n$, i.e. it satisfies

i)
$$\|x\| = 0 \iff x = 0;\ x \in \mathbb{K}^n$$

ii)
$$\|\lambda \cdot x\| = |\lambda| \cdot \|x\|;\ \lambda \in \mathbb{K}, x \in \mathbb{K}^n$$

iii)
$$\|x + y\| \le \|x\| + \|y\|;\ x, y \in \mathbb{K}^n\ (\textit{Triangle inequality}).$$

2. Consider the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Show that for matrices $A \in M(n \times n, \mathbb{K})$ the *operator norm*

$$\|A\| := sup\{\|Ax\| : x \in \mathbb{K}^n \textit{ and } \|x\| \le 1\}$$

is a norm on the vector space $M(n \times n, \mathbb{K})$, i.e. it satisfies

i)
$$\|A\| = 0 \iff A = 0;\ A \in M(n \times n, \mathbb{K})$$

ii)
$$\|\lambda \cdot A\| = |\lambda| \cdot \|A\|;\ \lambda \in \mathbb{K}, A \in M(n \times n, \mathbb{K})$$

iii)
$$\|A + B\| \le \|A\| + \|B\|;\ A, B \in M(n \times n, \mathbb{K})\ (\textit{Triangle inequality}).$$

In addition show

iv)
$$\|A \cdot B\| \le \|A\| \cdot \|B\|;\ A, B \in M(n \times n, \mathbb{K})$$

v)
$$\|\mathbb{1}\| = 1;\ \textit{unit matrix}\ \mathbb{1} \in M(n \times n, \mathbb{K}).$$

3. Determine the radius of convergence of the following power series:

i)

$$\sum_{\nu=0}^{\infty} \frac{1}{\nu!} \cdot z^{\nu}$$

ii)

$$\sum_{\nu=0}^{\infty} z^{\nu}$$

iii)

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu} \cdot z^{\nu}$$

iv)

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{(2\nu)!} \cdot z^{2\nu}$$

v)

$$\sum_{\nu=0}^{\infty} \nu! \cdot z^{\nu}$$

vi) Which functions do the power series i) - iv) represent?

4. Transform the following matrix to upper triangular form:

$$\begin{pmatrix} 3 & 4 & 3 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix} \in M(3 \times 3, \mathbb{R}).$$

Discussion: Tuesday, 25.10.2016, 12.15 p.m.

Problems 02

5. The geometric series

$$\sum_{\nu=0}^{\infty} z^\nu, z \in \mathbb{C},$$

has radius of convergence $R = 1$. Hence the series

$$\sum_{\nu=0}^{\infty} A^\nu \in M(n \times n, \mathbb{C})$$

is well-defined for a matrix $A \in M(n \times n, \mathbb{C})$ with $\|A\| < 1$.

Show that the matrix $\mathbb{1} - A \in M(n \times n, \mathbb{C})$ is invertible with

$$(\mathbb{1} - A)^{-1} = \sum_{\nu=0}^{\infty} A^\nu.$$

Hint: Imitate the proof of the analogous result for the complex series.

6. The logarithmic series

$$log(1+z) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu} \cdot z^\nu, z \in \mathbb{C},$$

has radius of convergence $R = 1$.
For a matrix $A \in M(n \times n, \mathbb{C})$ with $\|A\| < 1$ one defines

$$log(\mathbb{1} + A) := \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \cdot \frac{A^\nu}{\nu} \in M(n \times n, \mathbb{C}).$$

Consider an open subset $I \subset \mathbb{R}$ and a differentiable function

$$B : I \to M(n \times n, \mathbb{C})$$

with $\|B(t) - \mathbb{1}\| < 1$ and $[B'(t), B(t)] = 0$ for all $t \in I$.

Show: For all $t \in I$ the inverse $B(t)^{-1}$ exists and

$$\frac{d}{dt} log\ B(t) = B(t)^{-1} \cdot B'(t) = B'(t) \cdot B(t)^{-1}.$$

Hint: In order to compute $B(t)^{-1}$ apply problem 5 with $A := \mathbb{1} - B(t)$.

7. i) Consider an upper triangular matrix $A \in M(n \times n, \mathbb{C})$.

Show that a series of diagonalizable matrices $(A_\nu)_{\nu\in\mathbb{N}}$ exists with

$$A = \lim_{\nu\to\infty} A_\nu.$$

ii) Consider a matrix $A \in M(n \times n, \mathbb{C})$ with $\|A\| < log\ 2$.

Show

$$\|(exp\ A) - \mathbb{1}\| < 1$$

and

$$log(exp\ A) = A.$$

Hint: Prove the equality in different steps. First, consider the case of a diagonalizable matrix A. Secondly, generalize the result to an upper triangular matrix. Eventually, consider the general case.

8. Consider the endomorphism $f \in End(\mathbb{C}^2)$ defined with respect to the canonical basis by the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M(2 \times 2, \mathbb{C}).$$

i) Show that

$$A_s := \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \ (semisimple)$$

and

$$A_n := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \ (nilpotent)$$

are not the matrices of the Jordan decomposition of f.

ii) Compute the matrices of the Jordan decomposition of f.

Discussion: Tuesday, 8.11.2016, 12.15 p.m.

Problems 03

9. i) Show the Jacobi identity for matrices, i.e. for $A,B,C \in M(n \times n, \mathbb{K})$

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0.$$

ii) Show: The set of infinitesimal generators of all 1-parameter subgroups of the symplectic group is closed with respect to the commutator, i.e. for matrices $X,Y \in sp(m,\mathbb{K})$ holds

$$[X,Y] \in sp(m,\mathbb{K}).$$

10. Let $M := (\mathbb{R}^4, q_M)$ denote the Minkowski space with the quadratic form of signature (1,3)

$$q_M : \mathbb{R}^4 \to \mathbb{R}, q_M(x) := {x_0}^2 - ({x_1}^2 + {x_2}^2 + {x_3}^2), x = (x_0, ..., x_3).$$

Let $H := (Herm(2), q_H)$ denote the real vector space of Hermitian matrices

$$Herm(2) := \{X \in M(2 \times 2, \mathbb{C}) : X = X^*\}$$

equipped with the real quadratic form

$$q_H : Herm(2) \to \mathbb{R}, X \mapsto det\, X,$$

i.e.

$$q_H(X) = a \cdot d - |b|^2$$

for

$$X = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}, a,d \in \mathbb{R}, b \in \mathbb{C}.$$

i) Set $\sigma_0 := \mathbb{1} \in Herm(2)$ and denote by $\sigma_j \in Herm(2), j = 1,2,3,$ the Pauli matrices.

