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Lecture Winter term 2016/2017

Differentiable manifolds

Please note: These notes summarize the content of the lecture, many details and examples are omitted. Sometimes, but not always, we provide a reference for proofs, examples or further reading. Changes to this script are made without any further notice at unpredictable times. If you find any typos or errors, please let us know.

1. Lecture on Oct., 18. – **Submanifolds of Euclidean space**

- **References**: I do not follow specific books, possible references include [C, Jä-V, L, W].
- We give four equivalent definitions of the notion of submanifold of dimension $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. In all four of them, $M \subset \mathbb{R}^n$.
- **Condition (a) Local parametrizations:** For all $p \in M$ there is an open set $U \subset \mathbb{R}^k$, a neighbourhood $V \subset \mathbb{R}^n$ of p and a smooth map $\varphi : U \longrightarrow \mathbb{R}^n$ such that
  1. φ is a homeomorphism onto $V \cap M$, and
  2. for all $x \in U$ the differential $D_x\varphi : \mathbb{R}^k \longrightarrow \mathbb{R}^n$ is injective.
- **Condition (b) Locally flat:** For all $p \in M$ there are an open neigbourhood $V \subset \mathbb{R}^n$ of p and $W \subset \mathbb{R}^n$ of 0 and a diffeomorphism $\phi : V \longrightarrow W$ such that $\phi(p) = 0$ and $\phi(V \cap M) = \left(\mathbb{R}^k \times \{0 \in \mathbb{R}^{n-k}\}\right) \cap W$.
- **Condition (c) Locally regular level set:** For all $p \in M$ there is an open neighbourhood U and a smooth function $F : V \longrightarrow \mathbb{R}^{n-k}$ such that
  1. $F^{-1}(0) = (V \cap M)$, and
  2. for all $q \in M \cap V$ the differential $D_qF : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$ is surjective.
- **Condition (d) Locally a graph:** For all $p \in M$ there is an open neighbourhood $V \subset \mathbb{R}^n$ and a smooth function $U \subset \mathbb{R}^{n-k}$ defined on an open subset of $U \subset \mathbb{R}^k$ together with a permutation $\sigma \in S_n$ such that
$$V \cap M = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \,|\, (x, g(x)) \text{ with } x \in U\}.$$
- **Theorem:** For a given subset $M \subset \mathbb{R}^n$ the conditions (a)–(d) are equivalent.
- *Proof:* The proof of the implications $(b) \Rightarrow (c), (d) \Rightarrow (a)$ are trivial. For the proof of $(c) \Rightarrow (d)$ one uses the implicit function theorem, for the proof of $(a) \Rightarrow (b)$ one applies the inverse function theorem to a function Φ extending φ (the local inverse of Φ satisfies (b)).
- **Definition:** A subset $M \subset \mathbb{R}^n$ is a *submanifold* of dimension k if any of the conditions (a)–(d) is satisfied.
- **Remark:** When M is a non-empty submanifold the number k is then the same in all the conditions, in particular the dimension (a non-empty open subset of $\mathbb{R}^l$ is diffeomorphic to an open subset of $\mathbb{R}^m$ only if $m = l$ by the inverse function theorem, this remains true for homeomorphism but this is more difficult) of a non-empty submanifold is well defined. By convention, the empty subset is a submanifold of any dimension (including negative integers).

- **Examples :** $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n, k \le n$, $S^k = \{(x_1, \dots, x_{k+1}) \,|\, x_1^2 + \dots + x_n^2 = 1\} \subset \mathbb{R}^{k+1}$, and more interestingly

$$\mathrm{O}(n) = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \,|\, AA^T = E\}$$

are submanifolds. To prove this for $\mathrm{O}(n)$ verify condition (c) for

$$F : \mathrm{Mat}(n \times n, \mathbb{R}) \longrightarrow \mathrm{Sym}(n, \mathbb{R}) = \{B \in \mathrm{Mat}(n \times n, \mathbb{R}) \,|\, B = B^T\}$$
$$A \longmapsto AA^T - E.$$

The dimension of $O(n)$ is $\frac{n(n-1)}{2}$.

2. Lecture on Oct., 20. – **Continuation and abstract manifolds**

- **Proposition:** Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension k, $U, U' \subset \mathbb{R}^k$ open and

$$\varphi : U \longrightarrow M \qquad\qquad \varphi' : U' \longrightarrow M$$

local parametrizations of M. Then

$$\varphi^{-1} \circ \varphi' : \varphi'^{-1}(\varphi(U) \cap \varphi'(U')) \longrightarrow \varphi^{-1}(\varphi(U) \cap \varphi'(U'))$$

is a diffeomorphism.
- **Proof:** $\varphi^{-1} \circ \varphi'$ is a homeomorphism with inverse $\varphi'^{-1} \circ \varphi$. To show that $\varphi^{-1} \circ \varphi'$ is smooth near $\varphi'^{-1}(p) \in \varphi'^{-1}(\varphi(U) \cap \varphi'(U'))$ one constructs smooth map $F : V \longrightarrow \mathbb{R}^n$ on an open set in $\mathbb{R}^n$ containing $\varphi'^{-1}(p)$ such that $F \cap i = f$ near $\varphi'^{-1}(p)$ for a inclusion of $\mathbb{R}^k$ into $\mathbb{R}^n$ (as in the proof of $(a) \Rightarrow (b)$ last time) and F is a local diffeomorphism near φ'^{-1}.
- **Definition:** $f : M \longrightarrow \mathbb{R}$ is *smooth* near $p \in M$ if there is a local parametrization $\varphi : U \subset \mathbb{R}^k \longrightarrow M$ such that $p \in \varphi(U)$ and $f \circ \varphi$ is smooth.
- **Remark:** This is independent from the choice of φ by the above proposition.
- **Examples:** Restrictions of coordinate functions to submanifolds of $\mathbb{R}^n$ are smooth because of the following Lemma.
- **Lemma:** Let $M \subset \mathbb{R}^n$ be a submanifold. A function $f : M \longrightarrow \mathbb{R}$ is smooth near $p \in M$ if and only if there is an open neighbourhood U of p in $\mathbb{R}^n$ and a smooth map $F : U \longrightarrow \mathbb{R}$ such that $F|_{U \cap M} = f|_{U \cap M}$.
- **Proof:** Exercise.
- **Remark:** Maps into submanifolds of $\mathbb{R}^n$ can be viewed as a collection of n real valued functions. we therefore have defined what a smooth map between submanifolds of Euclidean spaces are.
- **Example:** Consider $\mathrm{O}(n) \subset \mathrm{Mat}(n \times n, \mathbb{R})$. The maps

$$\mathrm{inv} : \mathrm{O}(n) \longrightarrow \mathrm{O}(n)$$
$$A \longmapsto A^{-1} = A^T$$
$$\cdot : \mathrm{O}(n) \times \mathrm{O}(n) \longrightarrow \mathrm{O}(n)$$
$$(A, B) \longmapsto A \cdot B.$$

are all smooth (we view (somehow arbitrarily) $\mathrm{O}(n) \times \mathrm{O}(n)$ as submanifold of $\mathrm{Mat}(n \times n, \mathbb{R}) \times \mathrm{Mat}(n \times n, \mathbb{R})$. This makes $\mathrm{O}(n)$ a Lie group.
- We now start discussing manifolds without reference to an ambient space. The first attempt is preliminary.
- **Definition:** Let M be a set. A *smooth k-dim. atlas* $\mathcal{A}$ on M is a collection of maps $\varphi_i : U_i \longrightarrow M, i \in I$ (called *charts*) such that

1. $U_i \subset \mathbb{R}^k$ is open and $\varphi_i : U_i \longrightarrow \varphi_i(U_i)$ is bijective,
2. $\cup_i \varphi_i(U_i) = M$,
3. for all $i, j \in I$ such that $\varphi_i(U_i) \cap \varphi_j(U_j) \neq \emptyset$ the preimage under φ_i, φ_j are open in $\mathbb{R}^k$ and
$$\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(\varphi_i(U_i) \cap \varphi_j(U_j)) \longrightarrow \varphi_i^{-1}(\varphi_i(U_i) \cap \varphi_j(U_j))$$
is a diffeomorphism.

- **Definition:** Two such atlases $\mathcal{A}, \mathcal{A}'$ for M are *equivalent* if their union is still a smooth atlas.
- **Preliminary definition:** A manifold of dimension k is a set with an equivalence class of smooth k-dim atlases.
- **Example:** Submanifolds of $\mathbb{R}^n$ and products of such have natural smooth atlases.
- **Example:** For $k \geq 0$ let real projective space $\mathbb{RP}^k$ be the set of lines through the origin in $\mathbb{R}^{k+1}$, i.e.
$$\mathbb{RP}^k = (\mathbb{R}^{k+1} \setminus \{0\}) / \sim$$
where $(x_0, x_1, \dots, x_n) \sim (x_0', \dots, x_n')$ if and only if there is $\lambda \in \mathbb{R}$ such that $\lambda \cdot (x_0, \dots, x_n) = (x_0', \dots, x_n')$. Elements of this set are denoted by *homogeneous coordinates* $[x_0 : \dots : x_n]$. There is an atlas for $\mathbb{RP}^k$ with $k+1$ charts: For $i \in \{0, \dots, k\}$ let
$$\begin{aligned} \varphi_i : \mathbb{R}^k &\longrightarrow \mathbb{RP}^k \\ (x_1, \dots, x_k) &\longmapsto [x_1 : \dots : 1 : \dots : x_k] \end{aligned}$$
here the 1 occupies the ith slot of the homogenous coordinate. One obtains the complex projective space $\mathbb{CP}^n$ of dimension $2n$ when one replace $\mathbb{R}$ by $\mathbb{C}$. (Just to be clear: These are manifolds.)
- **Example:** Let $M = (\mathbb{R} \setminus \{0\}) \cup \{p, q\}$. We define a smooth 1-dim atlas containing exactly the two charts
$$\begin{aligned} \varphi_p : \mathbb{R} &\longrightarrow M \\ t &\longmapsto \begin{cases} t & t \neq 0 \\ p & t = 0 \end{cases} \end{aligned} \qquad \begin{aligned} \varphi_q : \mathbb{R} &\longrightarrow M \\ t &\longmapsto \begin{cases} t & t \neq 0 \\ q & t = 0. \end{cases} \end{aligned}$$
This is a smooth atlas.
- **Remark:** Every set with a smooth atlas carries a natural topology, this is the smallest topology on M such that all charts $\varphi_i : U_i \longrightarrow M$ are homeomorphisms onto their image. The topology induced in the previous example is not Hausdorff, i.e. every open neighbourhood of p intersects every open neighbourhood of q.
- **Remark:** We will occasionally review notions from point set topology. Good references include [Jä-T],[Q],[Y].

3. Lecture on Oct., 25. – **Abstract manifolds, smooth functions**

- We give list of constructions of topological spaces.
- Let $Y \subset (X, \mathcal{O})$. The *subspace topology* on Y is the smallest topology so that the inclusion $Y \hookrightarrow X$ is continuous, i.e. $V \subset Y$ is open if and only if there is an open set $U \subset X$ such that $X \cap U = V$.

- Let X be a topological space and $\sim$ an equivalence relation on X. Then the *quotient topology* on $X/\sim$ is the largest topology so that the projection $\pi : X \longrightarrow X/\sim$ is continuous, i.e. $V \subset X/\sim$ is open if and only if $\pi^{-1}(V)$ is open.
- Let $(X_i)_{i\in I}$ be a family of topological spaces. The *product topology* on $\prod_i X_i$ is the smallest topology so that for all $j \in I$ the projection $\prod_i X_i \longrightarrow X_j$ is continuous. Warning/Example: The subset $(-1,1)^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ is not open.
- A topological space is *Hausdorff* if for all $x \neq y$ there are disjoint open sets U_x, U_y such that $x \in U_x, y \in U_y$.
- A topological space is *compact* if for every covering $(U_i)_{i\in I}$ of X by open sets (i.e. $X = \cup_i U_i$)there is a finite subset $\{i_1, \dots, i_k\} \subset I$ so that $X = U_{i_1} \cup \dots \cup U_{i_k}$.
- A topological space is *paracompact* if for every open covering $(U_i)_{i\in I}$ there is a locally finite refinement, i.e. there is a collection $(V_j)_{j\in J}$ of open sets such that
  - $\cup_j V_j = X$,
  - for all j there is $i(j)$ so that $V_j \subset U_{i(j)}$,
  - every x has a neighbourhood V_x so that $V_x \cap V_j$ is empty for all but finitely many j.
- A justification for this requirement is a theorem of Stone saying that metric spaces are paracompact. Compact spaces are paracompact.
- Assume a topological space is Hausdorff and admits an atlas. Then the following conditions imply paracompactness:
  - X is second countable, i.e. there is a countable collection $U_n, n \in \mathbb{N}$ so that every open set can be obtained as union of these set.
  - There is a compact exhaustion of X, i.e. there is a family of compact sets $(K_i)_{i\in\mathbb{N}}$ which are nested (i.e. $K_i \subset \mathring{K}_{i+1} \subset K_{i+1} \subset \mathring{K}_{i+2} \dots$) and $X = \cup X_i$.
- **Definition**: A *smooth manifold* of dimension n is a topological space M which is Hausdorff and paracompact such that there is an atlas $\mathcal{A} = \{(U_i, \varphi_i)_{i\in I}\}$ such that
  - $U_i \subset \mathbb{R}^n$ is open and $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset M$ is a homeomorphism onto its image,
  - $\cup_i \varphi_i(U_i) = M$, and
  - for all $i, j \in I$ with $\varphi_i(U_i) \cap \varphi_j(U_j) \neq \emptyset$ the transition maps

    $$\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(\varphi_i(U_i) \cap \varphi_j(U_j)) \longrightarrow \varphi_i^{-1}(\varphi_i(U_i) \cap \varphi_j(U_j))$$

    are smooth
- **Examples:** Submanifolds of $\mathbb{R}^k$, finite products of manifolds, $\mathbb{RP}^n, \mathbb{CP}^n$ and many more.
- The next goal is the construction of sufficiently many smooth functions on smooth manifolds with positive dimension.
- **Reminder:** The function

  $$\begin{aligned} \lambda : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} 0 & t \leq 0 \\ \exp(-t^{-1}) & t > 0 \end{cases} \end{aligned}$$

  is smooth. The same holds for $\psi_\varepsilon(x) = \frac{\lambda(x)}{\lambda(x)+\lambda(\varepsilon-x)}$ for $\varepsilon > 0$. This function is nowhere negative and $\equiv 1$ on $\{x > \varepsilon\}$ while it is $\equiv 0$ on $\{x < 0\}$. Finally, the function f_ε on $\mathbb{R}^n$ defined by

$$f_\varepsilon(x) = 1 - \psi_\varepsilon(\|x\| - \varepsilon)$$

is smooth, it vanishes outside of a 2ε-ball around the origin and is $\equiv 1$ on the ε-ball around the origin.

