Ludwig-Maximilians-Universität München
Prof. Dr. Ivo Sachs, Prof. Dr. Thomas Vogel
Dr. Tomáš Procházka, Dr. Stephan Stadler

Differentiable Manifolds

Sheet 13

Due Tue., January 31, 10 am, in the letter box on the 1st floor.

1. Let N be a submanifold of the semi-Riemannian manifold (M, g) and ∇ the Levi-Civita connection on TM. We assume that g restricts to a semi-Riemannian metric on N.

   We denote the orthogonal projection of

   $$TM|_N = \{\text{tangent vectors of } M \text{ with base point in } N\}$$

   to TN by pr. Prove that

   $$\begin{aligned}\overline{\nabla} : \chi(N) \times \chi(N) &\longrightarrow \chi(N) \\ (X, Y) &\longmapsto \mathrm{pr}\left(\nabla_X Y\right)\end{aligned}$$

   is the Levi-Civita connection on N.

2. We keep the notation and setup from exercise 1. The second fundamental form of N at $p \in M$ in M is

   $$\begin{aligned}II_p : \chi(N) \times \chi(N) &\longrightarrow TN^{\perp} \\ (X, Y) &\longmapsto \mathrm{pr}^{\perp}(\nabla_X Y)\end{aligned}$$

   where $\mathrm{pr}^{\perp}$ is the orthogonal projection $T_pM \longrightarrow T_pN^{\perp}$. Show that II is a symmetric tensor (with values in $TN^{\perp} \subset TM$).

3. Let γ be a smooth curve in M and ∇ a connection on TM. Let $\frac{\nabla}{dt}$ be the operator on vector fields along γ induced by ∇. Show that if ∇ is metric with respect ot g, then

   $$\frac{d}{dt} g(X(t), Y(t)) = g\left(\frac{\nabla}{dt} X(t), Y(t)\right) + g\left(X(t), \frac{\nabla}{dt} Y(t)\right)$$

   for all vector fields X, Y along γ.

4. Let $\langle X, Y\rangle = -X^0Y^0 + \sum_{i=1}^n X^iY^i$ denote a Lorentzian metric on $\mathbb{R}^{n+1}$. We consider

   $$H^n = \{(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} \mid \langle x, x\rangle = -1 \text{ and } x^0 > 0\}.$$

   a) Show that H^n is a smooth, connected n-manifold and that $\langle \cdot, \cdot \rangle$ induces a Riemannian metric on H^n.

   b) Let $p = (1, 0, \ldots, 0) \in H^n$ and $X_0 = \frac{\partial}{\partial x^1} \in T_pH^n$. Find the unique geodesic $\gamma : \mathbb{R} \longrightarrow H^n$ in H^n with $\dot\gamma(0) = X_0$.

   c) We denote the isometry group of $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ by $\mathrm{O}(1, n)$. Prove that this group acts transitively and isometrically on H^n. Show that for all $Y_0 \in T_qH^n$ there is $A \in \mathrm{O}(1, n)$ such that $(DA)(Y_0) = X_0$.