Show: The family $(\sigma_j)_{j=0,...,3}$ is a basis of the vector space $Herm(2)$.

ii) Consider the map

$$\beta : M \to H, x = (x_0, ..., x_3) \mapsto X := \sum_{j=0}^{3} x_j \cdot \sigma_j.$$

Compute the components of the matrix $\beta(x) \in Herm(2)$ for $x = (x_0, \ldots, x_3) \in \mathbb{R}^4$.
Show: β is an isometric isomorphy, i.e. an isomorphism of vector spaces satisfying

$$q_H(\beta(x)) = q_M(x), x \in \mathbb{R}^4.$$

11. Use the notations introduced in Problem 10.

The Lorentz group is the matrix group of isometries of the Minkowski space M

$$O(1,3) := \{f \in GL(4, \mathbb{R}) : q_M(f(x)) = q_M(x) \ for \ all \ x \in \mathbb{R}^4\}$$

The group O(1,3) has 4 connected components. The connected component of $\mathbb{1} \in O(1,3)$ is the *proper orthochronous* Lorentz group $L_+^\uparrow$. The term indicates that elements from

$$L_+^\uparrow = \{B = (b_{ij})_{0 \leq i,j \leq 3} \in O(1,3) : det \ B = 1, b_{00} \geq 1\}$$

keep the orientation of vectors and the sign of their time component.

By means of the isometric isomorphism β we identify the group of isometries of H

$$O(H) := \{g \in GL(Herm(2)) : q_H(g(X)) = q_H(X) \ for \ all \ X \in Herm(2)\}$$

with $O(1,3)$ and denote by

$$L_+^\uparrow(H) \subset O(H)$$

the connected component of the neutral element $id_H \in O(H)$.

i) Show: The map

$$\Psi : SL(2, \mathbb{C}) \to L_+^\uparrow(H), B \mapsto \Psi_B,$$

with

$$\Psi_B : H \to H, X \mapsto B \cdot X \cdot B^*,$$

is a well-defined morphism of matrix groups.
Hint: The continous image of a connected set is connected.

ii) Denote by

$$o(H) \subset gl(Herm(2)) := (End(Herm(2)), [-,-])$$

the subalgebra of the infinitesimal generators of all 1-parameter subgroups of $O(H)$. And let

$$\psi := Lie\, \Psi : sl(2,\mathbb{C}) \to o(H)$$

be the tangent map of Ψ at $\mathbb{1} \in SL(2,\mathbb{C})$.

Show:

$$\psi(A)(X) = A \cdot X + X \cdot A^*, A \in sl(2,\mathbb{C}), X \in Herm(2).$$

iii) Show: The map

$$\psi : sl(2,\mathbb{C}) \to o(H)$$

is an isomorphism of real Lie algebras.
Hint: The family $(A_j)_{j=1,...,6}$ with

$$A_j := \begin{cases} \sigma_j & \text{if } j = 1,2,3 \\ i \cdot \sigma_{j-3} & \text{if } j = 4,5,6 \end{cases}$$

is basis of $sl(2,\mathbb{C})$ considered as real vector space. Compute explicitly the matrices representing $\psi(A_j), j = 1,...,6$ and show: They form a linearly independent family in the vector space $End(Herm(2))$.

12. Continue with the notations introduced in Problem 10 and 11.

i) Show

$$\Psi(SL(2,\mathbb{C})) \subset L_+^\uparrow(H)$$

is open and closed. Conclude:

$$\Psi : SL(2,\mathbb{C}) \to L_+^\uparrow(H)$$

is surjective.
Hint: First show that $\Psi(SL(2,\mathbb{C}))$ is open. Then use

$$L_+^\uparrow(H) = \bigcup_{g \in L_+^\uparrow(H)} g \cdot \Psi(SL(2,\mathbb{C})).$$

ii) Show $ker\, \Psi = \{\pm\mathbb{1}\} \subset SL(2,\mathbb{C})$.
Hint: Evaluate the condition $\Psi_B(X) = X$ for suitable basis elements $X \in Herm(2)$.

iii) Show:

$$\Psi : SL(2,\mathbb{C}) \to L_+^\uparrow(H)$$

is the universal covering space of the proper orthochronous Lorentz group. It is a two-fold covering space.

Discussion: Tuesday, 15.11.2016, 12.15 p.m.

Problems 04

13. i) Compute the descending central series of the Lie algebra of upper triangular matrices

$$\mathfrak{t}(2,\mathbb{K}) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in gl(2,\mathbb{K}) \right\}.$$

Consider a short exact sequence of Lie algebras

$$0 \to L_0 \to L_1 \to L_2 \to 0$$

and show:

ii) If L_1 is nilpotent then both L_0 and L_2 are nilpotent.

iii) L_1 is not necessarily nilpotent even if both L_0 and L_2 are nilpotent.

14. Show for a nilpotent Lie algebra $L \neq \{0\}$:

i) Any proper subalgebra $M \subsetneq L$ is properly contained in its normalizer, i.e.

$$M \subsetneq N_L(M).$$

ii) An ideal $I \subset L$ exists with $codim_L I := dim\, L - dim\, I = 1$.

iii) The centralizer of an ideal $I \subset L$ satisfies $C_L(I) \neq \{0\}$.

15. Consider a nilpotent $\mathbb{K}$-Lie algebra $L \neq \{0\}$.

i) Show: $ad(L) \subsetneq Der(L)$, i.e. not every derivation D of L is an inner derivation.