- Let $p \in M$, $\varphi : U \longrightarrow M$ a chart mapping the origin to p. Then for small enough $\varepsilon > 0$, the function

$$\begin{aligned} g : M &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} 0 & x \notin \varphi(U) \\ f_\varepsilon(\varphi^{-1}(x)) & x \in \varphi(U) \end{cases} \end{aligned}$$

is well defined and smooth. Using this construction it easy to show that the vector space of smooth functions on M has infinite dimension (provided that the dimension of M is positive (and M nonempty)).

4. Lecture on Oct., 27. – **Smooth functions on manifolds, embedding, partitions of unity**

- **Lemma:** Closed subsets of compact spaces are compact. (A set in a topological space is closed if its complement is open).
- **Lemma:** A compact subset of a Hausdorff space is closed.
- **Theorem:** Let X be a compact topological space and Y Hausdorff. A continuous, bijective map $f : X \longrightarrow Y$ is a homeomorphism.
- For the proof one uses the above Lemmas to show that f is open (i.e. maps open sets to open sets). In the situation at hand this is equivalent to showing that f is closed (i.e. closed sets are mapped to closed sets).
- **Theorem:** Let M^n be a compact manifold of dimension n. Then there is an embedding $F : M \longrightarrow \mathbb{R}^N$ onto a submanifold of $\mathbb{R}^n$.
- **Proof (main steps)**:
  1. For each $p \in M$ pick a local parametrization $\varphi_p : U_p \subset \mathbb{R}^n \longrightarrow M$ with $p \in \varphi_p(U_p)$ and a smooth function f_p which has support inside $\varphi_p(U_p)$ and $f_p \equiv 1$ on a neighbourhood V_p of p and $f_p < 1$ outside of V_p. Note that f_p and also the functions $f_p \cdot x_i$ extend (by 0) to smooth functions on M with support in U_p. The extensions are denoted by the same symbol.
  2. The sets V_p cover M. Since M is compact finitely many suffice. We denote them by $V_1, \dots, V_k$ and the associated functions are denoted by $f_1, \dots, f_p$.
  3. We show that

$$\begin{aligned} F : M &\longrightarrow (\mathbb{R} \times \mathbb{R}^n)^k \simeq \mathbb{R}^{k(n+1)} \\ q &\longmapsto ((f_1(q), (f_1 \cdot x_1)(q)), \dots, (f_k(q), (f_k \cdot x_n)(q))) \end{aligned}$$

  is the desired embedding. If $F(q) = F(q')$ then q, q' lie in the same set V_j. Since the coordinates (i.e. the functions $(f_j \cdot x_i), i = 1, \dots, n$ on this separate points of V_i we have $q = q'$. Local parametrizations of $F(M)$ are obtained from compositions of local parametrizations of M with F. We also use the above theorem to conclude that F is a homeomorphism onto its image and the subspace topology on $F(M) \subset \mathbb{R}^{k(n+1)}$. This concludes the proof.
- **Remark:** The corresponding theorem for non-compact manifolds is true. Before one can show that one should first show that $N = k(n+1)$ can be replaced by $2n + 1$ (this depends only on the dimension, not on some covering).

- **Definition:** Let $(U_i)_{i\in I}$ be an open covering of M. A partition of unity subordinate to the covering is a collection of smooth functions $(f_j)_{j\in J}$ such that
  1. for all x there is a neighbourhood V_x such that all but finitely many f_j vanish on V_x,
  2. for all $j \in J$ there is $i(j) \in I$ such that support$(f_j) \subset U_{i(j)}$, and
  3. $f_j \geq 0$ and $\sum_{j\in J} f_j = 1$.

  Because of the first condition one does not have to worry about convergence of the series in the third condition.
- The following statement is a corollary of the first proposition in the Lecture of Nov. 3, i.e. the existence of a partiatin of unitiy subordinate to a given open covering. The corresponding statement for a certain class of topological spaces and continuous (not smooth) functions is the Lemma of Urysohn.
- **Corollary:** Let A_0, A_1 be disjoint closed sets in a manifold. Then there is a smooth, nowhere negative function g such that $g \equiv 1$ on A_1 and $g \equiv 0$ on A_1.
- **Proof:** Pick a partition of unity $(f_j)_{j\in J}$ subordinate to the covering $U_0 = M \setminus A_0, U_1 = M \setminus A_1$. Then define

$$g = \sum_{\{j\in J | \text{supp}(f_j)\subset U_0\}} f_j.$$

5. Lecture on Nov., 3. – **Partition of unity (existence), tangent vectors**

- **Proposition:** Let $(U_i)_{i\in I}$ be an open covering of M. Then there exists a partition of unity subordinate to $(U_i)_i$.
- We will discuss three definitions for a tangent vector at $p \in M$ (M a smooth n-manifold).
- **Definition (geometric):** A smooth curve at p is a smooth map $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = p$. Two curves γ_0, γ_1 at p are equivalent ($\gamma_0 \sim \gamma_1$) if for a local parametrization $\varphi : U \subset \mathbb{R}^n \longrightarrow M$ around p we have

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \gamma_0)(t) = \left.\frac{d}{dt}\right|_{t=0} (\varphi \circ \gamma_1)(t).$$

- This is independent of φ and $\sim$ is an equivalence relation.
- The *(geometric) tangent space* at p is

$$T_pM := \{\text{smooth curves at } p\}/\sim .$$

  Elements of T_pM are tangent vectors.
- Let $C^\infty(M)$ be the ring (with pointwise addition and multiplication) of smooth real valued functions on M.
- **Definition (algebraic):** A *derivation at p* is a linear map

$$v : C^\infty(M) \longrightarrow \mathbb{R}$$

  which satisfies the Leibniz rule, i.e.

$$v(fg) = v(f)g(p) + f(p)v(g).$$

  The vector space T_pM of all derivations at p is the (algebraic) *tangent space* of M at p.
- **Remark:** Every derivation at p vanishes on constant functions. Moreover, if $f = g$ on a neighbourhood V of p then $v(f) = v(g)$. This is shown using a smooth function h with support in V which is $\equiv 1$ on a neighbourhood of p in V. (Then $0 = v(h(f-g)) = v(f) - v(g)$.) In particular, instead of $C^\infty(M)$ we

could have used $\mathcal{E}_p^\infty(M) = C^\infty(M)/\sim$ with $f \sim g$ if and only if f, g coincide on a neighbourhood of p. Elements of $\mathcal{E}_p^\infty(M)$ are *germs of functions at* p. From now on we will frequently consider smooth functions defined on neighbourhoods of p.

- **Lemma:** Let $U \subset \mathbb{R}^n$ be a ball around 0 and $f : U \longrightarrow \mathbb{R}$ smooth. Then there are smooth functions $f_i : U \longrightarrow \mathbb{R}$ such that
$$f(x) = f(0) + \sum_i x^i f_i(x)$$
and $f_i(0) = \frac{\partial f}{\partial x^i}(0)$. Here x^i is the i-th coordinate, not a power of something.
- Using this Lemma one shows that a derivation v at p is determined by $v(x^1), \ldots, v(x^n)$ where x^i are local coordinates from a local parametrization (φ, U) near p such that $x^i(p) = 0$ for all i. Then $\dim(T_pM) = n$. For $i = 1, \ldots, n$ the derivation v with $v(x^i) = \delta_{ij}$ is denoted by $\frac{\partial}{\partial x^i}$.
- **Definition:** A (physicists) tangent vector of M at p is a map
$$v : \mathcal{D}_p(M) = \{\text{local parametrizations around } p\} \longrightarrow \mathbb{R}^n$$
such that
$$v((\psi, V)) = D_p(\psi^{-1} \circ \varphi)v((\varphi, U)).$$
The vectorspace of such maps obviously dimension $\leq n$ and $= n$ because of the chain rule.
- To obtain an algebraic tangent vector from a geometric one:
$$\{\text{curves at } p\}/\sim \longrightarrow \{\text{derivation at } p\}$$
$$[\gamma] \longmapsto \left(f \longmapsto \frac{d}{dt}\bigg|_{t=0} (f \circ \gamma)(t) \right).$$
- To obtain a physicists tangent vector from an algebraic one:
$$\{\text{derivation at } p\} \longrightarrow \{\text{physicists tangent vectors}\}$$
$$v \longmapsto \big((\varphi, U) \longmapsto (v(x^i))_i\big)$$
where x^i are the coord. around p from φ.
- To obtain a geometric tangent vector from an physicists tangent vector:
$$\{\text{pysicists tangent vectors}\} \longrightarrow \{\text{curves at } p\}/\sim$$
$$v \longmapsto \left[t \longmapsto \varphi(\varphi^{-1}(p) + tv((\varphi, U)))\right].$$
- **Fact:** All these maps are well defined, bijective and the second map is a linear isomorphism. Moreover, passing from geometric to algebraic, then to the physicist version, then back to the geometric, then a geometric tangent vector gets mapped to itself, etc.

6. Lecture on Nov., 8. – **Differential, Example: Lie groups, tangent bundle**

- As we have three definitions of tangent vectors, there are three definitions of the differential of a smooth map $F : M \longrightarrow N$ between smooth manifolds at $p \in M$.

- **Definition (geometric, curves):** The differential D_pF is
$$D_pF : T_pM = \{\text{smooth curves at } p\} \longrightarrow T_pN$$
$$[\gamma] \longmapsto [F \circ \gamma].$$
- **Definition (algebraic, derivations):** The differential D_pF is
$$D_pF : T_pM = \{\text{derivations at } p\}/\sim \longrightarrow T_pN$$
$$v \longmapsto \big(g \longmapsto v(g \circ F) =: v(F^*g)\big).$$
- **Definition (physicists, transformation rule):** Let M be a manifold of dimension n and (V, φ) a local parametrization of M around $p \in M$. The differential D_pF is
$$D_pF : T_pM = \{v : \mathcal{D}_p(M) \longrightarrow \mathbb{R}^n + \text{ transformation rule}\} \longrightarrow T_pN$$
$$v \longmapsto \left((U, \varphi) \longmapsto D_{\varphi^{-1}(p)}\left(\psi^{-1} \circ F \circ \varphi\right)(v((U, \varphi)))\right).$$
- All these versions are well defined and compatible with the identifications of the various definitions of tangent spaces discussed last time.
- **Example:** $M = \mathbb{R}^n$ or an open set in $\mathbb{R}^n$. Then $T_pM \simeq \mathbb{R}^n$ canonically. The differential of a smooth map between open set of Euclidean space (viewed as manifolds) coincides with the usual definition where the differential is represented by the Jacobi matrix (with respect to the natural basis of Euclidean space.
- **Example:** Let $f : U \longrightarrow \mathbb{R}^m$ be a smooth map defined on an open set of $\mathbb{R}^n$ such that $0 \in \mathbb{R}^m$ is a regular value. Then for $p \in M = f^{-1}(0)$
$$T_pM \longrightarrow \ker\left(D_pf : \mathbb{R}^n = T_p\mathbb{R}^n \longrightarrow \mathbb{R}^m = T_{f(p)=0}\mathbb{R}^m\right)$$
$$[\gamma] \longmapsto \left.\frac{d}{dt}\right|_{t=0} \gamma(t)$$
is a natural isomorphism which we will use frequently to describe tangent spaces of submanifolds.
- **Lemma (chain rule):** Let $F : M \longrightarrow M'$ and $G : M' \longrightarrow M''$ be smooth maps between smooth manifolds. Then
$$D_p(G \circ F) = \left(D_{F(p)}G\right) \circ (D_pF).$$
- **Lengthy example about Lie groups, general case**: Let G be a Lie group and $g \in G$. Then
$$c_g : G \longrightarrow G$$
$$h \longmapsto ghg^{-1}$$
is smooth and we have the rule $c_g \circ c_{g'} = c_{gg'}$. In particular c_g is a diffeomorphisms with inverse $c_{g^{-1}}$. Moreover $c_g(e) = e$. Thus we can differentiate c_g at e and we get a linear map
$$\mathrm{Ad}_g := D_ec_g : T_eG \longrightarrow T_eG.$$
This is an isomorphism of vector spaces with inverse $Ad_{g^{-1}}$ and by the chain rule the map
$$Ad : G \longrightarrow \mathrm{Aut}(T_eG)$$
$$g \longmapsto Dc_g = \mathrm{Ad}_g$$

is a group homomorphism which is smooth. Note that $\mathrm{Aut}(T_eG)$ is an open subset of the vector space of all endomorphisms of T_eG. In particular $\mathrm{Aut}(T_eG)$ is a Lie group. The tangent space at E (the identity automorphism) is the space of all endomorphisms of T_eG. Moreover $Ad_e = \mathrm{id}_{T_eG}$. Thus we can differentiate Ad at e and we get a linear map

$$\begin{aligned} \mathrm{ad} : T_eG &\longrightarrow T_{\mathrm{id}}(\mathrm{Aut}(T_eG)) = \mathrm{End}(T_eG) \\ X &\longmapsto (Y \longmapsto (D_e\mathrm{Ad}(X))\,(Y) = \mathrm{ad}(X)(Y))\,. \end{aligned}$$

This map is called *adjoint representation of* T_eG.

