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Differentiable Manifolds

Sheet 2

Due wed., november 2nd, 10 am, in the letter box on the 1st floor. Please note that november 1st is a holiday.

1. The exam will be on february 15, from noon till 4pm.
2. Denote by S^n the euclidean unit sphere of dimension n, realized as
$$S^n := \{x \in \mathbb{R}^{n+1} |\ \|x\| = 1\}.$$
Set $N := (0, \ldots, 0, 1)$ be the north pole and define for every $x \in S^n - \{N\}$ the point $p_N(x)$ as the intersection of the line through N and x with the hyperplane $P := \{y \in \mathbb{R}^{n+1} |\ y_{n+1} = 0\}$. Show that
   (a) $p_N : S^n - \{N\} \to P$ is a homeomorphism.
   (b) One obtains a smooth atlas $\{p_N, p_S\}$ for S^n where p_S is defined as p_N with the north pole replaces by the south pole $S = -N$.
3. Let M be a smooth manifold. Suppose that $f : M \to M$ is a smooth involution without fixed points, i.e. for all points $x \in M$ the conditions $f \circ f(x) = x$ and $f(x) \neq x$ are satisfied. Define an equivalence relation on M by $x \sim y$ if and only if $f(x) = f(y)$. Show that the associated quotient space $M/\sim$ naturally inherits the structure of a smooth manifold such that the projection $\pi : M \to M/\sim$ is smooth.
4. Let S^n be the euclidean unit sphere.
   (a) Show that the real projective space $\mathbb{R}P^n := S^n/(x \sim -x)$ is a smooth manifold.
   (b) Set $H := \{y \in \mathbb{R}^6 |\ y_1 + y_2 + y_3 = 1\}$. Consider the map
$$\begin{aligned} v : \mathbb{R}P^2 &\longrightarrow H \\ [x : y : z] &\longmapsto (x^2, y^2, z^2, xy, yz, xz). \end{aligned}$$
   Show that v is a smooth embedding. Conclude that $\mathbb{R}P^2$ can be realized as submanifold of $\mathbb{R}^5$.