Hint: Set $L = I \oplus \mathbb{K} \cdot x_0$ for a suitable ideal $I \subset L$ and a suitable element $x_0 \in L \setminus I$. If $n \in \mathbb{N}$ is maximal with $C_L(I) \subset C^n L$ then choose $z_0 \in C_L(I) \setminus C^{n+1} L$ and define $D(I) := 0$ and $D(x_0) := z_0$.

ii) For $L := \mathfrak{n}(3,\mathbb{K})$ determine explicitly a derivation $D \in Der(L) \setminus ad(L)$.

16. Consider two $\mathbb{K}$-Lie algebras M and I, denoting their Lie brackets by respectively $[-,-]_M$ and $[-,-]_I$. Assume the existence of a morphism of Lie algebras

$$\alpha : M \to Der(I).$$

On the $\mathbb{K}$-vector space $L := I \oplus M$ define the $\mathbb{K}$-bilinear map

$$[-,-]_L : L \times L \to \mathbb{K}$$

by

$$[(i_1,m_1),(i_2,m_2)]_L := (\alpha(m_1)(i_2) - \alpha(m_2)(i_1) + [i_1,i_2]_I, [m_1,m_2]_M)$$

for $i_1, i_2 \in I$ and $m_1, m_2 \in M$.

Show:

i) The *semidirect sum* of I and M via α

$$I \rtimes_\alpha M := (L, [-,-]_L)$$

is a $\mathbb{K}$-Lie algebra.

ii) One has a short exact sequence of Lie algebras

$$0 \to I \xrightarrow{j} I \rtimes_\alpha M \xrightarrow{p} M \to 0$$

with $j(i) := (i,0)$ for all $i \in I$ and $p((i,m)) := m$ for all $m \in M$.

iii) The exact sequence from part ii) is also *split exact*, i.e. a morphism of Lie algebras

$$s : M \to I \rtimes_\alpha M$$

with $p \circ s = id_M$ exists.

Discussion: Tuesday, 22.11.2016, 12.15 p.m.

Problems 05

17. Consider a Lie algebra L. Show:

i) Each member $C^iL, i \in \mathbb{N}$, of the *descending central series* of L and $D^iL, i \in \mathbb{N}$, of the *derived series of* L is an ideal in L.

ii) For each $i \in \mathbb{N}$ one has a short exact sequence of Lie algebras

$$0 \to C^iL/C^{i+1}L \to L/C^{i+1}L \to L/C^iL \to 0$$

The exact sequence represents the middle term $L/C^{i+1}L$ as a central extension - i.e. $C^iL/C^{i+1}L \subset Z(L/C^{i+1}L)$ - of the Abelian Lie algebra L/C^iL.

18. i) Show as direct application of the definition of nilpotency: The Lie algebra $\mathfrak{n}(m,\mathbb{K})$ of strictly upper triangular matrices is nilpotent.

ii) Compute the derived algebra $D^1\mathfrak{t}(m,\mathbb{K})$.

iii) Show: The Lie algebra $\mathfrak{t}(m,\mathbb{K})$ of upper triangular matrices is solvable.

19. Consider a Lie algebra L and and ideal $I \subset L$. Assume: The Lie algebra L/I is nilpotent and for all $x \in L$ the restricted endomorphism

$$ad\ x : I \to I$$

is nilpotent.

Show: The Lie algebra L is nilpotent.

20. Consider a vector space V.
On one hand, each endomorphism $x \in gl(V)$ defines the endomorphism of the vector space $End(V)$

$$ad\ x : End(V) \to End(V), y \mapsto [x,y].$$

On the other hand, each automorphism $g \in GL(V)$ defines the automorphism of $End(V)$

$$Ad\ g : End(V) \to End(V), y \mapsto g \cdot y \cdot g^{-1}.$$

Denote by $exp : gl(V) \to GL(V)$ the exponential map.

i) Show for all $x \in gl(V), y \in End(V)$ by induction on $n \in \mathbb{N}$:

$$(ad\ x)^n(y) = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu \cdot y \cdot (-x)^{n-\nu}, n \in \mathbb{N}.$$

Hint: $\binom{n}{\nu-1} + \binom{n}{\nu} = \binom{n+1}{\nu}$.

ii) Why does the series

$$\sum_{\nu=0}^{\infty} \frac{1}{\nu!}(ad\ x)^\nu(y)$$

converge for all $x \in gl(V), y \in End(V)$? State an argument.

iii) Show for all $x \in gl(V), y \in End(V)$:

$$(Ad(exp\ x))(y) = e^{ad\ x}(y) := \sum_{n=0}^{\infty} \frac{1}{n!}(ad\ x)^n(y).$$

Discussion: Tuesday, 29.11.2016, 12.15 p.m.

Problems 06

21. Consider a nilpotent Lie algebra L.

Show: The Killing form κ of L is identically zero.

Hint: Apply the main theorem from the oral lecture about nilpotent Lie algebras.

22. Consider the Lie algebra $L := sl(2,\mathbb{K})$.

i) Compute the matrix

$$m(\kappa) = (\kappa(v_i, v_j)_{1 \le i,j \le 3}) \in M(3 \times 3, \mathbb{K})$$

of the Killing form κ of L with respect to the basis of L

$$\mathscr{B} = (v_1, v_2, v_3) := (h := E_{11} - E_{22}, x := E_{12}, y := E_{21}).$$

ii) Determine the rank of $m(\kappa)$.

23. Consider a Lie algebra $(L, [-,-])$ with $C^2L = 0$.

Show: The map

$$* : L \times L \to L, (x,y) \mapsto x + y + \frac{1}{2} \cdot [x,y],$$

defines a group $(L, *)$.