- **Terminology:** The tangent space at e of a Lie group G is often denoted by $\mathfrak{g}$, it is what is called a Lie algebra. In particular, $\mathrm{End}(T_eG)$ is a Lie algebra. The map ad is a Lie algebra homomorphism (once the notion of a Lie algebra is clear).
- **Same example, but more specific with $G = \mathrm{O}(n)$ (or any Lie group which is a subgroup of $\mathrm{Gl}(n,\mathbb{R})$):** We discussed $\mathrm{O}(n)$ in the first lecture, we showed that $\mathrm{O}(n)$ is a smooth submanifold of $\mathrm{Mat}(n\times n,\mathbb{R})$ as preimage of the regular value E of the map

$$\begin{aligned} F : \mathrm{Mat}(n\times n,\mathbb{R}) &\longrightarrow \mathrm{Sym}(n,\mathbb{R}) = \{\text{symmetric matrices}\} \\ A &\longmapsto AA^T. \end{aligned}$$

Then $T_E\mathrm{O}(n) = \ker(D_EF) = \{B \in \mathrm{Mat}(n\times n,\mathbb{R}) \,|\, B + B^T = 0\}$. (Recall $D_AF(B) = AB^T + BA^T$.) The map Ad is

$$\begin{aligned} \mathrm{Ad} : \mathrm{O}(n) &\longrightarrow \mathrm{Aut}(T_E\mathrm{O}(n)) \\ A &\longmapsto (X \longmapsto AXA^{-1}). \end{aligned}$$

Its differential at E is

$$\begin{aligned} \mathrm{ad} : T_E\mathrm{O}(n) =: \mathfrak{o}(n) &\longrightarrow \mathrm{End}(\mathfrak{o}(n)) \\ B &\longmapsto (X \longmapsto \mathrm{ad}(B)(X) = BX - XB)\,. \end{aligned} \tag{1}$$

In order to see this, recall that if $\gamma(t) = E + tB + t^2C(t)$ with $C(t)$ bounded as $t \to 0$, then

$$\left.\frac{d}{dt}\right|_{t=0} (\gamma(t))^{-1} = -B.$$

Hence if γ represents the tangent vector B in E of $\mathrm{O}(n)$, then

$$\mathrm{ad} = D\mathrm{Ad}(B)(X) = \left.\frac{d}{dt}\right|_{t=0} \left(t \longmapsto \gamma(t)X(\gamma(t))^{-1}\right).$$

Using the product rule we obtain (1). Except for the description of $\mathfrak{o}(n)$ as antisymmetric matrices the discussion above is valid for all Lie groups which are submanifolds of $\mathrm{Gl}(n,\mathbb{R})$ and subgroups at the same time.
- **Definition:** Let M be a smooth manifold of dimension n. The set

$$TM := \bigcup_{p\in M} T_pM$$

is the tangent bundle of M. There is an obvious map from TM to M taking a tangent vector in T_pM to $p \in M$. We denote this map by pr.
- Our goal now is to give TM the structure of a manifold. We would like pr to be a smooth map afterwards.

- Let $(U_i, \varphi_i)_{i\in I}$ be an atlas for M consisting of local parametrizations $\varphi_i : U_i \longrightarrow M$ of M. Then
$$\bigcup_{i\in I} \mathrm{pr}^{-1}(\varphi_i(U_i)) = TM.$$
We define a local parametrization of $\mathrm{pr}^{-1}(\varphi_i(U_i))$ by
$$\begin{aligned} \widehat{\varphi}_i : U_i \times \mathbb{R}^n &\longrightarrow \mathrm{pr}^{-1}(\varphi_i(U_i)) \subset TM \\ (x, w) &\longmapsto ((U_i, \varphi_i) \longmapsto w) \in T_{\varphi_i(x)}M. \end{aligned}$$
This is a bijective map. We used the physicist definitions of tangent vectors because using the transformation behavior inherent in this definition we will easily find the coordinate transformation for the atlas $((U_i \times \mathbb{R}^n), \widehat{\varphi}_i)_{i\in I}$. First note, that
$$\widehat{\varphi}_i^{-1}\left(\mathrm{pr}^{-1}(\varphi_i(U_i)) \cap \mathrm{pr}^{-1}(\varphi_j(U_j))\right) = \varphi_i^{-1}\left(\varphi_i(U_i) \cap \varphi_j(U_j)\right) \times \mathbb{R}^n.$$
The transition function $\widehat{\varphi}_j^{-1} \circ \widehat{\varphi}_i$ is then
$$\begin{aligned} \varphi_i^{-1}\left(\varphi_i(U_i) \cap \varphi_j(U_j)\right) \times \mathbb{R}^n &\longrightarrow \varphi_j^{-1}\left(\varphi_i(U_i) \cap \varphi_j(U_j)\right) \times \mathbb{R}^n \\ (x, w) &\longmapsto \left(\varphi_j^{-1} \circ \varphi_i(x), D_x(\varphi_j^{-1} \circ \varphi_i)(w)\right). \end{aligned} \tag{2}$$
We have expressed the new transition function in terms of the transition function of the atlas we started with. Hence $\widehat{\varphi}_j^{-1} \circ \widehat{\varphi}_i$ is smooth.

7. Lecture on Nov., 10. – **Tangent bundles, vector fields, commutators of vector fields**

- On TM we consider the smallest (sometimes people also say coarsest) topology such that $\varphi_i : U_i \times \mathbb{R}^n \longrightarrow TM$ is a homeoemorphism onto its image.
- **Lemma:** Let $\mathcal{A}, \mathcal{A}'$ be two equivalent atlases for X. Then the topologies on X induced by $\mathcal{A}, \mathcal{A}'$ coincide.
- **Proposition:** Let $\mathcal{A}$ be the smooth atlas for TM coming from a smooth atlas for M (cf. last item of the lecture on Nov., 8th). The topology on TM which is induced from $\mathcal{A}$ is Hausdorff and second countable, TM is a smooth manifold, the map $\mathrm{pr} : TM \longrightarrow M$ which maps tangent vectors in T_pM to p is smooth, surjective.

  Note that the transition function in (2) is of a particular form: The second component of $\widehat{\varphi}_j^{-1} \circ \widehat{\varphi}_i$ is a linear isomorphism (which depends on x). Thus the structure of T_pM as a vectorspace is preserved by the transition functions of our atlas.
- **Definition:** A *vector field* X on M is a smooth map $X : M \longrightarrow TM$ such that $\mathrm{pr} \circ X = \mathrm{id}_M$. The set $\mathcal{X}(M)$ of vector fields is a vector space over $\mathbb{R}$ (pointwise addition and scalar multiplication). Often one writes $\Gamma(TM)$ instead of $\mathcal{X}(M)$.
- **Lemma:** Given two vector fields X, Y on M there is a vector field $[X, Y]$ such that
$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{3}$$
for all smooth functions. Moreover, for all $X, Y, Z \in \mathcal{X}(M)$
$$\begin{aligned} [X, Y] &= -[Y, X] \qquad \text{(antisymmetry)} \\ 0 &= [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \qquad \text{(Jacobi identity)} \end{aligned}$$

and $[\cdot,\cdot]$ is bilinear over $\mathbb{R}$ (not over the ring of smooth funtions). If f is a smooth function and $X, Y \in \mathcal{X}(M)$, then

$$[X, fY] = f[X,Y] + (X(f))Y.$$

- **Definition:** A (real) *Lie algebra* is a vector space V (over $\mathbb{R}$) together with a bilinear pairing
$$[\cdot,\cdot] : V \times V \longrightarrow V$$
which is antisymmetric and satisfies the Jacobi identity.
- **Example:** The space of smooth vector fields is a real Lie algebra.

8. Lecture on Nov., 15. – **Vector fields, flows, Lie derivative, commutators**

- **Summary:** Let $I \subset \mathbb{R}$ be an interval and $X_t, t \in I$, be a smooth family of vector fields on M. For each $p_0 \in M$ and $t_0 \in I$ there is a interval $I_{max} \subset I$ containing t_0 and a smooth map $\gamma(p_0, t_0) : I_{max} \longrightarrow M$ such that

(4)
$$\begin{aligned} \gamma(p_0,t_0)(t_0) &= p_0 \\ \dot{\gamma}(p_0,t_0)(t) &= X_t(\gamma(p_0,t_0)(t)). \end{aligned}$$

and every smooth map $\alpha : J \longrightarrow M$ with the same properties has $J \subset I_{max}$ and coincides with γ on J.

For every compact interval $t_0 \in J \subset I$ there is a neighbourhood V of (p_0, t_0) in $M \times I$ such that $\gamma(q,s)$ is defined on the interval J for $(q,s) \in V$ and $\gamma(q,s)(t)$ depends smoothly on all variables $(q,s,t) \in V \times J$.

When M itself or the support

$$\overline{\{p \in M \,|\, X_t(p) \neq 0 \text{ for some } t \in I\}} \subset M$$

is compact then

(5)
$$I_{max} = I \text{ for all initial values } (p_0,t_0) \in M \times I.$$

We will assume this throughout the discussion to simplify the exposition. Property (5) is often referred to as completeness. For $t, t_0 \in I$ the maps

$$\begin{aligned} \psi_t : M &\longrightarrow M \\ q &\longmapsto \gamma(q,t_0)(t) \end{aligned}$$

is a diffeomorphisms. The inverse is

$$\begin{aligned} \psi_t^{-1} : M &\longrightarrow M \\ q &\longmapsto \gamma(q,t)(t_0). \end{aligned}$$

We obtain a (continuous with respect to every reasonable topology on Diff(M)) map

(6)
$$\begin{aligned} I &\longrightarrow \mathrm{Diff}(M) = \{\psi : M \longrightarrow M \,|\, \psi \text{ a diffeomorphism }\} \\ t &\longmapsto \psi_t. \end{aligned}$$

If X_t does not depend on time t, then we set $I = \mathbb{R}$ and $X_t = X$. All the above works (assuming completeness), $\gamma(p,t_0)(t)$ depends only on the difference $t - t_0$. We usually set $t_0 = 0$ and write $\gamma(p)(t)$ for solutions of (4). By uniqueness of the solutions of (4)

$$\gamma(\gamma(p)(s))(t) = \gamma(p)(s+t)$$

and (6) is a homomorphism of groups which is called the flow of X.

- **Definition:** Let ψ be a diffeomorphism of M and $Y \in \mathcal{X}(M)$. Then

  $(\psi^*Y)(p) := (D\psi^{-1})_{\psi(p)}(X(\psi(p)))$ is the pull back of Y with ψ

  $(\psi_*Y)(p) := (D\psi)_{\psi^{-1}(p)}(X(\psi^{-1}(p)))$ is the push forward of Y with ψ.

- **Definition:** Let $X, Y \in \mathcal{X}(M)$. The *Lie-derivative* of Y in direction X is

(7)
$$L_X Y := \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* Y$$

  where φ_t is the flow of X.
- $(\varphi_t^*Y)(p)$ is a smooth family of vectors in T_pM.
- **Lemma:** Let X, Y be smooth vector fields on M and $F \in \mathrm{Diff}(M)$. We write φ_t respectively ψ_s for the flow of X respectively Y. Then
  a) $F^*Y = \frac{d}{ds}\big|_{s=0} F^{-1} \circ \psi_s \circ F$.
  b) $F^*Y = Y \Leftrightarrow F \circ \psi_s = \psi_s \circ F$ for all $s \in \mathbb{R}$.
  c) $L_X Y = 0 \Leftrightarrow \varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$.
- The proof of a) is a direct computation with the chain rule. So is the part $\Leftarrow$ of b), the part $\Rightarrow$ of b) uses the uniqueness of solutions of initial value problems ($\psi_s(p_0)$ and $F^{-1} \circ \psi_s \circ F(p_0)$ solve the same initial value problem). Finally, c) follows from b) when one shows that $L_X Y = 0$ if and only if $\varphi_t^* Y = Y$ using (7).
- **Lemma:** Let X, Y be smooth vector fields on M and let φ_t respectively ψ_s be the flow of X respectively Y. Then
  a) $L_X Y = \frac{\partial^2}{\partial s \partial t}\Big|_{(0,0)} \varphi_{-t} \circ \psi_s \circ \varphi_t$.
  b) $(L_X Y)(f) = X(Y(f)) - Y(X(f))$ for all smooth functions f on M.
- The proof of a) relies on the previous Lemma. For b) use the item after the definition of $L_X Y$ to show that

$$L_X Y(f) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t.$$

  Then apply the chain rule to the composition of $t \longmapsto (-t, t)$ with $(\tau, \tau') \longmapsto f \circ \varphi_\tau \circ \psi_s \circ \varphi_{\tau'}$.
- **Remark:** In terms of local coordinates one can write

$$X(x^1, \ldots, x^n) = \sum_{\nu=1}^{n} a^\nu (x^1, \ldots, x^n) \frac{\partial}{\partial x^\nu} = a^\nu \partial_\nu$$
$$Y(x^1, \ldots, x^n) = \sum_{\mu=1}^{n} b^\mu (x^1, \ldots, x^n) \frac{\partial}{\partial x^\mu} = b^\mu \partial_\mu$$

