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Differentiable Manifolds

Sheet 12

Due Tue., January 24, 10 am, in the letter box on the 1st floor.

1. Let M be a compact, oriented, connected n-manifold with non-empty boundary and α a closed $(n-1)$-form with $\int_{\partial M} \alpha \neq 0$. Show that there is no closed form β on M which coincides with α on ∂M.

   Conversely, assume that ∂M is connected and $\int_{\partial M} \alpha = 0$. Prove that there is a closed $(n-1)$-form β on M whose restriction to ∂M is α.

2. a) Let ∇ be a covariant deriviative on TM which is torsion free and g a semi-Riemannian metric. Prove that

   $$\begin{aligned} R : \chi(M) \times \chi(M) \times \chi(M) \times \chi(M) &\longrightarrow \mathbb{R} \\ (X, Y, Z, W) &\longmapsto g(\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z, W) \end{aligned}$$

   is a tensor.

   b) Let G be a Lie group and g a bi-invariant metric (i.e. $g(X, Y)$ is constant when X, Y are both left invariant/right invariant). Show that the Levi-Civita connection of g on G satisfies $\nabla_X Y = [X, Y]/2$ for left invariant vector fields X, Y.

   Apology: The original version of this exercise assumed only that g is left invariant.

3. In this exercise we use the fact that through every pair of points $x, y \in S^n$ such that $x \neq y \neq -x$ there is a unique plane containing x, y and the origin. The intersection of this plane with S^n is the great circle through $x, y,$.

   a) Show that a map $f : S^n \longrightarrow S^n$ which has no fixed point is homotopic to the antipodal map $A : S^n \longrightarrow S^n$.

   b) Prove that every map $f : S^n \longrightarrow S^n$ whose degree is not $(-1)^{n+1}$ has a fixed point.

4. Let $k \in \mathbb{Z}$ and $n > 0$.

   a) Prove that for every closed oriented n-manifold there is a map $f : M \longrightarrow S^n$ of degree of k.

   Hint: Construct first a map g of the n-ball onto the sphere such that the map is constant on a neighbourhood of the boundary of the ball such that this map is an embedding on a neighbourhood of the origin and $g^{-1}(g(0)) = \{0\}$.

   b) Let M be a compact oriented connected manifold and $f : M \longrightarrow S^n$ a smooth map which is not surjective. Prove that f is homotopic to a constant map.

   Hint: If $p \notin f(M)$ choose $-p$ as the image of the constant map.

5. (Do not hand in, this is significantly more difficult.)

   Let M be a closed, oriented, connected n-manifold. Prove that two maps $f, g : M \longrightarrow S^n$ are homotopic if and only of $\deg(f) = \deg(g)$.

   Here are a few hints and useful facts for the difficult direction.

   - The goal is to construct a smooth map $H : M \times [0, 1] \longrightarrow S^n$ which coincides with f on $M \times \{0\}$ and with g on $M \times \{1\}$.
   - Solve exercise 4 first.
   - A connected compact 1-manifold with boundary is diffeomorphic to either a circle or a closed interval.
   - Assume H exists. Then H has a regular value p such that $N = H^{-1}(p) \subset M$ is a compact submanifold with boundary which has dimension 1 and whose interior is disjoint from $\partial(M \times [0, 1])$ such that $T_qN \oplus T_q\partial M = T_pM$ at boundary points $p \in H^{-1}(p)$. In particular, $\partial N \subset \partial(M \times [0, 1])$.
   - N has a neighbourhood $tube(N) \simeq N \times D^n$ (called tubular neighbourhood, think of the neighbourhood as a rope of dimension $n + 1$ with N in its interior) such that for each $q \in N$ the restriction of H to $\{q\} \times D^n$ is a diffeomorphisms onto its image. Moreover $(\partial N) \times D^n \subset \partial(M \times [0, 1])$.
   - You may assume that there is $p \in S^n$ which is a regular value for both f and g.
   - Construct $H^{-1}(p)$ first. Continue assuming that p is a regular value of H with all the nice properties mentioned above. Then define H on a tubular neighbourhood, then extend to $M \times [0, 1]$.