24. Denote by $L := heis_1$ the Heisenberg algebra of 1-dimensional quantum mechanics.

i) Show $C^2L = 0$.

ii) Consider exercise 20. For $x, y \in L, t \in \mathbb{R}$, show:

$$x \cdot e^{ty} = e^{ty} \cdot e^{-t \cdot ad(y)}(x) = e^{ty} \cdot (x - t[y,x]) = e^{ty} \cdot (x + t[x,y]).$$

iii) For arbitrary but fixed $x, y \in L$ consider the differentiable function

$$A : \mathbb{R} \to GL(3,\mathbb{R}), t \mapsto e^{tx} \cdot e^{ty} \cdot e^{-\frac{t^2}{2}[x,y]}.$$

Apply the product rule to decompose

$$\dot{A}(t) = \frac{dA(t)}{dt} = A_1(t) + A_2(t) + A_3(t).$$

For $A_1(t) = e^{tx} \cdot x \cdot e^{ty} \cdot e^{-\frac{t^2}{2}[x,y]}$ show:

$$A_1(t) = e^{tx} \cdot e^{ty} \cdot e^{-\frac{t^2}{2}[x,y]} \cdot (x + t[x,y])$$

and

$$\dot{A}(t) = A(t) \cdot (x+y).$$

iv) For arbitrary but fixed $x, y \in L$ consider the differentiable function

$$B : \mathbb{R} \to GL(3,\mathbb{R}), t \mapsto e^{tx+ty}.$$

Show

$$A(0) = B(0) \text{ and } \dot{B}(t) = B(t) \cdot (x+y) \text{ for all } t \in \mathbb{R}$$

and conclude

$$A(t) = B(t) \text{ for all } t \in \mathbb{R}.$$

Hint: Two solutions of the ordinary linear differential equation

$$\dot{F}(x,y,t) = F(x,y,t) \cdot (x+y)$$

are equal if they have the same initial value.

v) Show: The exponential map of the Heisenberg algebra $heis_1$ of 1-dimensional quantum mechanics

$$heis_1 \to GL(3,\mathbb{R}), x \mapsto e^x,$$

satisfies for all $x, y \in heis_1$ the functional equation

$$e^x \cdot e^y = e^{x*y}.$$

Discussion: Tuesday, 6.12.2016, 12.15 p.m.

Problems 07

25. Consider a $\mathbb{K}$-Lie algebra L and an ideal $I \subset L$.

Show: The Killing form of I

$$\kappa_I : I \times I \to \mathbb{K}$$

is the restriction of the Killing form of L to $I \times I$.

26. Consider a complex semisimple Lie algebra L, its Lie algebra $D := Der(L)$ of derivations and the subalgebra

$$M := ad(L) \subset D.$$

i) For $x \in L$ and $\delta \in D$ show

$$[\delta, ad\ x] = ad(\delta(x)).$$

and conclude

$$M \subset D$$

is an ideal.

ii) Denote by κ_D the Killing form of D and by

$$M^{\perp} := \{x \in D : \kappa_D(x, M) = 0\}$$

the orthogonal space of M with respect to κ_D. Note

$$dim\ M^{\perp} \geq dim\ D - dim\ M.$$

Show:

$$M \cap M^{\perp} = \{0\}.$$

Conclude:

$$[M, M^{\perp}] = \{0\} \text{ and } D = M \oplus M^{\perp}.$$

Hint: Reduce the first claim concerning $M \cap M^{\perp}$ to a statement involving the Killing form κ_M of $M \simeq L$.

iii) Consider a derivation $\delta \in M^{\perp}$. Show: For all $x \in L$

$$\delta(x) = 0.$$

Conclude: The adjoint map

$$ad : L \to Der(L)$$

is surjective, i.e. any derivation of L is an inner derivation.

27. Consider the Lie algebra $L := sl(m, \mathbb{K})$.

Show: $Z(L) = \{0\}$.

Hint: For $j \neq k$ set $h_{jk} := E_{jj} - E_{kk} \in L$. Assume

$$X = \sum_{r,s} x_{rs} \cdot E_{rs} \in Z(L).$$

From $0 = [h_{jk}, X]$ derive

$$X \in \mathfrak{d}(m, \mathbb{K})$$

using the linear independency of the family (E_{rs}). Then prove $Z(L) \cap \mathfrak{d}(m, \mathbb{K}) = \{0\}$.

28. Consider the Lie algebra $L := sl(m, \mathbb{C})$.

i) Show:

$$rad(L) = \{0\}.$$

Hint: According to Lie's theorem assume $rad(L)$ isomorphic to a subalgebra

$$B \subset (\mathfrak{t}(m, \mathbb{C}) \cap sl(m, \mathbb{C})).$$

Prove $X \in B \iff X^{\top} \in B$ and conclude

$$B \subset (\mathfrak{d}(m, \mathbb{C}) \cap sl(m, \mathbb{C})).$$

Conclude $rad(L) \subset Z(L)$.

ii) Show: L is semisimple.

Discussion: Tuesday, 13.12.2016, 12.15 p.m.

Selected Solutions 07

25 . Extend a base of the vector subspace $I \subset L$ to a base of L. For $x \in I$ the corresponding matrix representations of

$$ad\ x : L \to I \ and\ ad(x)|I : I \to I$$

satisfies

$$tr(ad\ x) = tr(ad(x)|I).$$

As a consequence, for $x, y \in I$

$$\kappa(x,y) = tr(ad(x)ad(y)) = tr((ad(x)ad(y)|I) = \kappa_I(x,y).$$

26 . i) For $x, y \in L, y \in D$:

$$[\delta, ad\ x](y) = \delta(ad(x)(y)) - (ad\ x)(\delta(y)) = \delta([x,y]) - [x, \delta\ y] =$$

$$[\delta(x), y] + [x, \delta\ y] - [x, \delta\ y] = [\delta(x), y] = ad(\delta(x))(y).$$

As a consequence $[D, M] \subset M$.

ii) To obtain the estimation

$$dim\ M^{\perp} \geq dim\ D - dim\ M$$

note: In

$$M^{\perp} := \{x \in D : \kappa_D(x, M) = 0\} = \bigcap_{m \in M} ker[\kappa_D(-, m) : D \to \mathbb{C}]$$

for each $m \in M$ the linear functional

$$\kappa_D(-, m) : D \to \mathbb{C}$$

reduces the dimension by at most one.