  (the rightmost expressions illustrate notation used in physics literature, in particular the summation convention where a sum sign is understood for indices which appear in upper and a lower index, the same applies to the second line in the next equation) and compute $[X, Y]$ from the definition (3) as follows

$$[X, Y](x^1, \ldots, x^n) = \sum_{\nu,\mu} \left(a^\nu \frac{\partial b^\mu}{\partial x^\nu} - b^\nu \frac{\partial a^\mu}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}$$
$$[X, Y]^\mu = a^\nu \partial_\nu b^\mu - b^\nu \partial_\nu a^\mu.$$

9. Lecture on Nov., 17. – **Left invariant vector fields, Lie algebras of Lie groups**

- **Lemma:** Let X, Y be smooth vector fields on M and $F : M \longrightarrow M$ a diffeomorphism. Then

$$DF([X,Y]) = [DF(X), DF(Y)]. \tag{8}$$

- This is a computation using the chain rule, φ_t is the flow of X:

$$\begin{aligned} DF([X,Y]) = DF(L_X Y) &= DF(\frac{d}{dt}\Big|_{t=0} \varphi_t^* Y) \\ &= \frac{d}{dt}\Big|_{t=0} D(\underbrace{F \circ \varphi_t \circ F^{-1}}_{\text{flow of } F_* X})^{-1} \underbrace{DF(Y)}_{=F_* Y} \\ &= [DF(X), DF(Y)]. \end{aligned}$$

- **Lemma:** The following map is a diffeomorphism whose restriction to $\{g\} \times T_e G$ is a linear isomorphism onto $T_g G$

$$\begin{aligned} \psi : G \times T_e G &\longrightarrow TG \\ (g, v) &\longmapsto (Dl_g)(v) \end{aligned}$$

  where $l_g : G \longrightarrow G$ denotes left multiplication with g, i.e. $l_g(h) = gh$. Moreover $\mathrm{pr} \circ \psi$ is the projection onto the first factor of $G \times T_e G$.
- Both the target and the domain of ψ are manifolds of dimension $2\dim(G)$. To see that $D\psi$ is surjective (hence an isomorphism) use the decomposition $T_{(g,v)}(G \times T_e G) = T_g G \times T_v(T_e G) = T_g G \times T_e G$.
- **Definition:** A vector field X on G is *left invariant* if $l_{g*} X = X$. (This is equivalent to $l_g^* X = X$.)
- A left invariant vector field is determined by its value at one point (for example e). Thus there is an isomorphism

$$\begin{aligned} T_e G &\longrightarrow \mathfrak{g} := \{\text{left-invariant vector fields on } G\} \\ X &\longmapsto (\underline{X} : h \longmapsto (Dl_h)(X)) \,. \end{aligned}$$

- **Theorem:** Left-invariant vector fields are complete, i.e. solutions of initial value problems are defined on $\mathbb{R}$.
- For the proof, start with a solution γ defined on $(-\varepsilon, \varepsilon)$ with $\gamma(h)(0) = h$. In order to extend this show that $l_{\gamma(e)(\varepsilon/2)} \circ \gamma$ solves the initial value problem $\alpha(h \cdot \gamma(e)(\varepsilon/2))(0) = h \cdot \gamma(e)(\varepsilon/2))$ and $\dot{\alpha} = \underline{X} \circ \alpha$.

  Combine these two curves to obtain a solution for the original initial value problem (starting at h) with domain $(-\varepsilon, 3\varepsilon/2)$, i.e. the size of the domain has increased by the amount $\varepsilon/2$. Doing this infinitely many times one obtains a solution with domain $\mathbb{R}$.
- **Lemma:** By (8) $\mathfrak{g}$ is closed under Lie brackets $[\cdot,\cdot]$, i.e. it is a Lie algebra. We say that $\mathfrak{g}$ is the Lie algebra of the Lie group G.
- **Definition:** The exponential map of a Lie group G is

$$\begin{aligned} \exp : T_e G = \mathfrak{g} &\longrightarrow G \\ X &\longmapsto \gamma_{\underline{X}}(e)(1) \end{aligned}$$

  where $\gamma_{\underline{X}}(e)(t)$ solves the initial value problem $\dot{\gamma} = \underline{X} \circ \gamma$ with initial value e and $\underline{X}$ is the left invariant vector field with $\underline{X}(e) = X$.

- **Example:** Let $G \subset \mathrm{Gl}(n, \mathbb{R})$ be a Lie subgroup (i.e. a submanifold and a subgroup) and $X \in \mathfrak{g}$. Then
$$\begin{aligned} \underline{X} : G &\longrightarrow TG \\ H &\longmapsto (Dl_H)(X) = H \cdot X. \end{aligned}$$
The solution of the initial value problem $\gamma(E)(t) = E$ and $\dot{\gamma} = \underline{X} \circ \gamma$ is
$$\gamma(E)(t) = \exp(tX) = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!}.$$
Note that $\exp(A)\exp(B) = \exp(A + B)$ only if $AB = BA$ but not in general. Because of left-invariance
$$\gamma(H)(t) = H \exp(tX).$$
Finally, using the Lemmas from the last lecture one obtains:
$$\underbrace{[\underline{X}, \underline{Y}]}_{\text{vector field}} (E) = XY - YX.$$
- Simple examples one can discuss include $G = \mathrm{SO}(3), S^1 = \mathrm{U}(1) \subset \mathbb{C}, \ldots$ with $\mathfrak{g} = \mathfrak{so}(3), \mathfrak{u}(1) \simeq i\mathbb{R}, \ldots$.

10. Lecture on Nov. 22. – **Multilinear Forms**

- **Definition:** If V is a vector space over some field $\mathbb{K}$ then a *linear form* is a linear function $\phi : V \to \mathbb{K}$. The set of linear forms on V form a vector space denoted by V^* with $\dim(V^*) = \dim(V)$.
- If $\{e_i\}$, $i = 1, \cdots n = \dim(V)$ is a basis of V then $\{e^{*i}\}$, $i = 1, \cdots n$ is the *dual basis* if $e^{*i}(e_j) = \delta^i_j$.
- A k-linear form (or simply k-form) is an alternating k-linear function
$$\begin{aligned} \phi \quad : \quad & \underbrace{V \times V \times \ldots \times V}_{k \text{ times}} \to \mathbb{K} \\ & (x_1, \cdots, x_k) \mapsto \phi(x_1, \cdots, x_k) = \mathrm{sgn}(\sigma)\phi(x_{\sigma(1)}, \cdots, x_{\sigma(k)}) \end{aligned}$$
for $\sigma \in \Sigma_k$, the symmetric group. In particular, for $\phi^i \in V^*$, $i = 1, \cdots k \leq n$, $\phi^1 \wedge \phi^2 \cdots \wedge \phi^k \in \wedge^k V^*$ defined through
$$\phi^1 \wedge \phi^2 \cdots \wedge \phi^k(x_1, \cdots, x_k) = \det(\phi^i(x_j))$$
$i, j = 1, \cdots k$ is a k-form.
- The above construction gives rises to a basis in $\wedge^k V^*$: **Proposition:** any k-linear exterior form $\omega \in \wedge^k V^*$ can be expanded as
$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \omega_{i_1, \cdots, i_k} e^{*i_1} \wedge e^{*i_2} \wedge \cdots \wedge e^{*i_k}$$
where $\omega_{i_1, \cdots, i_k} \in \mathbb{K}$.
- **Definition(exterior product)** For $\omega \in \wedge^k V^*$ and $\phi \in \wedge^p V^*$ the exterior product is defined through
$$\omega \wedge \phi = \sum_{\substack{1 \leq j_1 < j_2 < \cdots < i_p \leq n \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n}} \omega_{i_1, \cdots, i_k} \phi_{j_1, \cdots, j_k} e^{*i_1} \wedge e^{*i_2} \wedge \cdots \wedge e^{*i_k} \wedge e^{*i_1} \wedge e^{*j_2} \wedge \cdots \wedge e^{*j_p}$$

- **Proposition** For $\omega \in \wedge^k V^*$, $\psi \in \wedge^k V^*$, $\phi \in \wedge^p V^*$ and $\rho \wedge^q V^*$ we have
  a) $\phi \wedge (\omega + \psi) = \phi \wedge \omega + \phi \wedge \psi$
  b) $\phi \wedge (\omega \wedge \rho) = (\phi \wedge \omega) \wedge \rho$
  c) $\phi \wedge \omega = (-1)^{pk} \omega \wedge \phi$

11. Lecture on Nov. 24. – **Differential Forms on** $\mathbb{R}^n$[Ca]

- For $V = \mathbb{R}^n$ and $\{x^1, \cdots, x^n\}$ coordinates on $\mathbb{R}^n$ (or a subset thereof) we identify the elements of the dual basis $\{e^{*i}\}$ with the coordinate differentials $\{\mathrm{d}x^i\}$.
- **Definition** A *field of exterior forms* or *an exterior form of degree* k, $k \leq n$ is a map ω that associates to each point $p \in \mathbb{R}^n$ an element $\omega(p) \in \wedge^k V^*$. Furthermore, $\omega(p)$ can be expanded as

$$\omega(p) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1, \cdots, i_k}(p)\, \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

If, the real functions $a_{i_1, \cdots, i_k}(p)$ are differentiable, then ω is called a *differential* k*-form*. The set of differential k-forms forms a vector space, denoted by $\Omega^k(\mathbb{R}^n)$.
- **Definition** For $f \in C^1(\mathbb{R}^n)$ we denote by $\mathrm{d}f$ its differential. Then the map

$$\begin{aligned} \mathrm{d} \quad : \quad & \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n) \\ & \omega(p) \mapsto \mathrm{d}\omega(p) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathrm{d}a_{i_1, \cdots, i_k}(p) \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k} \end{aligned}$$

is well defined for $\{a_{i_1, \cdots, i_k}(p)\} \in C^1(\mathbb{R}^n)$. $\mathrm{d}\omega(p)$ is the *exterior derivative* of the differential form $\omega(p)$.
- **Proposition** For $\omega(p), \psi(p) \in \Omega^k(\mathbb{R}^n)$and $\phi \in \Omega^k(\mathbb{R}^n)$ we have
  a) $\mathrm{d}(\omega(p) + \psi(p)) = \mathrm{d}\omega(p) + \mathrm{d}\psi(p)$
  b) $\mathrm{d}(\omega(p) \wedge \phi(p)) = \mathrm{d}\omega(p) \wedge \phi(p) + (-1)^k \omega(p) \wedge \mathrm{d}\phi(p)$
  c) $\mathrm{dd}\omega(p) = 0$ assuming ω is twice differentiable
- **Definition** For $\omega(p), \psi(p) \in \Omega^k(\mathbb{R}^n)$ and $Z \in \mathfrak{X}(\mathbb{R}^n)$ we define the *interior derivative* or *interior product* as the map

$$\begin{aligned} \mathrm{i}_Z \quad : \quad & \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n) \\ & \omega(p) \mapsto ((\mathrm{i}_Z \omega)(p) : (x_1, \cdots, x_{k-1}) \mapsto \omega(Z, x_1, \cdots, x_{k-1})) \end{aligned}$$

is well defined for $(x_1, \cdots, x_{k-1}) \in \underbrace{V \times V \times \cdots \times V}_{k-1 \text{ times}}$ and $\{a_{i_1, \cdots, i_k}(p)\} \in C^1(\mathbb{R}^n)$. $\mathrm{d}\omega(p)$ is the *exterior derivative* of the differential form $\omega(p)$.
- **Proposition** For $\omega(p) \in \Omega^k(\mathbb{R}^n)$, $\phi \in \Omega^p(\mathbb{R}^n)$ and $Z \in \mathfrak{X}(\mathbb{R}^n)$ we have
  a) $\mathrm{i}_Z(\omega \wedge \phi) = (\mathrm{i}_Z \omega) \wedge \phi + (-1)^k \omega \wedge (\mathrm{i}_Z \phi)$
  b) $\mathrm{i}_Z(\mathrm{i}_Z \omega) = 0$

12. Lecture on Nov. 29. – **Pullback and Lie derivative** [Ca]

- **Definition** Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $f_* : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ be its differential, or *push forward*. Then the *pull back* of a differential form on $\mathbb{R}^m$ is given by the map

$$\begin{aligned} f^* \quad : \quad & \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n) \\ & \omega \mapsto ((f^*\omega)(p) : (x_1, \cdots, x_k) \mapsto \omega(f(p))(f_* x_1, \cdots, f_* x_k)) \end{aligned}$$

for $(x_1, \cdots, x_k) \in \underbrace{T_p\mathbb{R}^n \times T_p\mathbb{R}^n \times \cdots \times T_p\mathbb{R}^n}_{k \text{ times}}$.

- **Proposition** For $g \in C^\infty(\mathbb{R}^p, \mathbb{R}^n)$, $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, $h \in C^\infty(\mathbb{R}^n, \mathbb{R})$; $\omega, \psi \in \Omega^k(\mathbb{R}^m)$ and $\phi \in \Omega^k(\mathbb{R}^m)$ we have
  a) $f^*(\omega + \psi) = f^*\omega + f^*\psi$
  b) $f^*(h\omega) = f^*(h)g^*(\omega)$
  c) $f^*(\omega \wedge \phi) = (f^*\omega) \wedge (f^*\phi)$
  d) $(f \circ g)^*\omega = g^*(f^*\omega)$
  e) $\mathrm{d}f^*(\omega) = f^*(\mathrm{d}\omega)$
- **Remark** We choose coordinates $\{y^i\}$ on $\mathbb{R}^m$ and take

$$\omega(p) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1, \cdots, i_k}(p)\, \mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k}$$

Then we have, using b) and c)

$$\begin{aligned} (f^*\omega)(p) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (f^* a_{i_1, \cdots, i_k})(p)\, f^*\mathrm{d}y^{i_1} \wedge f^*\mathrm{d}y^{i_2} \wedge \cdots \wedge f^*\mathrm{d}y^{i_k} \\ &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1, \cdots, i_k}(f(p))\, \mathrm{d}f^{i_1} \wedge \mathrm{d}f^{i_2} \wedge \cdots \wedge \mathrm{d}f^{i_k} \end{aligned}$$

which gives a simple and intuitive expression for the pull back of a generic differential form.

