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# Differentiable Manifolds

## Sheet 11

Due Tue., January 17, 10 am, in the letter box on the 1st floor.

1. Cohomology of the torus.
   Consider the abstract torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We define $c_1$ and $c_2$ to be the closed curves $c_1(t) = (t, 0)$ and $c_2(t) = (0, t), t \in [0, 1]$. Any closed curve $\gamma : [a, b] \to T^2$ can be lifted to a continious curve $\tilde{\gamma} : [a, b] \to \mathbb{R}^2$ in $\mathbb{R}^2$, i.e. $\pi \circ \tilde{\gamma} = \gamma$, where $\pi : \mathbb{R}^2 \to T^2$ is the quotient map.
   (a) Show that $\tilde{\gamma}(b) - \tilde{\gamma}(a) \in \mathbb{Z}^2$ is independent of the chosen lift. Thus we can associate to every closed curve two integers called winding numbers.
   (b) Show that $\int_\gamma \beta = n \int_{c_1} \beta + m \int_{c_2} \beta$, where $\gamma$ is a closed curve on $T^2$, $\beta \in \Omega^1(T^2)$ is a closed 1-form and $n, m \in \mathbb{Z}^2$ are the winding numbers corresponding to $\gamma$.
   (c) Extend the 1-forms $dx_1, dx_2$ defined on $(0,1)^2$ uniquely to 1-forms $\alpha_1, \alpha_2$ on $T^2$. Show that every closed 1-form $\beta$ on $T^2$ can be written as $\beta = a_1\alpha_1 + a_2\alpha_2 + df$, where $a_1, a_2 \in \mathbb{R}$ and $f \in C^\infty(T^2)$.
   Hint: Use that a 1-form $\beta$ is exact, if and only if $\int_\gamma \beta = 0$ for all compact closed curves $\gamma$.

2. Green's theorem.
   On a compact Riemannian manifold $M$, prove the identity
   $$\int_M (f\triangle g - g\triangle f)\,\mathrm{vol} = \int_{\partial M} f * dg - g * df \tag{1}$$
   for $f, g \in C^\infty(M)$.

3. Harmonic functions on $S^2$.
   (a) Consider $S^2$ as submanifold of $\mathbb{R}^3$. Compute the metric induced on $S^2$ from the flat metric
   $$ds^2 = dx^2 + dy^2 + dz^2 \tag{2}$$
   in $\mathbb{R}^3$ (use the spherical coordinates).
   (b) Find the $\delta$ operator and Laplacian on $S^2$ with respect to the metric computed above.
   (c) Show that the vector field $y\partial_z - z\partial_y$ generating rotations of $\mathbb{R}^3$ restricts to vector field $V$ on $S^2$ and show that the Laplacian commutes Lie derivative $\mathcal{L}_V$.
   (d) Find all harmonic forms and identify the non-trivial de Rham cohomology groups of $S^2$.
   (e) Restrict the linear polynomials $x$, $y$ and $z$ from $\mathbb{R}^3$ to $S^2$ and show that they are eigenfunctions of Laplacian.

4. Vector analysis and Stokes theorem.
Consider $\mathbb{R}^3$ with cartesian coordinates $x^1, x^2, x^3$, equipped with the volume form vol $= dx^1 \wedge dx^2 \wedge dx^3$. To all vectors $\mathfrak{X}(\mathbb{R}^3) \ni A = A^i \partial_i$, we can associate a 1-form $\alpha^1$ and a 2-form $\alpha^2$ by

$$\alpha^1 = A^i dx^i \qquad \text{and} \qquad \alpha^2 = i_A vol. \tag{3}$$

In addition we have the usual inner product $\langle A, B \rangle = A^i B^i$ and the cross product $(A \times B)^i = \epsilon^{ijk} A_j B_k$.

(a) Show that $i_{A\times B}\mathrm{vol} = \alpha^1 \wedge \beta^1$ and $\mathrm{vol}(A, B, C) = \langle A \times B, C \rangle$.
We define grad, div and curl by

$$(df)^1 = \mathrm{grad} f, \qquad d\alpha^2 = (\mathrm{div} A)\,\mathrm{vol} \qquad \text{and} \qquad d\alpha^1 = i_{\mathrm{curl}A}\mathrm{vol}.$$

(b) Use forms to show the following relations for $A, B \in \mathfrak{X}(\mathbb{R}^3)$ and $f \in \mathcal{F}(\mathbb{R}^3)$:

$$\mathrm{div}\,(A \times B) = \langle \mathrm{curl} A, B \rangle - \langle A, \mathrm{curl} B \rangle \tag{4}$$

$$\mathrm{div}\,(fA) = f \mathrm{div} B + \langle \mathrm{grad} f, B \rangle \tag{5}$$

$$\mathrm{curl}\,(fA) = f \mathrm{curl} A + \mathrm{grad} f \times A \tag{6}$$

(c) classical Gauss Theorem in $\mathbb{R}^3$:
Consider $U, \partial U \subset \mathbb{R}^3$ parametrized by $\Phi : \mathbb{R}^3 \supset V_U \to U$ and $\phi : \mathbb{R}^2 \supset V_{\partial U} \to \partial U$. Use Stokes Theorem for forms to show

$$\int_U \mathrm{div} A \ \mathrm{vol} = \int_{\partial U} \langle A, N \rangle \ \mathrm{vol}_{\partial U} \tag{7}$$

$$\int_{V_U} \mathrm{div} A \ \det(D\Phi)\, dt_1 dt_2 dt_3 = \int_{V_{\partial U}} \langle A, N \rangle \, \|\partial_{s_1}\phi \times \partial_{s_2}\phi\| ds_1 ds_2, \tag{8}$$

where $N$ is the unit normal to the surface $\partial U$, $\|A\| = \langle A, A \rangle^{1/2}$ and $\mathrm{vol}_{\partial U} = i_N \mathrm{vol}$.

(d) classical Stokes Theorem in $\mathbb{R}^3$:
Consider $S, \partial S \subset \mathbb{R}^3$, parametrized by $\phi : \mathbb{R}^2 \supset V_S \to S$ and $\gamma : \mathbb{R} \supset V_{\partial S} \to \partial S$. Use Stokes Theorem for forms to show

$$\int_{V_S} \langle \mathrm{curl} A, \partial_{s_1}\phi \times \partial_{s_2}\phi \rangle \, ds_1 ds_2 = \int_{V_{\partial S}} \langle A, \partial_t \gamma \rangle \, dt \tag{9}$$