The orthogonal space $M^{\perp}$ of the ideal $M \subset D$ is an ideal of D. Because L is semisimple, its Killing form and also the Killing form κ_M is nondegenerate. Due to the previous exercise κ_M is the restriction of κ_D. For $x \in M \cap M^{\perp}$ we have

$$\kappa_D(x,M) = 0 \; due \; to \; x \in M^\perp$$

and

$$\kappa_D(x,M) = \kappa_M(x,M) \; due \; to \; x \in M.$$

Therefore

$$\kappa_M(x,M) = 0$$

which implies $x = 0$ by nondegenerateness of κ_M and proves

$$M \cap M^\perp = \{0\}.$$

Because $M \subset D$ and $M^\perp \subset D$ are ideals

$$[M, M^\perp] \subset M \cap M^\perp \subset \{0\}.$$

Hence

$$D = M \oplus M^\perp$$

as a vector space due to the dimension formula

$$dim\, D \geq dim\,(M + M^\perp) = dim\, M + dim\, M^\perp - dim(M \cap M^\perp) \geq$$

$$\geq dim\, M + (dim\, D - dim\, M) = dim\, D$$

and as a direct sum of Lie algebras due to $[M, M^\perp] = \{0\}$.

iii) According to part ii) any derivation $\delta \in D$ decomposes as

$$\delta = \delta_1 + \delta_2 \; with \; \delta_1 \in M, \delta_2 \in M^\perp.$$

Consider a derivation $\delta \in M^\perp$. For all $x \in L$ due to part i)

$$ad(\delta(x)) = [\delta, ad\; x] \in [\delta, M] \subset [M^\perp, M] = \{0\}.$$

Therefore $ad(\delta(x)) = 0$. Injectivity of ad implies

$$\delta(x) = 0.$$

As a consequence $M^\perp = \{0\}$ and $D = M = ad(L)$.

27 . i) Assume $X = (x_{rs}) \in Z(L)$. Set

$$X = \sum_{r,s} X_{rs} \; with \; X_{rs} := x_{rs} \cdot E_{rs}.$$

For arbitrary but fixed $j < k$

$$0 = [h_{jk}, X] = [E_{jj} - E_{kk}, \sum_{r,s} X_{rs}] = \sum_{r,s} [E_{jj}, X_{rs}] - \sum_{r,s} [E_{kk}, X_{rs}] =$$

$$= \sum_{r,s} \delta_{jr} \cdot x_{rs} \cdot E_{js} - \sum_{r,s} \delta_{js} \cdot x_{rs} \cdot E_{rj} - \sum_{r,s} \delta_{kr} \cdot x_{rs} \cdot E_{ks} + \sum_{r,s} \delta_{ks} \cdot x_{rs} \cdot E_{rk} =$$

$$= \sum_{s} x_{js} \cdot E_{js} - \sum_{r} x_{rj} \cdot E_{rj} - \sum_{s} x_{ks} \cdot E_{ks} + \sum_{r} x_{rk} \cdot E_{rk} = S_1 - S_2 - S_3 + S_4.$$

We compute each summand separately:

$$S_1 = X_{jj} + X_{jk} + \sum_{s \neq j,k} X_{js}$$

$$S_2 = X_{jj} + X_{kj} + \sum_{s \neq j,k} X_{sj}$$

$$S_3 = X_{kk} + X_{kj} + \sum_{s \neq j,k} X_{ks}$$

$$S_4 = X_{jk} + X_{kk} + \sum_{s \neq j,k} X_{sk}$$

We obtain

$$0 = S_1 - S_2 - S_3 + S_4 = 2X_{jk} - 2X_{kj} + \sum_{s \neq j,k} (X_{js} - X_{sj} - X_{ks} + X_{sk}).$$

Therefore $X_{rs} = 0$ for all $(r,s) \notin \{(j,j),(k,k)\}$. Varying the pairs $i < k$ implies

$$Z(L) \subset \mathfrak{d}(m, \mathbb{K}).$$

ii) For

$$X = \sum_{j} X_{jj} \in Z(L)$$

choose arbitrary but fixed $r \neq s$. Then

$$0 = [E_{rs}, X] = \sum_{j} [E_{rs}, X_{jj}] = \sum_{j} (x_{jj} \cdot E_{rs} \cdot E_{jj} - x_{jj} \cdot E_{jj} \cdot E_{rs}) =$$

$$= \sum_{j} (\delta_{js} \cdot x_{jj} \cdot E_{rj}) - \sum_{j} (\delta_{jr} \cdot x_{jj} \cdot E_{js}) = x_{ss} \cdot E_{rs} - x_{rr} \cdot E_{rs} = (x_{ss} - x_{rr}) \cdot E_{rs}.$$

Therefore $x_{jj} = const.$ independent from $j = 1, ..., m$. And $tr\, X = 0$ implies $X = 0$.

28 . Consider the Lie algebra $L = sl(m, \mathbb{C})$ and denote by $R := rad(L)$ its radical.

i) By definition $L \subset gl(m, \mathbb{C})$. Solvability of $R \subset gl(m, \mathbb{C})$ implies via Lie's theorem

$$R \subset (\mathfrak{t}(m, \mathbb{C}) \cap sl(m, \mathbb{C})).$$

Also the algebra $R^\top$ of transposed matrices is solvable. Hence $R^\top = R$ which implies

$$R \subset \mathfrak{d}(m, \mathbb{C}) \cap L,$$

all matrices from the radical are diagonal and have zero trace.

ii) Because $R \subset L$ is an ideal we have $[L,R] \subset R$.
Consider an arbitrary $X \in R$. Then X is a diagonal matrix according to part i)

$$X = \sum_j x_{jj} \cdot E_{jj}.$$

For arbitrary but fixed $r \neq s$

$$[X,E_{rs}] = \sum_j x_{jj} \cdot [E_{jj},E_{rs}] = \sum_j x_{jj} \cdot \delta_{jr} \cdot E_{js} - \sum_j x_{jj} \cdot \delta_{js} \cdot E_{rj} = x_{rr} \cdot E_{rs} - x_{ss} E_{rs} =$$

$$= (x_{rr} - x_{ss}) \cdot E_{rs} \in R \subset \mathfrak{d}(m,\mathbb{C})$$

which implies $[X,E_{rs}] = 0$. For a diagonal matrix

$$Y = \sum_k y_{kk} \cdot E_{kk} \in L$$

apparently

$$[X,Y] = 0.$$

As a consequence $R \subset Z(L)$, which due to part i) implies

$$R = 0.$$

ii) Now $R = rad(L) = 0$ implies $L = sl(m,\mathbb{C})$ semisimple.