- **Definition (Lie Derivative)** Let $Z \in \mathfrak{X}(\mathbb{R}^n)$ be a differentiable vector field, ϕ_t its flow and $\omega \in \Omega^k(\mathbb{R}^n)$, then the Lie derivative of ω is defined as

$$L_Z\omega = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^*\omega)\bigg|_{t=0}$$

In components we have

$$\begin{aligned} (L_Z\omega)(p) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (i_Z d a_{i_1, \cdots, i_k}) \wedge\, \mathrm{d}x^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k} \\ &\quad + \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} a_{i_1, \cdots, i_k}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}(i_Z \mathrm{d}x^{i_2}) \wedge \cdots \wedge \mathrm{d}x^{i_k} \end{aligned}$$

Useful formula: $L_Z\omega = (\mathrm{d}\, i_Z + i_Z\, \mathrm{d})\omega$.

13. Lecture on Dec. 1. – **Differential forms on smooth manifolds**

- **Definition** An exterior k-form ω on a smooth manifold M is a choice, for every point $p \in M$, of an element $\omega(p)$ in the vector space $\Lambda^k(T_pM)^*$ of alternating k-linear functions, $\omega(p) : \underbrace{T_pM \times T_pM \times \cdot \times T_pM}_{k \text{ times}} \to \mathbb{R}$.
- To continue we pull this form back to open sets U_α of the atlas of M. This naturally leads to
- **Definition** For a given parametrisation $f_\alpha : U_\alpha \to M$, the *representative* ω_α is given by

$$\omega_\alpha = f_\alpha^*\omega.$$

- **Remark** The independence of this definition on the choice of coordinates (parametrisations) follows from the observation that for $f_{\alpha/\beta} : U_{\alpha/\beta} \to M$ we have in $U_\alpha \cap U_\beta$

$$(f_\beta^{-1} \circ f_\alpha)^* \omega_\beta = f_\alpha^*((f_\beta^{-1})^* \omega_\beta) = f_\alpha^* \omega|_{f_\beta(U_\alpha \cap U_\beta)} = \omega_\alpha$$

- If ω_α is differentiable in one parametrisation (and therefore in all by the above remark) then ω is a differential form on M. The vector space of differential k-forms is denoted by $\Omega^k(M)$.
- **Remark** The operations (d, Λ, i_X, L_X) are naturally extended to $\Omega^k(M)$. In particular,

$$\mathrm{d}\omega_\alpha = \mathrm{d}f_\alpha^* \omega =: f_\alpha^* \mathrm{d}\omega|_{f_\alpha(U_\alpha)}$$

- If $\omega \in \Omega^n(M)$, M oriented, $n = \dim(M)$ has compact support, $K \subset M$, then this form can be integrated over M as follows: Suppose first that $K \subset f_\alpha(U_\alpha)$ for some $\alpha \in I$ and $\{x^i\}$, $i = 1, \cdots, n$, cartesian coordinates on $\mathbb{R}^n$, then we define

$$\int_M \omega = \int_{U_\alpha} \omega_\alpha = \int_{U_\alpha} a_\alpha \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \int_{U_\alpha} a_\alpha \mathrm{d}x^1 \mathrm{d}x^2 \cdots \mathrm{d}x^n$$

where the last expression is the Lebesgue integral defined for continuous functions on U_α. The last step in the above definition proceeds through evaluation of ω_α on an infinitesimal hypercube in $\mathbb{R}^n$ spanned by the vectors $\mathrm{dx^i}\partial_{x^i}$ where the $\mathrm{dx^i}$ are the coordinate differentials.
- **Remark** Under a change of coordinates, $f = (f_\alpha^{-1} \circ f_\beta) : U_\beta \to U_\alpha$ such that f_* has positive determinant with $\{x^i = f^i(y)\}$ coordinates on U_α and $\{y^i\}$ coordinates on U_β we have

$$\begin{aligned}
\int_{U_\beta} a_\beta \mathrm{d}y^1 \mathrm{d}y^2 \cdots \mathrm{d}y^n &= \int_{U_\beta} \omega_\beta = \int_{U_\beta} f^* \omega_\alpha \\
&= \int_{U_\beta} a_\alpha(f(y)) f^* \mathrm{d}x^1 \wedge \cdots f^* \mathrm{d}x^n \\
&= \int_{U_\alpha} a_\alpha(f) \mathrm{d}f^1 \wedge \cdots \mathrm{d}f^n \\
&= \int_{U_\alpha} a_\alpha(x) \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^n \\
&= \int_{U_\alpha} a_\alpha(x) \mathrm{d}x^1 \cdots \mathrm{d}x^n.
\end{aligned}$$

- If the support K is not contained in any coordinate nbhd $f_\alpha(U_\alpha)$ we construct a *partition of unity* $\{\phi_i\}$ *subordinate to* the covering $\{U_\alpha\}$. That is (see part I)
  - a) $\sum_{i=1}^{m} \phi_i = 1$
  - b) $0 \leq \phi_i \leq 1$ and $\text{sup}\phi_i \in U_\alpha$ for some $\alpha \in I$

  and define $\int_M \omega := \sum_{i=1}^{m} \int_M \phi_i \omega$.
- **Remark** The convergence of the above sum is guaranteed by the assumption of paracompactness (see part I)

14. Lecture on Dec. 6. – **Manifolds with boundary**

- In order to parametrise manifolds with boundary we consider maps from open sets in $H^n = \{x^1, \cdots, x^n \in \mathbb{R}^n | x_1 \leq 0$.
- **Definition** An open set, V, in H^n is the intersection of an open set $U \subset \mathbb{R}^n$ with H^n. A function $f : V \to \mathbb{R}$ is differentiable if there exists an open set $U \subset \mathbb{R}^n$ such that $V \subset U$ together with a differentiable function $\bar{f} : U \to \mathbb{R}$ such that $\bar{f}|_V = f|_V$.
- A smooth manifold with boundary is then defined in complete analogy with a smooth manifold without boundary (see 3rd lecture) by replacing $\mathbb{R}^n$ by H^n everywhere.
- A point $P \in M$ is on the boundary ∂M if for some parametrisation $f : V \subset H^n \to M$ we have $f(0, x^2, \cdots, x^n) = P$.
- **Lemma** This definition of a point on ∂M is independent of the choice of parametrisation.
- **Proposition** The boundary ∂M of an n-dimensional smooth manifold with boundary is an $(n-1)$-dimensional smooth manifold. Furthermore,the orientation on M induces an orientation on ∂M.
- Let ω be an $(n-1)$-form on a smooth manifold M of dimension n with boundary. Then $\mathrm{d}\omega$ can be integrated on $M/$
- **Theorem (Stokes)** Let M be a smooth, compact, oriented manifold of dim n with boundary and $i : \partial M \to M$ be the inclusion map of the boundary into M. Then for $\omega \in \Omega^{(n-1)}$ we have

$$\int\limits_{\partial M} i^*\omega = \int\limits_{M} \mathrm{d}\omega.$$

15. Lecture on Dec. 8. – **Poincare Lemma**

- $\omega \in \Omega^k(M)$ is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\gamma$ globally for some $\gamma \in \Omega^{k-1}$. Since $\mathrm{d}^2 = 0$ every exact form is closed.The converse is not true but we want to show that every closed form is nevertheless exact in the nbhd of some point.
- **Defiition** A smooth manifold M is *contractible* to some point $p_0 \in M$ if there exists a differentiable map

$$\begin{aligned} H \quad &: \quad M \times \mathbb{R} \to M \\ &(p, t) \mapsto H(p, t) \in M \end{aligned}$$

  such that $H(p, 1) = p$ and $H(p, 0) = p_0 \; \forall p \in M$.
- To every $\omega \in \Omega^k(M)$ we can the associate a k-form $\bar{\omega} \in \Omega^k(M \times \mathbb{R})$ as

$$\bar{\omega} = H^*\omega$$

  On the other hand, any $\bar{\omega} \in \Omega^k(M \times \mathbb{R})$ has a unique decomposition of the form

$$\bar{\omega} = \omega_1 + \mathrm{d}t \wedge \eta$$

  with $i_{\partial_t}\omega_1 = 0$ and $i_{\partial_t}\eta_1 = 0$
- Conversely we can associate a k-form $\omega \in \Omega^k(M)$ to each $\bar{\omega} \in \Omega^k(M \times \mathbb{R})$ with the help of the inclusion map

$$\begin{aligned} i_t \quad &: \quad M \to M \times \mathbb{R} \\ &i_t(p) = (p, t) \in M \times \mathbb{R} \end{aligned}$$

  Then $i_t^*\bar{\omega} \in \Omega^k(M)$ provided $\bar{\omega} \in \Omega^k(M \times \mathbb{R})$.

- Let us furthermore define the map

$$I \; : \; \Omega^k(M\times\mathbb{R}) \to \Omega^{k-1}(M)$$
$$(I\eta)(z_1,\cdots,z_{k-1}) = \int_0^1 \eta(p,t)(\partial_t, i_{t*}z_1,\cdots,i_{t*}z_{k-1})dt.$$

16. Lecture on Dec. 13. – **Poincare lemma, deRham cohomology and Riemannian manifolds**

The key result which then establishes local exactness is the

- **Lemma**

$$i_1^*\bar\omega - i_0^*\bar\omega = d(I\bar\omega) + I(d\bar\omega)$$

Indeed, since $H\circ i_1 = \text{id}$ and $H\circ i_1 = p_0, \forall p\in M$ we have

$$\omega = (H\circ i_1)^*\omega = i_1^*\bar\omega$$

and

$$0 = (H\circ i_0)^*\omega = i_0^*\bar\omega$$

From this the desired result he follows:

- **Theorem** Let M be a contractible, smooth manifold and $\omega\in\Omega^k(M)$ with $\mathrm{d}\omega = 0$. Then there exists a $k-1$ form $\alpha\in\Omega^{k-1}(M)$ such that $\omega = \mathrm{d}\alpha$.
- $\Omega^k(M)$ is a vector space over $\mathbb{R}$ whose elements form a group with respect to addition. It turns out, however that there are invariant sub group which we will now review.
- **Definition** Let M be a smooth manifold of dimension n. The the set of
  a) closed k-form is the k-th *cocycle group*, with real coefficients $Z^k(M,\mathbb{R})$
  b) exact k-form is the k-th *coboundary group*, with real coefficients $B^k(M,\mathbb{R})$
  c) $H^k(M,\mathbb{R}) = Z^k(M,\mathbb{R})/B^k(M,\mathbb{R})$ is the k-th *deRham cohomology group* with real coefficients.
- Let us now assume that M is a smooth manifold endowed with a (pseudo) Riemannian metric

$$g \; : \; \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{F}(M)$$
$$(x,y)(p) \mapsto g_p(x,y)$$

for $p\in M$. In particular, in an open set V containing p with coordinates $\{x^i\}$ the vectors $\{\partial_{x^i}\}$ form a basis of $\mathfrak{X}(M)$. Then $g_{ij}(p) := g(\partial_{x^i},\partial_{x^j})$ is a set of smooth functions on $V\subset M$.

- If g is non-degenerate, then $g^{ij} = (g^{-1})_{ij}$ gives rise to an isomorphism between T_pM and T_p^*M through

$$T_p^*M \ni \mathrm{d}x^i = g^{ij}g(\partial_{x^j},\cdot)$$

- **Definition** $\{\partial_{x^j}\}$ is called the *coordinate basis* of T_pM. The *orthonormal* or *non-coordinate* basis $\{e_a\}$ of of T_pM is defined by the condition $g(e_a,e_b) = \delta_{ab}$. Note that for a given Riemannian metric g the set $\{e_a\}$ is unique only up $e'_a = A_a^b e_b$ where A is an orthogonal transformation.

17. Lecture on Dec. 15. – **Volume form, Hodge $*$ operation**

- After picking an orientation the *volume form* vol on M is a differential form of maximal degree s.t $\mathrm{vol}(e_1, \cdots, e_n) = 1$. This, in turn, gives rise to a multilinear map $* : \Omega^m(M) \to \Omega^{n-m}(M)$, the *Hodge star operation* point wise defined as

$$\begin{aligned} * \quad &: \Omega^m(M) \to \Omega^{n-m}(M) \\ &e^{a_1} \wedge \cdots \wedge e^{a_m} \mapsto \delta^{a_1 b_1} \cdots \delta^{a_m b_m} i_{e_{b_m}} \cdots i_{e_{b_1}} \mathrm{vol} \end{aligned}$$

where δ^{ab} is replaced by η^{ab} for a pseudo Riemannian manifold. e^i denotes the dual basis of the orthonormal basis $e_1, \ldots$. In particular, $*1 = \mathrm{vol}$. In terms of the coordinate basis the volume form takes the form

(9) $$\mathrm{vol} = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \frac{\sqrt{|g|}}{n!}\epsilon_{i_1 \cdots i_n} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$$

where $\sqrt{|g|}$ is positive evaluation of the absolute value of the determinant of g. Accordingly

$$*\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = \frac{\sqrt{|g|}}{(n-m)!} g^{i_1 j_1} \cdots g^{i_m j_m} \epsilon_{j_1 \cdots j_m j_{m+1} \cdots j_n} \mathrm{d}x^{i_{m+1}} \wedge \cdots \wedge \mathrm{d}x^{j_n}$$

Taking the Hodge star operation twice produces the identity up to a sign. Concretely, for $\omega \in \Omega^m$

$$* * \omega = \begin{cases} (-1)^{m(n-m)}\omega & \text{Riemannian} \\ (-1)^{m(n-m)+1}\omega & \text{Lorentzian} \end{cases}$$

where the extra minus sign in the Lorentzian case is due to the absolute value of the determinant entering in the definition of the Hodge $*$ operation.

