Ludwig-Maximilians-Universität München
Prof. Dr. Ivo Sachs, Prof. Dr. Thomas Vogel
Dr. Tomáš Procházka, Dr. Stephan Stadler

Differentiable Manifolds

Sheet 14

Due Tue., February 7, 10 am, in the letter box on the 1st floor.

1. The goal of this exercise is to describe the so called Berger spheres and to compute some parts of the curvature tensor. It is not as bad as its length suggests.

   a) Show that the following map is a well defined diffeomorphism

   $$\mathrm{SU}(2) = \{A \in \mathrm{Mat}(2,\mathbb{C}) \mid A\overline{A}^T = E, \det(A) = 1\} \longrightarrow S^3 = \{(z,w) \in \mathbb{C}^2 \mid z\overline{z} + w\overline{w} = 1\}$$
   $$\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \longmapsto (z,w).$$

   b) The Lie-algebra $\mathfrak{su}(2)$ of $\mathrm{SU}(2)$ is generated by

   $$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   Let g be the left-invariant metric on $\mathrm{SU}(2)$ such that X_1, X_2, X_3 is an orthonormal basis. Show that this metric is bi-invariant, i.e. if $A \in \mathrm{SU}(2)$, then $c_A^* g = g$ (c_A denotes conjugation by A). Compute $[X_1, X_2], [X_2, X_3], [X_3, X_1]$.

   c) We now replace the metric g by the left-invariant metric $g_\varepsilon, \varepsilon > 0$, such that $\varepsilon^{-1}X_1, X_2, X_3$ is an orthonormal basis of g_ε. Let ∇^ε be the Levi-Civita connection of g_ε. Apply the formula (sometimes called the Koszul formula)

   $$\begin{aligned} 2g(\nabla_X Y, Z) &= L_X g(Y,Z) + L_Y g(X,Z) - L_Z g(X,Y) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \end{aligned}$$

   to left-invariant vector fields X_1, X_2, X_3 on $(\mathrm{SU}(2), g_\varepsilon)$ to show that

   $$\begin{aligned} \nabla^\varepsilon_{X_1} X_2 &= (2-\varepsilon^2) X_3 & \qquad \nabla^\varepsilon_{X_2} X_1 &= -\varepsilon^2 X_3 \\ \nabla^\varepsilon_{X_2} X_3 &= X_1 & \nabla^\varepsilon_{X_3} X_2 &= -X_1 \\ \nabla^\varepsilon_{X_3} X_1 &= \varepsilon^2 X_2 & \nabla^\varepsilon_{X_1} X_3 &= (\varepsilon^2 - 2) X_2. \end{aligned}$$

   It is convenient to use vanishing torsion to obtain the second column from the first. Moreover show $\nabla^\varepsilon_{X_i} X_i = 0$.

   d) Using the results from the previous exercise verify that

   $$\begin{aligned} &R(X_i, X_j)X_k = 0 \text{ when } i,j,k \text{ are pairwise distinct} \\ &R(X_1, X_2)X_2 = \varepsilon^2 X_1 \\ &R(X_1, X_3)X_3 = \varepsilon^2 X_1 \\ &R(X_2, X_3)X_3 = (4 - 3\varepsilon^2) X_2 \end{aligned}$$

e) Compute $\mathrm{Ric}(\varepsilon^{-1}X_1, \varepsilon^{-1}X_1), \mathrm{Ric}(X_2, X_2), \mathrm{Ric}(X_3, X_3)$ and the scalar curvature of $(\mathrm{SU}(2), g_\varepsilon)$. Show that the sectional curvature of the X_2, X_3-plane converges to 4 as $\varepsilon \to 0$.

f) Let $\gamma(t) = \exp(tX_1)$ be the flow line of the left-invariant vector field X_1 on $\mathrm{SU}(2)$. Show that this is a geodesic whose image is a circle of length $2\pi\varepsilon$.

g) Show that the vector field

$$Z(t) = \cos((\varepsilon^2 - 2)t)X_2(\gamma(t)) + \sin((\varepsilon^2 - 2)t)X_3(\gamma(t))$$

is a parallel vector field along γ. Conclude that the parallel transport along $\gamma : [0, 2\pi] \longrightarrow (\mathrm{SU}(2), g_\varepsilon)$ from $\gamma(0)$ to $\gamma(2\pi)$ is a rotation around $X_1(\gamma(0))$ and compute the angle.

2. Let ∇ be the Levi-Civita connection of a Riemannian metric. Prove the following identities for the curvature tensor.

a) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

b) $g(R(X, Y)Z, W) = g(R(Y, X)W, Z)$.