Problems 08

29. Consider a complex Lie algebra L.

Show: If L is semisimple and solvable then $L = \{0\}$.

30. Consider a short exact sequence

$$0 \to L_0 \to L_1 \to L_2 \to 0$$

of complex Lie algebras.

Show: Semisimplicity of L_1 implies semisimplicity of L_2.

31. Consider a finite-dimensional vector space V and an endomorphism $f \in End(V)$ which splits V as a direct sum of eigenspaces

$$V = \bigoplus_{\lambda} V_{\lambda}(f).$$

Let $W \subset V$ be an f-stable subspace, i.e.$f(W) \subset W$.

i) Show: If an element

$$w = v_1 + ... v_k \in W$$

decomposes as the sum of eigenvectors of f with corresponding, pairwise distinct eigenvalues $(\lambda_i)_{i=1,...,k}$, then $v_i \in W$ for all $i = 1, ..., k$.

Hint: Induction on k. For the induction step consider $f(w) - \lambda_1 \cdot w$.

ii) Show:

$$W = \bigoplus_{\lambda} (W \cap V_{\lambda}(f)).$$

iii) Show: The assumption $f(W) \subset W$ is necessary for the conclusion of part ii).

32. Consider a complex simple Lie algebra L and two symmetric, nondegenerate bilinear forms

$$\gamma, \delta : L \times L \to \mathbb{C},$$

which are "associative" in the sense

$$\gamma([x,y],z) = \gamma(x,[y,z]) \textit{ and } \delta([x,y],z) = \delta(x,[y,z]), x,y,z, \in L.$$

Show: A constant $\mu \in \mathbb{C}^*$ exists such that

$$\gamma = \mu \cdot \delta : L \times L \to \mathbb{C}.$$

Hint: For $x, y \in L \setminus \{0\}$ use the linear maps

$$L \to L^*, x \mapsto \gamma(x,-), \textit{ and } L \to L^*, y \mapsto \delta(-,y),$$

to define an endomorphism $f : L \to L, x \mapsto y$. Relate the behaviour of f to the adjoint representation.

Discussion: Tuesday, 20.12.2016, 12.15 p.m.

Problems 09

33. Consider a $\mathbb{K}$-Lie algebra L and two finite dimensional L-modules V and W. Consider the induced L-modules V^*, $V^* \otimes W$ and $Hom_{\mathbb{K}}(V,W)$.

Show: The canonical isomorphism of $\mathbb{K}$-vector spaces

$$V^* \otimes W \to Hom_{\mathbb{K}}(V,W), \lambda \otimes w \mapsto f_{\lambda,w},$$

with

$$f_{\lambda,w}(v) := \lambda(v) \cdot w, v \in V,$$

is a morphism of L-modules.

34. Consider an Abelian Lie algebra L.

Show: The Lie algebra of derivations of L equals the Lie algebra of linear endomorphisms of the vector space of L, i.e.

$$Der(L) = gl(L).$$

35. Consider a Lie algebra S and a vector space V, considered as an Abelian Lie algebra. According to Excercise 34 any representation

$$\rho : S \to gl(V)$$

satisfies $\rho(S) \subset Der(V)$. Therefore the semidirect product

$$V \rtimes_\rho S$$

is a well-defined Lie algebra, fitting into the exact sequence of Lie algebras

$$0 \to V \to V \rtimes_\rho S \to S \to 0.$$

Assume S semisimple and $\rho : S \to V$ nonzero and irreducible. Show for $L := V \rtimes_\rho S$:

i) *Derived algebra*: $L = [L,L]$

Hint: Consider $S \subset L$ as subalgebra and $V \subset L$ as ideal with $L = S + V$.
Verify $\rho(S)(V) = [S,V]_L$. Conclude $V = [S,V]_L$ and $[S,S]_L = S$. Show $L \subset [L,L]$.

ii) *Center*: $Z(L) = \{0\}$

iii) *No direct product*: There do not exist Lie algebras L_1 semisimple and L_2 solvable with $L \simeq L_1 \times L_2$. In particular, L is not semisimple.

36. Consider a complex semisimple Lie algebra L. Using Weyl's theorem on complete reducibility give a direct proof for

$$ad(L) = Der(L),$$

cf. Exercise 26.

Hint: Check that any derivation $\delta \in Der(L)$ defines an L-module structure on the vector space $\mathbb{C} \oplus L$ according to

$$x.(a,y) := (0, a \cdot \delta(x) + [x,y]_L), x, y \in L, a \in \mathbb{C}.$$

Discussion: Tuesday, 10.1.2017, 12.15 p.m.

Problems 10

37. Consider the semisimple Lie algebra $M := sl(3,\mathbb{C})$ and its subalgebra

$$L := span_{\mathbb{C}} < h := E_{11} - E_{22}, x := E_{12}, y := E_{21} > \simeq sl(2,\mathbb{C}).$$

The restriction of the adjoint representation $ad : M \to gl(M)$ to the subalgebra L defines an L-module structure

$$L \times M \to M, (x,m) \mapsto x.m := ad(x)(m).$$

i) Compute the vector space dimension of M and of the direct sum of irreducible $sl(2,\mathbb{C})$-modules:

$$V := V(0) \oplus V(1) \oplus V(1) \oplus V(2).$$

ii) Show: Both $sl(2,\mathbb{C})$-modules M and V are isomorphic.

iii) Specify a primitive element e and the derived family $(e_i := \frac{1}{i!} \cdot (y^i.e))_{i \in \mathbb{N}}$ for each irreducible summand of M.