18. Lecture on Dec. 20. – **Inner product, adjoint to** d

- An important application of the Hodge $*$ operation is the definition of an inner product on Ω^m. For $\omega, \eta \in \Omega^m(M)$ we define

$$\begin{aligned} (\cdot, \cdot) \quad &: \quad \Omega^m \times \Omega^m \to \mathbb{R} \\ (\omega, \eta) \quad &\mapsto \quad \int_M \omega \wedge *\eta \end{aligned}$$

For (M, g) Riemannian the inner product $(\cdot, \cdot)$ is positive definite, $(\omega, \omega) > 0$, $\omega \neq 0$.
- Let $\mathrm{d} : \Omega^{m-1} \to \Omega^m$ be the exterior derivative on the deRham complex (Ω, d). Then the adjoint exterior derivative $\delta : \Omega^m \to \Omega^{m-1}$ is defined by

$$\delta = \begin{cases} (-1)^{n(m+1)+1} * \mathrm{d} * & \text{Riemannian} \\ (-1)^{n(m+1)} * \mathrm{d} * & \text{Lorentzian} \end{cases}$$

- **Proposition** Let (M, g) be a compact orientable, (pseudo) Remannian manifold without boundary and $\alpha \in \Omega^m(M)$, $\beta \in \Omega^{m-1}(M)$. Then

$$(\mathrm{d}\beta, \alpha) = (\beta, \delta\alpha).$$

19. Lecture on Dec. 22. – Laplacian, Hodge decomposition

- The Laplace operator on differential forms is defined as

$$\begin{aligned}\Delta : \Omega^m &\to \Omega^m \\ \Delta &= (\mathrm{d}+\delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}\end{aligned}$$

If (M, g) is a compact Riemannian manifold without boundary, then Δ is a semi-positive definite operator since

$$(\omega, \Delta\omega) = (\omega, (\mathrm{d}+\delta)^2\omega) = (\delta\omega, \delta\omega) + (\mathrm{d}\omega, \mathrm{d}\omega) \geq 0$$

- **Definition**
  1) $\omega \in \Omega^m$ is *harmonic* if $\Delta\omega = 0$, *closed* if $\mathrm{d}\omega = 0$ and *co-closed* if $\delta\omega = 0$
  2) $\omega \in \Omega^m$ is *co-exact* if $\omega = \delta\lambda$ for some $\lambda \in \Omega^{m+1}$ everywhere on M.
  3) The set of harmonic form of degree m is denoted by $Harm^m(M)$.
- An m-form is harmonic if it is closed and co-closed.
- **Hodge decomposition Theorem** Let (M, g) be a compact, orientable Remannian manifold without boundary. Then $\Omega^m(M)$ can be decomposed uniquely as

$$\begin{aligned}\Omega^m &= \mathrm{d}\Omega^{m-1} \oplus \delta\Omega^{m+1} \oplus Harm^m \\ \omega_m &= \mathrm{d}\alpha_{m-1} \oplus \delta\beta_{m+1} \oplus \gamma\end{aligned}$$

- **Theorem** Let (M, g) be a compact, orientable Riemannian manifold without boundary. Then

$$H^m(M) \cong Harm^m(M)$$

The isomorphism is provided by identifying $[\omega] \in H^m$ with $P\omega_m$ where P is the projection to the harmonic subspace.

20. Lecture on Jan. 10. – $H^n(M)$ for $\dim(M) = n$ and applications

- **Brouwer's fixed point theorem:** Let $B_1(0) \subset \mathbb{R}^{n+1}$ be the closed unit ball around the origin and $f : B_1(0) \longrightarrow B_1(0)$ a smooth map. Then f has a fixed point, i.e. there is a point $x \in B_1(0)$ such that $f(x) = x$.
- **Remark:** The theorem holds for continuous maps.
- **Proof:** By contradiction. $B_1(0)$ is an $n+1$-manifold with boundary. If there is no fixed point, then

$$\begin{aligned}\psi : B_1(0) &\longrightarrow \partial B_1(0) \\ x &\longmapsto \psi(x) = \text{the intersection of the line through } f(x), x \\ &\qquad \text{with } \partial B_1(0) \text{ which is closer to } x \text{ than to } f(x)\end{aligned}$$

is smooth and satisfies $\psi(x) = x$ for all $x \in \partial B_1(0)$. Let ω be a n-form on $\partial B_1(0) = S^n$ so that $\int_{\partial B_1(0)} \omega \neq 0$.

Let $\iota : \partial B_1(0) \longrightarrow B_1(0)$ denote the inclusion (this is a smooth map). Then $\iota \circ \psi = id_{\partial B_1(0)}$. Hence

$$0 \neq \int_{S^n} \omega = \int_{S^n} (\psi \circ \iota)^* \omega = \int_{S^n} \iota^*(\psi^*\omega).$$

By the Poincaré-Lemma, $\psi^*\omega$ is exact, i.e. there is a $n-1$-form λ such that $\psi^*\omega = d\lambda$. Then

$$0 \neq \int_{S^n} \iota^*(d\lambda) = \int_{S^n} d(\iota^*\lambda) = \int_{\partial S^n = \emptyset} \lambda = 0.$$

yields a contradiction (we used Stokes theorem and the naturality of d). More details can be found in [M]

- **Theorem:** Let M be a closed, oriented, connected manifold of dimension n. Then $H^n(M) \simeq \mathbb{R}$.
- **Proof:** We use the Hodge decomposition, for his we fix a positive definite Riemannian metric on M. The following diagram summarizes fact discussed previously.

$$\begin{array}{ccc} \mathrm{Harm}^k(M) & \xrightarrow{\simeq} & H^k(M) \\ \simeq \Big\downarrow * & & \Big\downarrow \simeq \\ \mathrm{Harm}^{n-k}(M) & \xrightarrow{\simeq} & H^{n-k}(M). \end{array}$$

We apply this to $k = 0$: Then $H^0(M) = \ker(d : \Omega^0(M) = C^\infty(M) \longrightarrow \Omega^1(M))$ consists of functions with vanishing differential. On connected manifolds such functions are constant. Thus $H^n(M) \simeq \{\text{constant functions on } M\} = \mathbb{R}$.

- **Lemma:** Let M be closed manifold, oriented, connected and of dimension n. Then

$$\int_M : H^n(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega$$

is a well defined isomorphism.

- **Proof:** The map is well defined by Stokes theorem: If ω, η represent the same cohomology class, then $\omega - \eta = d\lambda$ and

$$\int_M \omega - \int_M \eta = \int_M d\lambda = \int_{\partial M = \emptyset} \lambda = 0.$$

Linearity is clear, it is surjective since $\int_M (*1) \neq 0$ and we already know that $\dim(H^n(M)) = 1$.

- Under the current assumptions on M this means that ω is exact if and only if $\int_M \omega = 0$.
- **Definition:** Let M, N be closed oriented manifolds of dimension n and $f : M \longrightarrow N$ a smooth map. For $\omega \in \Omega^n(N)$ with $\int_N \omega \neq 0$ define

$$\deg(f) = \frac{\int_M f^*\omega}{\int_N \omega}.$$

This is called the *degree* of f.

- **Remark:** This does not depend on the choice of ω since the integrals only depend on the cohomology class of ω. Because of $H^n(N) \simeq \mathbb{R}$ the choice of a non-vanishing cohomology class is unique up to a non-vanishing factor which is canceled in the definition of the degree.
- **Theorem:** $\deg(f)$ is an integer, it can be computed in terms of the behavior of Df at all points of $f^{-1}(p)$ for a regular value p of f.
- **Fact:** A theorem by Sard implies that regular values of f exist.
- **Proof of Theorem:** Pick a regular value p, consider $f^{-1}(p)$. For each $q \in f^{-1}(p)$ there is a neighborhood $U(q)$ such that $f|_{U(q)}$ is a diffeomorphism onto its image (which is a neighborhood of p). In particular, $f^{-1}(p)$ is finite and one can choose the $U(q)$ pairwise disjoint. We pick ω with support in $\bigcap_q f(U(q))$ such that $\int_N \omega \neq 0$.

Then $\int_M f^*\omega = \sum_{q\in f^{-1}(p)} \int_{U(q)} f|_{U(q)}^*\omega$. By the transformation rule

$$\int_{U(q)} f|_{U(q)}^*\omega = \begin{cases} \int_{f(U(q))}\omega & \text{if } Df_q \text{ is an orientation preserving isomorphism} \\ -\int_{f(U(q))}\omega & \text{if } Df_q \text{ is an orientation reversing isomorphism.} \end{cases}$$

Then $\int_{f(U(q))}\omega = \int_N \omega$ implies

$$\deg(f) = \left|\left\{ \begin{array}{c} q \in f^{-1}(p) \\ Df_q \text{ or. preserving} \end{array} \right\}\right| - \left|\left\{ \begin{array}{c} q \in f^{-1}(p) \\ Df_q \text{ or. reversing} \end{array} \right\}\right| \in \mathbb{Z}.$$

21. Lecture on Jan. 12. – **Mapping degree - Examples**

- **Lemma:** Let M_1, M_2, M_3 be smooth orientable closed connected manifolds of the same dimension and $f : M_1 \longrightarrow M_2, g : M_2 \longrightarrow M_3$ are smooth maps. Then $\deg(g\circ f) = \deg(g)\deg(f)$.
- **Proof:** Let $\omega \in \Omega^n(M_3)$ with $\int_{M_3}\omega \neq 0$ If $\deg(g) \neq 0$, then the proof is
$$\deg(g\circ f) = \frac{\int_{M_1} f^*(g^*\omega)}{\int_{M_3}\omega} = \frac{\int_{M_1} f^*(g^*\omega)}{\int_{M_2} g^*\omega}\cdot\frac{\int_{M_2} g^*\omega}{\int_{M_3}\omega} = \deg(f)\deg(g)$$
where we used $g^*\omega$ to compute $\deg(f)$. That is legitimate since $\deg(g)\neq 0$ implies $\int_{M_2} g^*\omega \neq 0$. If $\deg(g) = 0$, then $g^*\omega = d\lambda$ for some $\lambda \in \Omega^{n-1}(M_2)$. Then
$$\deg(g\circ f) = \frac{\int_{M_1} f^*(g^*(\omega))}{\int_{M_3}\omega} = \frac{\int_{M_1} f^*(d\lambda)}{\int_{M_3}\omega} = \frac{\int_{M_1} d(f^*\lambda)}{\int_{M_3}\omega} = 0 = \deg(f)\deg(g)$$
by Stokes theorem.
- **Examples:** The identity map has degree 1, the antipodal map $A : S^n \longrightarrow S^n, A(x) = -x$ has degree $(-1)^{n+1}$, it has to satisfy $1 = \deg(A^2) = (\deg(A))^2$. The map $\varphi_k : S^1 \longrightarrow S^1, z \mapsto z^k$ has degree k where k is a given integer.
- **Lemma:** The wedge product of closed forms is exact, the wedge product of a closed form with an exact form is exact.
- The proof is a direct computation.
- **Consequence:** If M is a smooth manifold, then $H^*(M)$ is not only a $\mathbb{R}$-vector space. It is a ring! We apply this ring structure to show:
- **Proposition:** Every map of $f : S^2 \longrightarrow T^2$ has degree 0. (Here $T^2 = S^1\times S^1$ is a torus carrying the product orientation.)
- **Proof:** There are two projection maps $\mathrm{pr}_1, \mathrm{pr}_2 : T^2 \longrightarrow S^1$ (on the first/second factor). Let $\alpha \in \Omega^1(S^1)$ such that $\int_{S^1}\alpha \neq 0$. Then $\omega = \mathrm{pr}_1^*\alpha \wedge \mathrm{pr}_2^*\alpha = \alpha_1\wedge\alpha_2$ satisfies
$$\int_{T^2}\omega = \left(\int_{S^1}\alpha\right)^2 \neq 0.$$
We now assume the fact that $H^1(S^2) = \{0\}$ (this will be proved later). The degree of f is then
$$\deg(f) = \frac{\int_{S^2} f^*\alpha_1 \wedge f^*\alpha_2}{\int_{T^2}\omega}.$$
Since $f^*\alpha_1$ is exact ($H^1(S^2) = 0$) it follows that $f^*\alpha_1\wedge f^*\alpha_2$ is also exact. The integral of an exact form over a closed manifold vanishes by Stokes theorem. Therefore $\deg(f) = 0$.
- **Lemma:** $H^1(S^2) = \{0\}$.

- **Proof:** Let $D_H = S^2 \setminus \{(0,0,-1)\}$ and $D_L = S^2 \setminus \{(0,0,1)\}$. These sets are discs, there intersection is connected. Let $\eta \in \Omega^1(S^2)$ be closed. We want to show that η is exact.