38. The vector space $\mathbb{C}[u,v]$ of complex polynomials in two variables has a basis of monomials $(u^\mu \cdot v^\nu)_{\mu,\nu \in \mathbb{N}}$. A *homogeneous polynomial* of degree $n \in \mathbb{N}$ is an element

$$P(u,v) = \sum_{\mu+\nu=n} a_{\mu\nu} \cdot u^\mu \cdot v^{n-\mu} \in \mathbb{C}[u,v], a_{\mu\nu} \in \mathbb{C}.$$

Denote by

$$Pol^n \subset \mathbb{C}[u,v]$$

the subspace of homogeneous polynomials of degree n.

i) For $n \in \mathbb{N}$ determine the vector space dimension $dim\ Pol^n$.

ii) Set $L := sl(2,\mathbb{C})$. The tautological L-module $V(1) \simeq \mathbb{C}^2$ has the L-operation

$$L \times \mathbb{C}^2 \to \mathbb{C}^2, (z,w) \mapsto z.w := z(w).$$

Identify the elements of the canonical basis of $\mathbb{C}^2$ with the variables u and v

$$\begin{pmatrix}1\\0\end{pmatrix} \simeq u \text{ and } \begin{pmatrix}0\\1\end{pmatrix} \simeq v.$$

Show: The vector space $Pol^n, n \in \mathbb{N}$, is an irreducible L-module of highest weight n with respect to the L-operation

$$L \times Pol^n \to Pol^n, (z, P(u,v)) \mapsto z.P(u,v) := (z.u)\frac{\partial P(u,v)}{\partial u} + (z.v)\frac{\partial P(u,v)}{\partial v}.$$

Determine a primitive element $e \in Pol^n$.

39. Set $L := sl(2,\mathbb{C})$.

i) Consider the two irreducible L-modules $V(3)$ and $V(7)$.

Show: The tensor product

$$V := V(7) \otimes V(3)$$

decomposes as the direct sum of irreducible L-modules

$$V \simeq V(10) \oplus V(8) \oplus V(6) \oplus V(4).$$

Hint: Consider primitive elements $e \in V(7)$ and $f \in V(3)$. Use their derived families $(e_i)_{i=0,...,7}$ and $(f_j)_{j=0,...,3}$ to obtain bases of the tensor product V. Determine primitive elements for each of the supposed summands.

ii) Make a conjecture for the general case: How does the tensor product

$$V(n) \otimes V(m), n \geq m,$$

decompose as a sum of irreducible L-modules?

40. Set $L = sl(3,\mathbb{C})$.

i) Consider the subalgebra of traceless diagonal matrices

$$H := \mathfrak{d}(3,\mathbb{C}) \cap L.$$

Prove that $H \subset L$ is a maximal toral subalgebra.

ii) Consider the basis of H

$$(h_1 := E_{11} - E_{22}, h_2 := E_{22} - E_{33})$$

Compute the Cartan decomposition of L with respect to H, i.e. determine a basis of each root space L_α of L and determine for the corresponding root $\alpha \in \Phi$ the values $\alpha(h_1)$ and $\alpha(h_2)$.

iii) Which linear relations exist between the roots from Φ?

iv) Show: There exist three roots $\alpha_i \in \Phi, i = 1, 2, 3$, with elements

$$h_i \in H, x_i \in L_{\alpha_i}, y_i \in L_{-\alpha_i}$$

such that

$$L_i := span_{\mathbb{C}} < h_i, x_i, y_i > \simeq sl(2, \mathbb{C})$$

and

$$L = H \oplus \bigoplus_{i=1,2,3} (L_{\alpha_i} \oplus L_{-\alpha_i}).$$

Discussion: Tuesday, 17.1.2017, 12.15 p.m.

Problems 11

41. A Cartan subalgebra H of a Lie algebra L is a nilpotent subalgebra $H \subset L$ equal to its normalizer, i.e. $H = N_L(H)$.

Show: Any maximal toral subalgebra of a complex semisimiple Lie algebra L is a Cartan subalgebra of L.

Hint: Use the Cartan decomposition of L.

42. Consider a root system Φ of a real finite-dimensional vector space V.

Show: For any root $\alpha \in \Phi$ the required symmetry σ_α of V with vector α satisfying

$$\sigma_\alpha(\Phi) \subset \Phi$$

is uniquely determined.

Hint: Assume the existence of σ_1 and σ_2. Consider $u := \sigma_2 \circ \sigma_1$. On one hand, prove $u(x) \equiv x \text{ } mod \text{ } \mathbb{R} \cdot \alpha$ and conclude: All eigenvalues of u are $= 1$. On the other hand: Show the existence of an exponent $n \in \mathbb{N}$ with $u^n = id$. From both results derive $u = id$.

43. Consider the Lie algebra $L = sl(3, \mathbb{C})$ and the maximal toral subalgebra

$$H := \mathfrak{d}(3, \mathbb{C}) \cap L.$$

i) For the root set Φ of (L, H) verify the axioms (R1)-(R4) of a root system of the vector space $V := \mathbb{R}^2$.

ii) Determine a base $\Delta = \{\alpha, \beta\}$ of Φ. To which type of the classification (see Lemma 7.7 of the lecture) does Φ belong?

iii) Show: The Weyl group $\mathscr{W}$ of Φ is isomorphic to the symmetric group Sym_3.

Hint: Both groups are generated by two elements. Determine the relations.

44. Consider the following definitions relating real and complex structures:

- Elements of a complex vector space V can be considered elements of a *real* vector space $V_{\mathbb{R}}$ by restricting the scalars from $\mathbb{C}$ to $\mathbb{R}$. Similarly, if L is a complex Lie algebra then by restricting scalars from $\mathbb{C}$ to $\mathbb{R}$ the Lie algebra L can be considered a *real* Lie algebra $L_{\mathbb{R}}$.