  By the Poincaré-Lemma there are 0-forms/smooth functions λ_L respectively λ_H on D_L respectively D_H such that

  $$d\lambda_H = \eta|_{D_H} \qquad\qquad d\lambda_L = \eta|_{D_L}.$$

  If $\lambda_H \equiv \lambda_L$ on $D_L \cap D_H$, then these two forms can be glued to a global form λ such that $d\lambda = \eta$. We modify λ_L to make sure that his works. Note that on $D_H \cap D_L$

  $$d\left(\lambda_H|_{D_H\cap D_L} - \lambda_L|_{D_H\cap D_L}\right) = \eta|_{D_H\cap D_L} - \eta|_{D_H\cap D_L} \equiv 0.$$

  Hence $\lambda_H - \lambda_L$ is constant on $D_H \cap D_L$ (we use that $D_H \cap D_L$ is connected). Let C be the constant and replace λ_L by $\lambda_L + C$.
- **Consequence:** For no Riemannian metric there is a non-trivial harmonic 1-form on S^2.
- **Fact:** After the next theorem, you will have all means needed to prove that for $n > 0$

  $$H^k(S^n) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } 1 \le k \le n-1 \\ \mathbb{R} & \text{if } k = n. \end{cases}$$

- **Theorem:** Let $f, g : M \longrightarrow N$ be smooth maps between manifolds which are homotopic, i.e. there is smooth map $h : M \times \mathbb{R} \longrightarrow N$ such that $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. Then $f^* = g^* : H^*(N) \longrightarrow H^*(M)$.
- **Proof:** The proof is almost the same as the proof of the Poincaré-Lemma. Consider

  $$\Omega^*(N) \xrightarrow{h^*} \Omega^*(M \times \mathbb{R}) \underset{\iota_1^*}{\overset{\iota_0^*}{\rightrightarrows}} \Omega^*(M)$$

  where $\iota_j : M \longrightarrow M \times \mathbb{R}$ is the inclusion $p \mapsto (p, j)$ for $j = 0, 1$. Then $h \circ \iota_0 = f$ and $h \circ \iota_1 = g$. We use the operator $I : \Omega^k(M, \mathbb{R}) \longrightarrow \Omega^{k-1}(M)$ from the proof of the Poincaré-Lemma (with the property $d \circ I + I \circ d = i_1^* - i_0^*$). If $\eta \in \Omega^*(N)$ is closed, then

  $$g^*\eta - f^*\eta = (d \circ I + I \circ d)(h^*\eta) = d(Ih^*\eta).$$

  Hence $[f^*\eta] = [g^*\eta]$.
- This concludes our discussion of the differential topology of manifolds for some time.
- **Theorem:** Let M be a smooth manifold. Then M admits a Riemannian (i.e. positive definite) metric.
- **Proof**: Pick a covering of M by charts (U_i, φ_i) and a subordinate partition of unity ρ_j. Then

  $$g(X,Y) := \sum_j \rho_j(\varphi^*_{i(j)} g_i)(X,Y) = \sum_j \rho_j g_{i(j)}(D\varphi_{i(j)}X, D\varphi_{i(j)}Y)$$

  is positive definite.
- This does not work for Lorentzian metrics.

22. Lecture on Jan. 17. – **Tensors,Connections**

- **Remark:** Formally, a tensor field (r, s) on a manifold is a section of the bundle of **multilinear** maps

$$\underbrace{T^*M \times \ldots \times T^*M}_{r \text{ times}} \times \underbrace{TM \times \ldots \times TM}_{s \text{ times}} \longrightarrow \mathbb{R}.$$

  In many cases, the definition of a tensor will involve several summands some of which are not tensorial in the sense that they take (locally defined) vector fields as input. It is then important to check that the number/vector the tensor returns for a collection of vector fields depends only on the value of the vector fields at a given point.

  Examples include differential forms (a s-form is a $(0, s)$-tensor).

- **Criterion:** Tensors are not only $\mathbb{R}$-linear but linear over functions.
- **Definition:** A *connection* or *covariant differential* on TM is a map

$$\begin{aligned} \chi(M) \times \chi(M) &\longrightarrow \chi(M) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

  such that for all smooth functions f
  - ∇ is $\mathbb{R}$-bilinear,
  - ∇ is linear over smooth functions in the first factor, i.e. $\nabla_{fX} Y = f \nabla_X Y$ and
  - $(\nabla_X (fY))(p) = f(p)(\nabla_X Y)(p) + (Df)(X(p))Y(p)$.

  The same definition applies to the bundle T^*M in defining $\nabla_X \alpha$ with α a 1-form.
- **Definition:** Let ∇ be a connection on TM. The *torsion* of ∇ is the antisymmetric tensor

$$\begin{aligned} T : \chi(M) \times \chi(M) &\longrightarrow \chi(M) \\ (X, Y) &\longmapsto \nabla_X Y - \nabla_Y X - [X, Y]. \end{aligned}$$

- **Definition:** A connection on a manifold with a (possibly indef.) metric g is *metric* if

$$L_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

- **Theorem:** On each semi-Riemannian manifold M there is a unique connection ∇ which is metric and has vanishing torsion. This connection is called the *Levi-Civita connection*. It satisfies

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & L_X g(Y, Z) + L_Y g(X, Z) - L_Z g(X, Y) \\ & - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

- **Examples:** Let M be a submanifold of $\mathbb{R}^n$ and g the restriction of the standard metric to M. Let pr denote the orthogonal projection of $T_p\mathbb{R}^n$ to T_pM (for some $p \in M$). Then the Levi-Civita connection on M is

$$(\nabla_X Y)(p) := \mathrm{pr}(D_X Y)(p) = \mathrm{pr}\left(x^\mu (\partial_\mu y^\nu) \partial_\nu\right)(p).$$

- **Definition:** Let g be a semi-Riemannian metric, ∇ the Levi-Civita connection and x^μ local coordinates near a point. The Christoffel symbols of g are defined by

$$\nabla_{\partial_\nu} \partial_\mu = \Gamma^\kappa_{\nu\mu} \partial_\kappa.$$

- **Lemma:** The Christoffel symbols can be computed by

$$\Gamma^{\kappa}_{\nu\mu} = \frac{1}{2} g^{\kappa m} \left(\frac{\partial g_{\mu m}}{\partial x^{\nu}} + \frac{\partial g_{\nu m}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{m}} \right).$$

23. Lecture on Jan. 19. – **Parallel transport**

- **Definition:** A *connection on* T^*M is a map

$$\nabla : \chi(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$$

which is $\mathbb{R}$-bilinear, $\mathbb{C}^{\infty}$-linear in the first variable and satisfies

$$\nabla_X(f\alpha) = (L_X f)\alpha + f\nabla_X \alpha.$$

- **Definition:** Let ∇ be a connection on TM. Then there is a unique connection ∇' on T^*M such that

$$L_X(\alpha(Y)) = (\nabla'_X \alpha)(Y) + \alpha(\nabla_X Y).$$

Usually, one writes again ∇ instead of ∇'.
- More generally, of A is a (r,s)-tensor and ∇ a connection on TM, then there is a unique operator $\mathcal{D}$ such that $\mathcal{D}$ defined by

$$L_X \left(A(\alpha_1, \ldots, \alpha_r, Y_1, \ldots, Y_s) \right) = \left(\mathcal{D}_X A \right) \left(\alpha_1, \ldots, \alpha_r, Y_1, \ldots, Y_s) \right) + \sum_i A(\ldots, \nabla_X \alpha_i, \ldots, Y_1, \ldots, Y_s) + \sum_j A(\alpha_1, \ldots, \alpha_r, \ldots, \nabla_X Y_j, \ldots)$$

and $\mathcal{D}_X A$ is again a (r,s)-tensor for each vector field X, $\mathcal{D}A$ is a $(r, s+1)$-tensor.
- **Example:** If ∇ is the Levi-Civita connection of (M,g), then $\nabla g \equiv 0$. If ∇ is any connection on M, then the Hessian of $f \in C^{\infty}(M)$ is defined as

$$\mathrm{Hess}(f)(X,Y) = (\nabla'_X(df))Y.$$

This bilinear form is symmetric for all f if and only if ∇ is torsion free.
- **Definition:** Let A be a $(0,s)$-tensor and $f : M \longrightarrow N$ smooth. Then f^*A defined by $(f^*A)(X_1, \ldots, X_s) := A(Df(X_1), \ldots, Df(X_s))$ is a $(0,s)$-tensor, the pull back of A.
- **Definition:** Let M be a manifold and $\gamma : (a,b) \longrightarrow M$ a continuous curve. A *vector field along* γ is a continuous map $X : (a,b) \longrightarrow TM$ such that $X(t) \in T_{\gamma(t)}M$.
- **Definition:** Let ∇ be a connection on TM and γ a smooth curve. Then there is a unique operator

$$\frac{\nabla}{dt} : \{\text{smooth vector fields along } \gamma\} \longrightarrow \{\text{smooth vector fields along } \gamma\}$$

which is linear over $\mathbb{R}$, $\frac{\nabla}{dt}(f_\gamma) = \frac{d}{dt}(f(t))X_\gamma + f(t)\frac{\nabla}{dt}X_\gamma(t)$, and

$$\frac{\nabla}{dt} X_\gamma(t) = \nabla_{\dot\gamma}(t) X$$

for every of X_γ which is the restriction of a local vector field near $\gamma(t)$ to γ.
- **Definition:** A vector field X along γ is called parallel if $\frac{\nabla}{dt} F_\gamma \equiv 0$.

- **Theorem:** Let γ be a smooth curve and $X_0 \in T_{\gamma(0)}M$. Then there is a unique parallel vector field X along γ such that $X(0) = X_0$. If ∇ is metric, then the map which assigns $X(s)$ to X_0 is an isometry.

  The solution of this initial value problem is called parallel transport of X_0 along γ.
- **Definition:** Let $\gamma : [0,1] \longrightarrow M$ be a smooth curve and ∇ a connection on TM. Then

$$\begin{aligned} P_\gamma : T_{\gamma(0)}M &\longrightarrow T_{\gamma(1)}M \\ X_0 &\longmapsto X(1) \end{aligned}$$

  where X is the unique parallel vector field along γ which coincides with X_0 at 0. P_γ is an isomorphism of vector spaces who depends on γ.

24. Lecture on Jan. 24. – **Geodesics, curvature**

- **Remark:** If ∇ is a metric connection on TM and X, Y are vector fields along γ, then

$$\frac{d}{dt}g(X(t), Y(t)) = g\left(\frac{\nabla}{dt}X(t), Y(t)\right) + g\left(X(t), \frac{\nabla}{dt}Y(t)\right).$$

  In particular, parallel transport along γ is an isometry when ∇ is metric.
- **Definition:** Let ∇ be a connection on M. A curve $\gamma : I \longrightarrow M$ is a geodesic when $\dot\gamma$ is parallel along γ, i.e. $\frac{\nabla}{dt}\dot\gamma = 0$.

  This condition is often written as $\nabla_{\dot\gamma}\dot\gamma = 0$ Strictly speaking, one has to extend $\dot\gamma$ to a vector field on M defined on a neighbourhood of $\gamma(t)$.
- **Theorem:** Let $X_0 \in T_pM$ and ∇ a connection. Then there is a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\dot\gamma(0) = X_0$.
- **Proof:** In local coordinates $x^1, \dots, x^n$ around p a curve $\gamma(t) = (c^1(t), \dots, c^n(t))$ is a geodesic if and only if

$$\ddot c^k(t) + \Gamma^k_{ij}(c^1(t), \dots, c^n(t))\dot c^i(t)\dot c^j(t) = 0 \text{ for all } k = 1, \dots, n.$$

  This is a system of ordinary differential equations of order 2 and all coefficients are smooth (we assumed that connections are smooth implicitly by requiring that $\nabla_X Y$ is smooth for smooth X, Y, it would have been more explicit to require that the Christoffel symbols are smooth).

  Standard theorems from the theory of ordinary differential equations then finish the proof.
- **Example:** Straight lines in $(\mathbb{R}^n, g_{st})$ and great circles on $S^n \subset (\mathbb{R}^{n+1}, g_{st})$ (both parametrized by arc length) are geodesics.
- **Remark:** If ∇ is metric and ∇ is metric, $\|\dot\gamma\|$ is constant.
- **Lemma:** Let $\nabla, \overline{\nabla}$ be two connections on TM which have the same geodesics, i.e. γ is a ∇-geodesic if and only if it is a $\overline{\nabla}$-geodesic (we talk about *parametrized* curves).

  Then $\nabla - \overline{\nabla}$ is a *antisymmetric* $(0,2)$-tensor with values in TM. Conversely, if $\nabla - \overline{\nabla}$ is antisymmetric, then both connections have the same geodesics.
- **Proof:** Let $X \in TM$. Then there is a geodesic γ with $\dot\gamma(0) = X$ (at the same time for ∇ and $\overline{\nabla}$. Then

$$\begin{aligned} (\nabla - \overline{\nabla})(X, X) &= \nabla_{\dot\gamma}\dot\gamma - \overline{\nabla}_{\dot\gamma}\dot\gamma \\ &= 0. \end{aligned}$$

i.e. $\nabla - \overline{\nabla}$ is antisymmetric. The converse is simpler.

- **Lemma:** Let ∇ be a connection and A a antisymmetric $(0,2)$-tensor with values in TM. Then the torsion of $\overline{\nabla} = \nabla + A$ is $T^{\overline{\nabla}} = T^{\nabla} + 2A$ where T^{∇} is the torsion of ∇ (A could be $-T^{\nabla}/2$).
- **Proof:** Compute.
- **Consequence:** Define an equivalence relation on the set of connections on TM as follows: $\nabla \sim \overline{\nabla}$ if and only if these connections have the same geodesic. Then each equivalence class contains precisely one torsion free connection.
- **Definition:** The curvature of a connection ∇ is the $(0,3)$-tensor with value in TM

$$
\begin{aligned}
R : \chi(M) \times \chi(M) \times \chi(M) &\longrightarrow \chi(M) \\
(R, Y, Z) &\longmapsto R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\end{aligned}
$$

- **Theorem (Riemann)**: Let ∇ be the Levi-Civita connection of (M, g). Then $R \equiv 0$ if and only if for every $p \in M$ there is a chart $\varphi : U \longrightarrow (\mathbb{R}^n, g_{st})$ around p such that φ is an isometry (and g_{st} is the standard (semi)-Riemannian metric on $\mathbb{R}^n$).
- **Proof:** Step 0: Pick a orthonormal basis $X_1, \ldots, X_n$ of T_pM and a local coordinate system $(y^1, \ldots, y^n)$ around p. We use the Levi-Civita connection througout.