- If M is a real Lie algebra then the *complexification* of M is the complex Lie algebra $\mathbb{C} \otimes_{\mathbb{R}} M$ with Lie bracket

$$[z_1 \otimes m_1, z_2 \otimes m_2] := (z_1 \cdot z_2) \otimes [m_1, m_2], z_1, z_2 \in \mathbb{C}, m_1, m_2 \in M.$$

- A *real form* of a complex Lie algebra L is a real subalgebra $M \subset L_{\mathbb{R}}$ such that the complex linear map

$$j : \mathbb{C} \otimes_{\mathbb{R}} M \to L, 1 \otimes m \mapsto m, i \otimes m \mapsto i \cdot m,$$

  is an isomorphism of complex Lie algebras.

i) Show: The Lie algebra $su(n)$ is a real form of the Lie algebra algebra $sl(n, \mathbb{C})$.

Hint: The decomposition

$$z = x + i \cdot y = \frac{z + \bar{z}}{2} + i \cdot \frac{z - \bar{z}}{2i}$$

of complex numbers induces a similar decomposition of elements from $sl(n, \mathbb{C})$ and an inverse of the map j

$$sl(n, \mathbb{C}) \to \mathbb{C} \otimes_{\mathbb{R}} su(n).$$

ii) Let M be a real Lie algebra and $L := M_{\mathbb{C}}$ its complexification. Consider a complex vector space V.
Show: Any real representation of M on the real vector space $V_{\mathbb{R}}$ has a unique extension to a complex representation of L on the complex vector space V, i.e. for the real-linear M-module structure $\mu_{\mathbb{R}} : M \times V_{\mathbb{R}} \to V_{\mathbb{R}}$ exists a unique complex linear L-module structure

$$\mu_{\mathbb{C}} : L \times V \to V$$

such that for all $m \in M, x \in \mathbb{R}, v \in V$

$$\mu_{\mathbb{C}}(j(x \otimes m), v) = \mu_{\mathbb{R}}(x \cdot m, v).$$

Hint: The definition of $\mu_{\mathbb{C}}$ reduces to the definition of $\mu_{\mathbb{C}}(j(i \otimes m), v)$.

Discussion: Tuesday, 24.1.2017, 12.15 p.m.

Problems 12

45. Consider a root system Φ of a vector space V.

Show that any base $\Delta = \{\alpha_1, ..., \alpha_r\}$ of Φ can be obtained by a linear functional, i.e.

i) A linear functional $t \in V^*$ exists such that

$$\Delta \subset \Phi_t^+ := \{\alpha \in \Phi : t(\alpha) > 0\}.$$

ii)

$$\Delta = \{\alpha \in \Phi_t^+ : \alpha \textit{ indecomposable}\}.$$

Hint: $\Phi^+ \subset \Phi_t^+$, $\Phi^- \subset \Phi_t^- := \{\alpha \in \Phi : t(\alpha) < 0\}$ and $\Phi^+ \dot\cup \Phi^- = \Phi = \Phi_t^+ \dot\cup \Phi_t^-$ imply $\Phi^+ = \Phi_t^+, \Phi^- = \Phi_t^-$.

46. Consider a root system Φ of a vector space V and denote by $(-,-)$ a scalar product on V invariant with respect to the Weyl group of Φ.

Show:

i) Two roots $\alpha, \beta \in \Phi$ are orthogonal with respect to $(-,-)$ iff their Cartan integer satisfies $< \alpha, \beta >= 0$.

ii) If $(\alpha, \beta) = 0$ for two roots $\alpha, \beta \in \Phi$ then

$$\sigma_\alpha \circ \sigma_\beta = \sigma_\beta \circ \sigma_\alpha.$$

iii) For a symmetry σ_α of V with vector $\alpha \neq 0$ the fixed hyperplane H_α is the orthogonal space of α with respect to $(-,-)$.

47. Consider a root system Φ of a vector space V and denote by $(-,-)$ a scalar product on V invariant with respect to the Weyl group $\mathscr{W}$ of Φ. The root system Φ is *reducible* if a decomposition

$$\Phi = \Phi_1 \dot\cup \Phi_2, \Phi_1 \neq \emptyset, \Phi_2 \neq \emptyset,$$

exists with $(\Phi_1, \Phi_2) = 0$. Otherwise Φ is *irreducible*. Analogously defined are the terms *reducible* and *irreducible* for a base Δ of Φ.

Show for an arbitrary base Δ of Φ:

i) Reducibility of Φ implies reducibility of Δ.

Hint: $span\ \Delta = V$.

ii) Irreducibility of Φ implies irreducibility of Δ.

Hint: If $\Delta = \Delta_1 \dot{\cup} \Delta_2$ then define $\Phi_i := \mathscr{W}(\Delta_i), i = 1, 2$. Use that the symmetries $\alpha \in \Delta$ generate $\mathscr{W}$ and use Exercise 46 to show

- $\alpha_1 \in \Delta_1, \alpha_2 \in \Delta_2$ implies $\sigma_{\alpha_2}(\alpha_1) = \alpha_1$
- $\alpha_1, \beta_1 \in \Delta_1$ implies $\sigma_{\alpha_1}(\beta_1) \in span\ \Delta_1$

and to conclude $\Phi_1 \subset span\ \Delta_1$. Analogously $\Phi_2 \subset span\ \Delta_2$. From $(\Delta_1, \Delta_2) = 0$ follows $(\Phi_1, \Phi_2) = 0$. Without restriction $\Phi_1 = \emptyset$ which implies $\Delta_1 = \emptyset$.

48. Consider the root system from Lemma 7.7, no. 5.

i) Determine all bases of Φ.

ii) How many unordered pairs of distinct roots exist? How many unordered pairs with one short root and one long root exist?

Hint: Use the fact that the Weyl group can be generated by two elements.

Discussion: Tuesday, 7.2.2017, 12.15 p.m.