  Step 1: Extend $X_1, \ldots, X_2$ to parallel vector fields along the y^1-coordinate axis $(\cdot, 0, \ldots, 0)$.

  Step 2: Extend the result to parallel vector fields along the curves $(y^1, \cdot, 0, \ldots, 0)$ which are parallel to the y^2-coordinate axis in our coordinates.

  Step 3: Extend the result to parallel vector fields along the curves $(y^1, y^2, \cdot, 0, \ldots, 0)$ which are parallel to the y^3-coordinate axis in our coordinates.

  Step 4: Iterate, obtain vector fields $\hat{X}_1, \ldots, \hat{X}_n$ on a neighbourhood of p. Because the connection is metric, parallel transport is an isometry. Hence $\hat{X}_1, \ldots, \hat{X}_n$ is everywhere orthonormal.

  Step 5: From $R \equiv 0$ we conclude $\frac{\nabla}{dt} \hat{X}_i = 0$ along $(y^1, \cdot, 0, \ldots, 0)$: This uses $[\partial_{y^i}, \partial_{y^j}] = 0$ and

$$
0 = \nabla_{\partial_{y^1}} \nabla_{\partial_{y^2}} \hat{X}_i = \nabla_{\partial_{y^2}} \nabla_{\partial_{y^1}} \hat{X}_i.
$$

  Hence $\nabla_{\partial_{y^1}} \hat{X}_i$ is parallel along $(y^1, \cdot, 0, \ldots, 0)$. By construction $\nabla_{\partial_{y^1}} \hat{X}_i = 0$ at $(y^1, 0, 0, \ldots, 0)$. Hence $\nabla_{\partial_{y^1}} \hat{X}_i = 0$ along $(y^1, \cdot, 0, \ldots, 0)$.

  Iterating this argument we obtain $\nabla_{y^k} \hat{X}_i = 0$ for all i, k on the domain of $\hat{X}_i$. Hence $\nabla_Z \hat{X}_i = 0$ for all Z.

  Step 6: Because ∇ is torsion free

$$
\begin{aligned}
0 &= \nabla_{\hat{X}_i} \hat{X}_j - \nabla_{\hat{X}_j} \hat{X}_i - [\hat{X}_i, \hat{X}_j] \\
&= 0 - 0 - [\hat{X}_i, \hat{X}_j].
\end{aligned}
$$

  Therefore the local flows φ_i of $\hat{X}_i$ and φ_j of $\hat{X}_j$ commute for all i, j.

  Step 7: We construct the coordinates. The following map is well defined for $\varepsilon > 0$ small enough.

$$
\begin{aligned}
\psi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times \ldots (-\varepsilon, \varepsilon) &\longrightarrow M \\
(x^1, x^2, \ldots, x^n) &\longmapsto \varphi_1(x^1) \left(\varphi_2(x^2) \left(\ldots \varphi_n(x^n)(p) \right) \ldots \right).
\end{aligned}
$$

Step 8: These are coordinates with the desired properties. Because all flows commute with each other

$$\begin{aligned}(D\psi)(\partial_{x^i}) &= \frac{d}{dx^i}\psi(x^1,\dots,x^n)\\ &= \frac{d}{dx^i}\varphi_i(t_i)\big(\varphi_1(x^1)\big(\dots\varphi_n(x^n)(p)\big)\dots\big)\\ &= \hat{X}_i(\psi(x^1,\dots,x^n)).\end{aligned}$$

By the inverse function theorem ψ is a diffeomorphism from a neighbourhood of 0 to a neighbourhood of p. The last computation also shows that the coefficients of g in the coordinates given by ψ are constant and coincide with the coefficients of the standard metric.

25. Lecture on Jan. 26. – **Curvature Tensors, Bianchi identities, Examples**

- **Proposition:** The curvature tensor of a Riemannian manifold (M, g) has the following properties:
  (i) $R(X,Y)Z = -R(Y,X)Z$
  (ii) $g(R(X,Y)Z,W) = g(R(Y,X)W,Z)$
  (iii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
  (iv) $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$
- **Proof:** (i) is obvious, (ii) follows from a computation using that ∇ is metric, (iii) uses the fact that ∇ is torsion free (and the Jacobi identity) and (iv) follows from (i),(ii),(iii), cf. [M2], p.54.
- **Definition:** TM valued $(0,3)$-tensors are called curvature tensors if they satisfy the properties in the previous proposition. (iii) is called the first Bianchi identity.
- **Proposition:** The curvature tensor of a Riemannian manifold satisfies the second Bianchi identity

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0.$$

- **Proof:** computation
- It is notoriously laborious to compute the curvature tensor unless the example in question is very symmetric.
- **Definition:** Let (M,g) be a Riemannian manifold and $X, Y \in T_pM$ linearly independent. Then

$$K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - (g(X,Y))^2}$$

- **Warning:** The lecture contained a typo, the above definition is correct. See also (10) below.
- **Example:** The Levi-Civita connection on $\mathbb{R}^n$ with the standard Lorentzian/Riemannian metric has

$$\nabla_X Y = (L_X h^i)\frac{\partial}{\partial x^i}$$

where h^i are the components of Y. The curvature tensor vanishes.
- **Example:** The orthogonal group acts by isometries on (S^n, g) (viewed as submanifold of $\mathbb{R}^{n+1}$) and using this action one can move every 2-plane on TS^n to

any other such plane. Therefore the sectional curvature of S^n is constant, and we have seen that this determines the curvature tensor.

A curvature tensor which yields constant sectional curvature on S^n is

$$g(R(X,Y)Z,W) = K\left(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\right).$$

Since the sectional curvature determines the curvature tensor, this is the right answer (you have to determine K).

- **Example:** Let G be a Lie-group and g a bi-invariant metric. Then $\nabla_X Y = [X,Y]/2$ where X, Y are left-invariant vector fields. This is enough to determine the sectional curvature on G: By the definition of R and the Jacobi identity

(10) $$R(X,Y)Z = \frac{1}{4}[Z,[X,Y]].$$

Because g is Ad-invariant $g(\mathrm{Ad}_{\exp(tZ)*}X, \mathrm{Ad}_{\exp(tZ)}Y) = g(X,Y)$ where φ_t is the flow of the left-invariant vector field Z and $Ad_h : G \longrightarrow G$ maps g to hgh^{-1} (see p. 13 at the bottom). Differentiating with respect to t one gets

$$g([Z,X],Y) + g(X,[Z,Y]) = 0$$

where all vector fields are left invariant. Then

$$K(X,Y) = \frac{g([X,Y],[X,Y])}{4(g(X,X)g(Y,Y) - (g(X,Y))^2)}$$

when g is positive definite.

- **Example:** When $(M,g) = (M_1,g_1) \times (M_2,g_2)$ then the curvature tensor of M decomposes accordingly.

26. Lecture on Jan. 31. – Ricci curvature, divergence, Bochner's theorem

- **Definition:** Let (M,g) be a (semi)-Riemannian manifold and ∇ the Levi-Civita connection and fix an Orthonormal basis E_i of T_pM. Then

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \sum_i g(R(E_i,X)Y,E_i) \text{ respectively}\\ \mathrm{scal} &= \sum_j \mathrm{Ric}(E_j,E_j)\\ &= \sum_{i,j} g(R(E_i,E_j)E_j,E_i)\end{aligned}$$

is the *Ricci*-curvature respectively the scalar curvature. The Ricci curvature is a symmetric $(0,2)$-tensor field, the scalar curvature a smooth function which do not depend on the choice of ONB.

- **Definition:** Let X be a smooth vector field on (M,g) a semi Riemannian manifold of dimension n and *vol* the Riemannian volume form (c.f. (9) on p. 20). Then $L_X\mathrm{vol}$ is a n-form, hence there is a unique function $\mathrm{div}(X)$ such that

$$\mathrm{div}(X)\cdot vol = L_X\mathrm{vol} = di_X\mathrm{vol}.$$

This is the *divergence* of X.

- **Lemma:** With the notation from above $\mathrm{div}(X) = \sum_i g(\nabla_{E_i}X, E_i)$.

- **Proof:** This follows from the definitions and the fact that vol is parallel. This means that $\nabla_Z \mathrm{vol} 0$ for all Z: This can be seen from the definition of vol by extending Z to a local vector field and E_i to a local framing which is parallel along flow lines of Z. (Note that g is parallel because it is metric.)
- **Lemma:** Let X be as above and $\alpha(\cdot) = g(X, \cdot)$ the dual 1-form of X. Then $\delta\alpha = \mathrm{div}(X)$.
- **Proof:** Although this is a local statement, it is convenient to assume that M is closed, oriented. Then for every smooth function f

$$\begin{aligned}(\delta\alpha, f) &= \int_M g(\alpha, df)\mathrm{vol} = \int_M (L_X f)\mathrm{vol} \\ &= -\int_M f(L_X \mathrm{vol}) = \int_M f \mathrm{div}(X)\mathrm{vol}\end{aligned}$$

by Stokes theorem. This implies the claim.
- **Lemma:** In the situation above, α is closed if and only if $V \longmapsto \nabla_V X$ is symmetric.
- **Proof:** Computation using the fact $d\alpha(X, Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X, Y])$.
- **Lemma:** Let $X \in \chi(M)$ so that $g(X, \cdot) = \alpha$ is closed and $f = \|X\|^2/2$ Then
  1. $\mathrm{grad}(f) = \nabla_X X$.
  2. $\nabla_V(\mathrm{grad}(f)) = R(V, X)X + (\nabla_X S)(V) + S(S(V))$ with $S(V) = \nabla_V X$.
- **Proof:** Recall $g(\mathrm{grad}(f), \cdot) = df$. Part 1 is elementary. For part 2 compute using the fact that ∇ is torsion free.
- **Lemma:** In the situation from the previous Lemma:

$$-\Delta f = \|\nabla X\|^2 + g(X, \mathrm{grad}(\mathrm{div}(X))) + \mathrm{Ric}(X, X). \tag{11}$$

- **Poof:** Take traces of the summands in the second part of the previous lemma. Use $\mathrm{trace}(\nabla_X S) = \nabla_X(\mathrm{trace} S)$. This is proved as follows using orthonormal frames $(E_1, \ldots, E_n)$ defined on the neighbourhood of a point.

$$\begin{aligned}\mathrm{trace}(\nabla_X S) &= \sum_i g((\nabla_X S)E_i, E_i) \\ &= \sum_i (L_X(g(S(E_i), E_i)) - g(S(E_i), \nabla_X E_i) - g(S(\nabla_X E_i), E_i)) \\ &= \nabla_X \left(\sum_i g(S(E_i), E_i) \right) - \sum_i (g(S(E_i), \nabla_X E_i) + g(S(\nabla_X E_i), E_i)) \\ &= \nabla_X(\mathrm{trace}(S)) - 2\sum_i g(S(E_i), \nabla_X E_i) \\ &= \nabla_X(\mathrm{trace}(S)) - 2\sum_i \left(g\left(\sum_j g(S(E_i), E_j)E_j, \nabla_X E_i \right) \right) \\ &= \nabla_X(\mathrm{trace}(S)) - 2\sum_{i,j} g(S(E_i), E_j) g(E_j, \nabla_X E_i) \\ &= \nabla_X(\mathrm{trace}(S)).\end{aligned}$$

TO get to the second line use the definition of ∇_S, rearrange to the third, use symmetry of S to get to the fourth. To get to the fifth line write $S(E_i)$ in terms of the orthonormal basis E_j, rearrange to get to the sixth. The last equality follows from the symmetry of S on the one hand, and the fact that $0 = L_X(g(E_i, E_j)) = g(\nabla_X E_i, E_j) + g(E_i, \nabla_X E_j)$ (symmetric matrices are orthogonal to antisymmetric matrices).

- **Theorem (S. Bochner, 1946)**: Let (M, g) be a connected positive definite Riemmanian manifold such that $\mathrm{Ric}(V, V) \geq 0$ for all $V \in TM$.

  If α is a harmonic 1-form, then the vector field dual to it is parallel. There are at most n linearly independent harmonic 1-forms.

  If there is a point p where $\mathrm{Ric}(V, V) > 0$ for all $0 \neq V \in T_pM$, then all harmonic 1-forms vanish.
- **Proof:** Let α be harmonic, hence closed and coclosed. Use (11). Note that $\int_M (\Delta f) \cdot 1\mathrm{vol} = 0$ because Δ is symmetric. $\mathrm{div}(X)$ vanishes because $\delta\alpha = 0$ by a previous lemma. Therefore

$$\begin{aligned} 0 &= \int_M \Delta f \mathrm{vol} \\ &= \int_M \left(\|\nabla X\|^2 + \mathrm{Ric}(X, X)\right) \mathrm{vol}. \end{aligned}$$

  Hence $\nabla X \equiv 0$, i.e. X is parallel. Because M is connected X is determined by its value at one point. The space of parallel vector fields is at most n-dimensional. If $\mathrm{Ric}(V, V) > 0$ for all $0 \neq V \in T_pM$, then X has to vanish at that point and X vanishes every where. Then there is no non-trivial harmonic 1-form.
- **Example:** The sphere satisfies the assumptions of the theorem and we know $H^1(S^n) = 0$ when $n \geq 2$. The torus $T^n = S^1 \times \ldots S^1$ shows that n linearly independent 1-forms can indeed arise. T^n has no metric with $\mathrm{Ric} \geq 0$ such that the inequality is strict in one point.

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